640:350: LINEAR ALGEBRA HIGHLIGHTS OF LECTURES FOR WEEK 1

Sections to be covered: 1.1–1.3.

The following is a list of **key points** of sections 1.1–1.3.

(1) Quick review of vector additions and scalar multiplication of vectors, their geometric interpretations (entities having a magnitude and direction, and represented by arrows, or directed line segments; parallelogram laws for vector additions).

(2) Any point in a plane or space can be identified as a vector (once an origin has been chosen). Given two distinct points A and B, then any point lying on straightline through A and B can be represented either as

$$(1-t)A + tB,$$

or as

$$A + t(B - A)$$

for some scalar t. These two are consistent, as in the second case, $A + t(B - A) = (1 - t)A + tB$.

(3) Likewise, given three non-collinear points A, B, and C, let $\mathbf{u} = A - C$ denote the vector pointing from C to A, $\mathbf{v} = B - C$ denote the vector pointing from C to B, then any point lying on the plane through A, B, and C can be written as

$$C + s\mathbf{u} + t\mathbf{v} \quad \text{for some scalars } s \text{ and } t.$$

Thus if we take C to be the origin, then the set of points $\{sA + tB : s, t \in \mathbb{R}\}$ forms the plane containing A, B, and the origin; and among all points $sA + tB$ in the plane containing A, B, and the origin, $sA + tB$ lies on the straightline through A and B iff $s + t = 1$.

(4) Many mathematical objects do not have a simple interpretation as having a magnitude and direction, yet have natural operations which behave the same as the vector additions and scalar multiplication of vectors. These properties can be summarized as the 8 properties on p. 7. Any collection of mathematical objects, with two operations defined, called additions and scalar multiplications, obeying the 8 properties on p. 7, is called a *vector space*. In addition of the usual vector space consisting of vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, we will often encounter

$\mathbb{M}_{m\times n}(\mathbb{R})$: the vector space of all $m \times n$ matrices with entries in $\mathbb{R}$.

$\mathbb{M}_{m\times n}(\mathbb{C})$: the vector space of all $m \times n$ matrices with entries in $\mathbb{C}$.

$\mathcal{F}(\mathbb{R}^n, \mathbb{R})$: the vector space of all $\mathbb{R}$-valued functions defined on all of $\mathbb{R}^n$.

$\mathcal{F}(S, \mathbb{R})$: the vector space of all $\mathbb{R}$-valued functions defined on S, where S is a given set (often a subset of $\mathbb{R}^n$).

$C(S, \mathbb{R})$: the vector space of all $\mathbb{R}$-valued *continuous* functions defined on S, where S is a given subset of $\mathbb{R}^n$.

$C^k(\mathbb{R}^n, \mathbb{R})$: the vector space of all $\mathbb{R}$-valued functions defined on $\mathbb{R}^n$ which have *continuous derivatives* of order up to and including k.

$P(\mathbb{R})$: the vector space of all polynomials with coefficients in $\mathbb{R}$.

$P_n(\mathbb{R})$: the vector space of all polynomials of degree n or less and with coefficients in $\mathbb{R}$.

(5) Make sure to understand the convention for the *degree* of a polynomial, in particular for polynomials of degree 0 and -1.

(6) Learn how some familiar rules in arithmetic carry out to vector additions and *how* they are derived from the 8 properties on p. 7: **Theorems 1.1** and **1.2, Corollaries 1** and **2**.
(7) Learn the definition of a *subspace* of a vector space and **Theorems 1.3**, which is often used to verify that a given subset of a vector space is a subspace.
(8) Understand that the set of vectors lying on a straightline through the origin forms a subspace, but not the set of vectors lying on a straightline which does not pass through the origin. The latter is called an affine subspace.
(9) The following subspaces are often encountered.
   - The set of $n \times n$ diagonal (or symmetric, or upper triangular) matrices with entries in $\mathbb{R}$ forms a subspace of $\mathbb{M}_{n\times n}(\mathbb{R})$.
   - $C^k(\mathbb{R}^n, \mathbb{R})$ is a subspace of $C(\mathbb{R}^n, \mathbb{R})$, which in turn is a subspace of $\mathcal{F}(\mathbb{R}^n, \mathbb{R})$.
   - The set of all *even* (or *odd, T-periodic*) functions in $\mathcal{F}(\mathbb{R}^n, \mathbb{R})$.
   - $P_n(\mathbb{R})$ is a subspace of $P(\mathbb{R})$.
   - The set of solutions to the linear differential equation $3x''(t) - 2x'(t) + 5x(t) = 0$ for $t \in \mathbb{R}$.