Name:_______________ ID Number (last 4 digits only): _________

MIDTERM II — Math 350:01

Please be advised that, while you are working on one part of a problem, you may use the information from the other parts, including assertions made in the statement of the other parts.

1. (20 points) Answer True or False to each of the following questions. If you answer "Yes" to a question, please provide a brief supporting argument; if you answer "False" to a question, please provide a counter-example to the statement, or demonstrate why it is false.

(i) Elementary row operations preserve the null space and column space of the matrices.

F

Null Space is preserved, but not column space.

$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathrm{Col}(A) = \mathrm{Span}\{\begin{pmatrix}1\\1\end{pmatrix}\}$, $\mathrm{Col}(\mathrm{rref}(A)) = \mathrm{Span}\{\begin{pmatrix}1\\0\end{pmatrix}\}$

(ii) Similar matrices have the same ranks. T

If $A = PBP^{-1}$ for invertible P, P will be product of elementary matrices, and P^{-1} also product of elementary matrices, multiplication by elementary matrices preserve ranks.

(iii) The sum of two eigenvectors of a linear operator T is always an eigenvector of T. F

$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\vec{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ are eigenvectors of L_A, yet

$\vec{v}_1 + \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is not: $L_A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

(iv) If A is an $n \times n$ matrix such that $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b} \in \mathbb{R}^n$, then $\det A \neq 0$.

T

The given info $\Rightarrow$ A has a pivot in each row. Since A is $n \times n$,

$\Rightarrow$ A is invertible $\Rightarrow$ $\det(A) \neq 0$

(v) If λ is an eigenvalue for a linear operator T, then λ^2 is an eigenvalue for T^2.

$\exists\ \vec{v}_0 \neq \vec{0}$, s.t. $T(\vec{v}_0) = \lambda \vec{v}_0$. Applying T one more time, $T(T(\vec{v}_0)) = T(\lambda \vec{v}_0) = \lambda T(\vec{v}_0) = \lambda^2 v_0$, i.e. $T^2(\vec{v}_0) = \lambda^2 \vec{v}_0$. T

2. Let β denote the standard ordered basis for $\mathbb{R}^2$:

$$\left\{e_1 = \begin{pmatrix}1\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\end{pmatrix}\right\},$$

and

$$\beta' = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right\}$$

be another ordered basis for $\mathbb{R}^2$, $L_A : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be the linear operator defined by

$$L_A(\mathbf{v}) = A\mathbf{v}, \quad \text{for } \mathbf{v} \in \mathbb{R}^2, \text{ where } \; A = \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}.$$

(i) (6 points) Find the change of coordinate matrix Q from β to β'.

$\vec{v}_1 = \begin{pmatrix}1\\1\end{pmatrix} = 1 \cdot e_1 + 1 \cdot e_2$, $\vec{v}_2 = \begin{pmatrix}1\\-1\end{pmatrix} = 1 \cdot e_1 + (-1)\, e_2$.

$\therefore [\vec{v}_1 \;\; \vec{v}_2] = [e_1 \;\; e_2] \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$,

and $Q = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$ is the change of coordinate matrix.

(ii) (6 points) Given $[\mathbf{v}]_\beta = \begin{pmatrix}-2\\1\end{pmatrix}$, and $[\mathbf{w}]_{\beta'} = \begin{pmatrix}-1\\2\end{pmatrix}$. Find $[\mathbf{v}]_{\beta'}$ and $[\mathbf{w}]_\beta$.

From $[v]_\beta = Q\,[v]_{\beta'}$, $\Rightarrow \begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}[v]_{\beta'}$. $\Rightarrow$

$[v]_{\beta'} = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}^{-1}\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}$. Check: $-\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{3}{2}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-2\\1\end{pmatrix}$

$[w]_\beta = Q\,[w]_{\beta'} = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}1\\-3\end{pmatrix}$

Or: $[w]_{\beta'} = \begin{pmatrix}-1\\2\end{pmatrix} \Rightarrow \vec{w} = (-1)\begin{pmatrix}1\\1\end{pmatrix} + (2)\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\-3\end{pmatrix} = 1 \cdot e_1 + (-3)\, e_2$

$\therefore [w]_\beta = \begin{pmatrix}1\\-3\end{pmatrix}$

turn over to continue.

Continuation of # 2 from previous page.

(ii) (8 points) Find the matrix representation $[L_A]_{\beta'}$ of L_A in the basis β'.

$$[L_A]_\beta = A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$Q\,[L_A]_{\beta'} \cdot Q^{-1} = [L_A]_A = A, \Rightarrow$$

$$[L_A]_{\beta'} = Q^{-1} A Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

This indicates that $L_A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $L_A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Check: $L_A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

$L_A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Or directly compute

$L_A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$L_A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$= 0\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

So $[L_A]_{\beta'} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$.

3. Define $T : P_2(\mathbb{R}) \mapsto \mathbb{R}^3$ by $T(f(x)) = (f(0), f(0), f(1))$.

(i) (6 points) Determine the rank of T.

$\{1, x, x^2\}$ is an ordered basis for $P_2(\mathbb{R})$.

$T(1) = (1, 1, 1)$, $T(x) = (0, 0, 1)$, $T(x^2) = (0, 0, 1)$

So $R(T) = \text{span}\{(1,1,1), (0,0,1), (0,0,1)\} = \text{span}\{(1,1,1), (0,0,1)\}$,

$\{(1,1,1), (0,0,1)\}$ is a basis for ~~$P_2(\mathbb{R})$~~ $R(T)$ & $\text{rank}(T) = 2$.

(ii) (8 points) Determine the null space of T— you need to describe elements in the null space of T as a subset of $P_2(\mathbb{R})$, not just in terms of their coordinates in some basis.

Let $f = a + bx + cx^2$, $T(f) = a\,T(1) + b\,T(x) + c\,T(x^2)$

$= a(1,1,1) + b(0,0,1) + c(0,0,1)$

$= (a, a, a+b+c) = 0$

iff $a = 0$ & $a+b+c = 0$. This system has sol'ns

$\begin{pmatrix} 0 \\ -c \\ c \end{pmatrix} = c \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, for any c.

Thus $N(T) = \{a + b + cx^2 : a = 0, b = -c\} = \{-c + cx^2 \mid c \in \mathbb{R}\} = \text{span}\{-1 + x^2\}$.

(iii) (6 points) Determine the range $R(T)$ of T.

From (i), $R(T) = \text{span}\{(1,1,1), (0,0,1)\}$

4. Suppose that the linear operator $T : V \to V$ on an n dimensional vector space V has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(E_{\lambda_1}) = n - 1$, where E_{λ_1} is the eigenspace of T corresponding to the eigenvalue λ_1.

(a). (8 points) Prove that T is diagonalizable.

From $\dim(E_{\lambda_1}) = n-1$, we can choose a basis $\{\vec{v}_1, \cdots \vec{v}_{n-1}\} \subseteq E_{\lambda_1}$ for E_{λ_1}. Since λ_2 is another e.v. for T, we can choose an eigenvector $\vec{v}_n$ for T corresponding to λ_2. A Theorem we learned implies that $\{\vec{v}_1, \cdots, \vec{v}_{n-1}, \vec{v}_n\}$ remains l.i. But $\dim(V) = n$, thus $\{\vec{v}_1, \cdots, \vec{v}_{n-1}, \vec{v}_n\}$ forms a basis for V, and T is diagonalizable.

(b). (6 points) Prove that the characteristic polynomial of T is $f_T(t) = (-1)^n(t-\lambda_1)^{n-1}(t-\lambda_2)$.

Working with the basis β in (a), we find

$$[T]_\beta = \begin{bmatrix} \lambda_1 & & & 0 \\ & \ddots & & \\ & & \lambda_{n-1} & \\ 0 & & & \lambda_n \end{bmatrix}$$

Thus the char. poly. of T is given by

$$\det([T]_\beta - tI) = \det\begin{pmatrix} \lambda_1 - t & & & 0 \\ & \ddots & & \\ & & \lambda_{n-1}-t & \\ 0 & & & \lambda_n - t \end{pmatrix} = (\lambda_1 - t)\cdots(\lambda_{n-1}-t)(\lambda_n - t)$$
$$= (-1)^n (t-\lambda_1)\cdots(t-\lambda_{n-1})(t-\lambda_n)$$

(c). (6 points) Let $g(t) = (t - \lambda_1)(t - \lambda_2)$. Prove that $g(T) = T_0$. (Hint: Choose a basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ for V and prove that $g(T)(\mathbf{v}_i) = \mathbf{0}$ for each $i = 1, \cdots, n$.)

Working with the basis β in (a), for any $\vec{v}_i$, $1 \le i \le n-1$

$$T(\vec{v}_i) = \lambda_1 \vec{v}_i, \Rightarrow g(T)(\vec{v}_i) = (T - \lambda_1 I)(T - \lambda_2 I)(\vec{v}_i)$$
$$= (T - \lambda_2 I)\{(T - \lambda_1 I)(\vec{v}_i)\}$$
$$= (T - \lambda_2 I)\{\vec{0}\} = \vec{0},$$

for $\vec{v}_n$, $T(\vec{v}_n) = \lambda_2 \vec{v}_n$, $\Rightarrow g(T)(\vec{v}_n) = (T - \lambda_1 I)(T - \lambda_2 I)(\vec{v}_n) = 0$.

Thus, for any $\vec{v} \in V$, we have $\vec{v} = c_1 \vec{v}_1 + \cdots + c_{n-1}\vec{v}_{n-1} + c_n \vec{v}_n$ for some scalars $c_1, \cdots c_n$. $\Rightarrow$

$$g(T)(\vec{v}) = c_1 g(T)(\vec{v}_1) + \cdots + c_{n-1} g(T)(\vec{v}_{n-1}) + c_n g(T)(\vec{v}_n) = \vec{0},$$

5. Let T be the linear operator of $P_3(\mathbb{R})$ defined by

$$T(f(x)) = f(x) + 2f'(x) + 3f''(x),$$

and W be the T-cyclic subspace of $P_3(\mathbb{R})$ generated by $f(x) = x$.

(a). (8 points) Find the matrix representation of T under the standard ordered basis of $P_3(\mathbb{R})$.

$T(1) = 1, \quad T(x) = x + 2, \quad T(x^2) = x^2 + 4x + 6, \quad T(x^3) = x^3 + 6x^2 + 18x$

So $[T(1), \ T(x) \ \ T(x^2) \ \ T(x^3)] = (1 \ \ x \ \ x^2 \ \ x^3)\begin{bmatrix} 1 & 2 & 6 & 0 \\ 0 & 1 & 4 & 18 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

and $\begin{bmatrix} 1 & 2 & 6 & 0 \\ 0 & 1 & 4 & 18 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is the matrix rep. of T under the standard basis of $P_3(R)$.

(b). (8 points) Find the characteristic polynomial $f_T(t)$ of T and determine whether T is diagonalizable.

We can use the matrix rep. in (a) to find the char. poly of T as

$$\det \begin{bmatrix} 1-t & 2 & 6 & 0 \\ 0 & 1-t & 4 & 18 \\ 0 & 0 & 1-t & 6 \\ 0 & 0 & 0 & 1-t \end{bmatrix} = (1-t)^4.$$

The only eigenvalue of T is $t = 1$, with multiplicity 4.

The eigenspace E_1 of T corresponding to $t=1$ is given by

$\begin{bmatrix} 0 & 2 & 6 & 0 \\ 0 & 0 & 4 & 18 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. This system has 1 free-variable, so the sol'n space is 1-dim'l, which is < 4. Thus T is not diagonalizable.

turn over to continue.

Continuation of #5 from previous page.

(c). (10 points) Find a basis for W and the matrix representation under this basis for T_W, the restriction of T to W.

$W = \text{Span}\{x, T(x), T^2(x), \cdots\}$.

Since $T(x) = x+2$, $T^2(x) = T(x+2) = T(x) + 2T(1)$

$= x+4 \in \text{Span}\{x, T(x)\}$,

as $x+4 = (-1)x + 2\cdot(x+2) = (-1)x + 2T(x)$,

$\beta_W = \{x, T(x) = x+2\}$ forms a basis for W.

$T_W(x) = x+2$, $T_W(x+2) = (-1)x + 2(x+2)$, so

$[T_W]_{\beta_W} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ is the matrix rep. of T_W

under the basis β_W.

(d). (4 points) Find the the characteristic polynomial of T_W.

The char. poly of T_W is found by

$$\det\begin{bmatrix} 0-t & -1 \\ 1 & 2-t \end{bmatrix} = -t(2-t)+1 = t^2-2t+1$$
$$= (t-1)^2.$$