640:350: LINEAR ALGEBRA HIGHLIGHTS OF LECTURES FOR WEEK 10

Sections to be covered: 7.1, 7.2.

We will not have time to do a full discussion of these two sections. The main objectives are to learn: (i) what Jordan canonical forms are; (ii) that Jordan canonical forms may arise in attempting to diagoanalize a matrix; (iii) how the forms and parameters in the Jordan canonical forms of a matrix relate to the eigenvalues (and other data) of the matrix, and (iv) how the Jordan canonical forms are used in applications.

The discussion in these sections of the text is a difficult one. We will not follow it closely and will provide an alternative discussion in a handout. Full understanding of the detailed discussions will not be required, except for the following concepts and properties.

(1) The definition of Jordan canonical forms (blocks) and Jordan canonical basis.
(2) The concept of generalized eigenvectors and how they appear in a Jordan canonical basis.
(3) The concept of generalized eigenspace and its properties, **Theorem 7.1** and **Theorem 7.2.**
(4) possible forms (# of blocks and block sizes) and parameters in the Jordan canonical forms of a matrix.
(5) The algorithm in the handout to construct a basis for each generalized eigenspace consisting of cycles of generalized eigenvectors.
(6) How Jordan canonical forms are used in applications—see # 19 of 7.2.

For item (2), suppose that the following $p \times p$ Jordan block appears in the matrix representation of the linear operator T in some ordered basis β:

$$J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

Then β contains an ordered portion $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$, such that

$$T(\mathbf{v}_1) = \lambda \mathbf{v}_1, \quad T(\mathbf{v}_2) = \mathbf{v}_1 + \lambda \mathbf{v}_2, \quad \cdots, \quad T(\mathbf{v}_p) = \mathbf{v}_{p-1} + \lambda \mathbf{v}_p,$$

which can be rewritten as

$$[T - \lambda I]\, \mathbf{v}_1 = 0, \quad [T - \lambda I]\, \mathbf{v}_2 = \mathbf{v}_1, \quad \cdots, \quad [T - \lambda I]\, \mathbf{v}_p = \mathbf{v}_{p-1},$$

which shows that each of $\mathbf{v}_i$ is obtained from applying $T - \lambda I$ to $\mathbf{v}_{i+1}$, and can be further written as

$$[T - \lambda I]^p\, \mathbf{v}_p = 0, \quad \mathbf{v}_1 = [T - \lambda I]^{p-1}\, \mathbf{v}_p, \quad \cdots \quad \mathbf{v}_{p-1} = [T - \lambda I]\, \mathbf{v}_p.$$

Thus β contains an ordered portion of the form $\{[T - \lambda I]^{p-1}\, \mathbf{v}_p, \cdots, [T - \lambda I]\, \mathbf{v}_p, \mathbf{v}_p\}$ with $[T - \lambda I]^p\, \mathbf{v}_p = \mathbf{0}$, namely, $\mathbf{v}_p \in K_\lambda$. So we see that β contains a cycle of generalized eigenvectors of T corresponding to λ.

Note that if we reverse the ordering in $\{[T - \lambda I]^{p-1}\, \mathbf{v}_p, \cdots, [T - \lambda I]\, \mathbf{v}_p, \mathbf{v}_p\}$ to $\{\mathbf{v}_p, [T - \lambda I]\, \mathbf{v}_p, \cdots, [T - \lambda I]^{p-1}\, \mathbf{v}_p\}$, then the matrix representation of T in this ordered basis becomes

$$\begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \lambda \end{pmatrix},$$

which is J^t, the transpose of J. This proves that J is similar to J^t.