640:350:01 Homework 18-20

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7.1.2(b)

Find a basis for each generalize eigenspace of L_A consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of A. Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

Letting γ be the standard ordered basis for $\mathbb{R}^2$, the characteristic polynomial of L_A is

$$f(t) = \det([L_A]_\gamma - tI) = \det \begin{bmatrix} 1-t & 2 \\ 3 & 2-t \end{bmatrix} = (1-t)(2-t) - (2)(3) = t^2 - 3t - 4$$
$$= (t+1)(t-4)$$

Thus the eigenvalues of L_A are $\lambda_1 = -1$ and $\lambda_2 = 4$, each having multiplicity 1. Thus, by Theorem 7.4, we know $\dim(K_{\lambda_1}) = \dim(K_{\lambda_2}) = 1$ and hence $K_{\lambda_1} = E_{\lambda_1}$ and $K_{\lambda_2} = E_{\lambda_2}$. So we can find the generalized eigenspaces K_{λ_1} and K_{λ_2} as follows.

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow K_{\lambda_1} = E_{\lambda_1} = \left\{ t \begin{bmatrix} -1 \\ 1 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$$
$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \to \begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow K_{\lambda_2} = E_{\lambda_2} = \left\{ t \begin{bmatrix} 2 \\ 3 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$$

We see that $\beta_1 = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ and $\beta_2 = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ are ordered bases for the generalized eigenspaces. Thus, by Theorem 7.4, β_1 and β_2 are disjoint and their union $\beta = \beta_1 \cup \beta_2$ is an ordered basis for $\mathbb{R}^2$. Therefore the Jordan canonical form J of A is thus

$$J = [A]_\beta = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$$

since

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

7.1.3(c)

Find a basis for each generalized eigenspace of T consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of T. Given $T: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ and $T(A) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A$ for all A in $M_{2\times 2}(\mathbb{R})$.

Letting γ be the standard ordered basis for $M_{2\times 2}(\mathbb{R})$, we see that $[T]_\gamma$ is

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$[T]\gamma = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial of T is

$$f(t) = \det\left([T]_\gamma - tI\right) = \det\begin{bmatrix} 1-t & 0 & 1 & 0 \\ 0 & 1-t & 0 & 1 \\ 0 & 0 & 1-t & 0 \\ 0 & 0 & 0 & 1-t \end{bmatrix} = (1-t)^4$$

Thus the only eigenvalue of T is $\lambda_1 = 1$, having multiplicity 4. So, by Theorem 7.4, we know $\dim\left(K_{\lambda_1}\right) = 4$ and thus $K_{\lambda_1} = M_{2\times 2}(\mathbb{R})$. Prior to finding the generalized eigenspace, we find the eigenspace.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_1} = \left\{ s\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \middle| s,t \in \mathbb{R} \right\}$$

Thus $u_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $v_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ are the eigenvectors of T. We now find the cycles of generalized eigenvectors having u_1 and v_1 as their respective initial vectors.

$$\text{Solutions to } (T - \lambda_1 I)(u_2) = u_1: \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow u_2 = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \text{ where } a,b \in \mathbb{R}$$

$$\text{Solutions to } (T - \lambda_1 I)(u_3) = u_2: \begin{bmatrix} 0 & 0 & 1 & 0 & a \\ 0 & 0 & 0 & 1 & b \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{No solutions}$$

Thus, arbitrarily letting $a = b = 0$, a cycle of generalized eigenvectors having u_1 as its initial vector is $\beta_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$.

$$\text{Solutions to } (T-\lambda_1 I)(v_2) = v_1: \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \text{ where } a, b \in \mathbb{R}$$

$$\text{Solutions to } (T-\lambda_1 I)(v_3) = v_2: \begin{bmatrix} 0 & 0 & 1 & 0 & a \\ 0 & 0 & 0 & 1 & b \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{No solutions}$$

Thus, arbitrarily letting $a = b = 0$, a cycle of generalized eigenvectors having v_1 as its initial vector is $\beta_2 = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. We see that $\beta = \beta_1 \cup \beta_2$ is an ordered basis for K_{λ_1} consisting of a union of disjoint cycles of generalized eigenvectors. Thus the Jordan canonical form J of T is

$$J = [T]_\beta = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

7.1.7(d)

Let T be a linear operator on V, and let λ be an eigenvalue of T. Prove that if $\operatorname{rank}((T-\lambda I)^m) = \operatorname{rank}((T-\lambda I)^{m+1})$ for some integer m, then $K_\lambda = N((T-\lambda I)^m)$.

Proof:

Let m be a positive integer. Assume $\operatorname{rank}((T-\lambda I)^m) = \operatorname{rank}((T-\lambda I)^{m+1})$.

(i) Let $x \in K_\lambda$. Then, by definition, there exists some positive integer p such that $(T-\lambda I)^p(x) = 0$, that is, $x \in N((T-\lambda I)^p)$. If $p < m$, then $(T-\lambda I)^m(x) = (T-\lambda I)^{m-p} \circ (T-\lambda I)^p(x) = (T-\lambda I)^{m-p}(0) = 0$ and thus $x \in N((T-\lambda I)^m)$. If $p \geq m$, then, by part (c) we know $N((T-\lambda I)^m) = N((T-\lambda I)^p)$ and thus $x \in N((T-\lambda I)^m)$. In either case, $x \in N((T-\lambda I)^m)$, thus $K_\lambda \subseteq N((T-\lambda I)^m)$.

(ii) Let $x \in N((T-\lambda I)^m)$. Then $(T-\lambda I)^m(x) = 0$. So, by definition, $x \in K_\lambda$ and thus $K_\lambda \supseteq N((T-\lambda I)^m)$.

Therefore we can conclude that $K_\lambda = N((T-\lambda I)^m)$.

7.2.3(d)

Let T be a linear operator on a finite-dimensional vector space V with Jordan canonical form

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

For each eigenvalue λ_i, find the smallest positive integer p_i for which $K_{\lambda_i} = N((T - \lambda_i I)^{p_i})$.

The eigenvalues of T are $\lambda_1 = 2$ and $\lambda_2 = 3$. From the first two Jordan blocks, we see that the eigenvalue $\lambda_1 = 2$ has two corresponding cycles of generalized eigenvectors in the Jordan basis, containing 3 vectors and 1 vector, respectively. Thus $p_1 = \max\{3, 1\} = 3$. From the last two Jordan blocks, we can conclude that the eigenvalue $\lambda_2 = 3$ has two corresponding cycles of generalized eigenvectors in the Jordan basis, each containing 1 vector. Thus $p_2 = \max\{1, 1\} = 1$.

7.2.5(a)

Find a Jordan canonical form J of T and a Jordan canonical basis β for T, where V is the real vector space of functions spanned by the set of real-valued functions $\{e^t, te^t, t^2e^t, e^{2t}\}$ and T is the linear operator on V defined by $T(f) = f'$.

Let $\gamma = \{e^t, te^t, t^2e^t, e^{2t}\}$ be a basis for V. Then the matrix representation $[T]_\gamma$ is

$$T(e^t) = e^t = [e^t \quad te^t \quad t^2e^t \quad e^{2t}] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad T(te^t) = te^t + e^t = [e^t \quad te^t \quad t^2e^t \quad e^{2t}] \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(t^2e^t) = t^2e^t + 2te^t = [e^t \quad te^t \quad t^2e^t \quad e^{2t}] \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$[T]_\gamma = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$T(e^{2t}) = 2e^{2t} = [e^t \quad te^t \quad t^2e^t \quad e^{2t}] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

We now find the characteristic polynomial f_T of T.

$$f_T(t) = \det([T]_\gamma - tI) = \det\begin{bmatrix} 1-t & 1 & 0 & 0 \\ 0 & 1-t & 2 & 0 \\ 0 & 0 & 1-t & 0 \\ 0 & 0 & 0 & 2-t \end{bmatrix} = (1-t)^3(2-t)$$

Thus the eigenvalues of T are $\lambda_1 = 1$ and $\lambda_2 = 2$, having multiplicities 3 and 1, respectively. We now find the eigenspaces corresponding to each eigenvalue.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_1} = \left\{ [e^t \quad te^t \quad t^2e^t \quad e^{2t}]s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| s \in \mathbb{R} \right\}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_2} = \left\{ [e^t \quad te^t \quad t^2e^t \quad e^{2t}]s \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}$$

Thus $f_1(t) = e^t$ and $g_1(t) = e^{2t}$ are eigenvectors corresponding to eigenvalues λ_1 and λ_2, respectively. We now find cycles of generalized eigenvectors having f_1 and g_1 as their respective initial vectors. Note that their lengths must equal the respective multiplicities of λ_1 and λ_2, since there is only one such cycle per eigenvalue.

$$\text{Solutions to } (T - \lambda_1 I)(f_2) = f_1\text{:} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow f_2(t) = ae^t + te^t \text{ where } a \in \mathbb{R}$$

$$\text{Solutions to } (T - \lambda_1 I)(f_3) = f_2\text{:} \begin{bmatrix} 0 & 1 & 0 & 0 & a \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 0 & a \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow f_3(t) = be^t + ate^t + \frac{1}{2}t^2e^t \text{ where } b \in \mathbb{R}$$

Thus, arbitrarily letting $a = b = 0$, the cycle of generalized eigenvectors having f_1 as its initial vector is $\beta_1 = \left\{e^t, te^t, \frac{1}{2}t^2e^t\right\}$, which is a basis for K_{λ_1}. Since the cycle of generalized eigenvectors having g_1 as its initial vector has length 1, we know it is $\beta_2 = \{e^{2t}\}$, which is a basis for K_{λ_2}. We see that $\beta = \beta_1 \cup \beta_2$ is a Jordan canonical basis for V and that the Jordan canonical form J of T is

$$J = [T]_\beta = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$