640:350:01 Homework 15

Timothy J. Shields

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5.1.3(b)

Determine all the eigenvalues of A, and, for each eigenvalue λ, find the set of eigenvectors corresponding to λ. If possible, find a basis for $\mathbb{R}^3$ consisting of eigenvectors of A. If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

$$A = \begin{bmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{bmatrix}$$

The eigenvalues of A are the roots of its characteristic polynomial.

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -2 & -3 \\ -1 & 1-\lambda & -1 \\ 2 & 2 & 5-\lambda \end{bmatrix}$$

$$= \left((-\lambda) \det \begin{bmatrix} 1-\lambda & -1 \\ 2 & 5-\lambda \end{bmatrix} \right) - \left((-2) \det \begin{bmatrix} -1 & -1 \\ 2 & 5-\lambda \end{bmatrix} \right) + \left((-3) \det \begin{bmatrix} -1 & 1-\lambda \\ 2 & 2 \end{bmatrix} \right)$$

$$= -\lambda[(1-\lambda)(5-\lambda)+2] + 2[(-1)(5-\lambda)+2] - 3[-2-(1-\lambda)(2)]$$

$$= (-\lambda^3 + 6\lambda^2 - 7\lambda) + (2\lambda - 6) + (-6\lambda + 12)$$

$$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6$$

$$= (-1)(\lambda - 1)(\lambda - 2)(\lambda - 3)$$

Thus the eigenvalues of A are 1, 2, and 3. We now find the set of vectors S_i satisfying the equation $(A - \lambda I)x = 0$ for the ith eigenvalue. Note $S_i - \{0\}$ is the set of eigenvectors corresponding to the ith eigenvalue.

$$\lambda = 1: \begin{bmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow S_1 = \left\{ t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$$

$$\lambda = 2: \begin{bmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow S_2 = \left\{ t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$$

$$\lambda = 3: \begin{bmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow S_3 = \left\{ t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$$

Clearly[†] $\beta = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$ consisting of eigenvectors of A. Let

$$Q = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Then

$$Q^{-1}AQ = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$$

5.1.4(g)

Find the eigenvalues of T and an ordered basis β for $P_3(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix, where $T(f(x)) = xf'(x) + f''(x) - f(2)$.

Let γ be the standard ordered basis for $P_3(\mathbb{R})$. Then

$$T(1) = 0 + 0 - 1 = -1, \qquad T(x) = x + 0 - 2 = -2 + x,$$
$$T(x^2) = 2x^2 + 2 - 4 = -2 + 2x^2, \qquad T(x^3) = 3x^3 + 6x - 8$$
$$[T]_\gamma = \begin{bmatrix} [T(1)]_\gamma & [T(x)]_\gamma & [T(x^2)]_\gamma & [T(x^3)]_\gamma \end{bmatrix} = \begin{bmatrix} -1 & -2 & -2 & -8 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The eigenvalues of T are the roots of the characteristic polynomial of T.

$$\det([T]_\gamma - \lambda I) = \det \begin{bmatrix} -1-\lambda & -2 & -2 & -8 \\ 0 & 1-\lambda & 0 & 6 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix} = (-1-\lambda)(1-\lambda)(2-\lambda)(3-\lambda)$$
$$= (\lambda+1)(\lambda-1)(\lambda-2)(\lambda-3)$$

Thus the eigenvalues of T are -1, 1, 2, and 3. We now find an arbitrary eigenvector x corresponding to each eigenvalue.

$$\lambda = -1: \quad \begin{bmatrix} 0 & -2 & -2 & -8 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow [x]_\gamma = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = 1: \quad \begin{bmatrix} -2 & -2 & -2 & -8 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow [x]_\gamma = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

[†] At this point we know T is diagonalizable by Theorem 5.9(a). That the union of some bases for the eigenspaces is a basis for $\mathbb{R}^3$ is stated in Theorem 5.9(b).

$$\lambda = 2\colon \begin{bmatrix} -3 & -2 & -2 & -8 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2/3 & 2/3 & 8/3 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow [x]_\gamma = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}$$

$$\lambda = 3\colon \begin{bmatrix} -4 & -2 & -2 & -8 \\ 0 & -2 & 0 & 6 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 & 1/2 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 7/2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow [x]_\gamma = \begin{bmatrix} -7 \\ 6 \\ 0 \\ 2 \end{bmatrix}$$

Let $\beta = \{1, 1 - x, 2 - 3x^2, -7 + 6x + 2x^3\}$. Clearly[‡] β is a basis for $P_3(\mathbb{R})$, and we see that

$$T(1) = 0 + 0 - 1 = -1, \qquad T(1 - x) = -x + 0 - (1 - 2) = 1 - x$$

$$T(2 - 3x^2) = -6x^2 - 6 - (2 - 12) = 4 - 6x^2,$$

$$T(-7 + 6x + 2x^3) = 6x + 6x^3 + 12x - (-7 + 12 + 16) = -21 + 18x + 6x^3$$

$$[T]_\beta = \big[[T(1)]_\beta \quad [T(x)]_\beta \quad [T(x^2)]_\beta \quad [T(x^3)]_\beta \big] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Thus we have found a basis β such that $[T]_\beta$ is diagonal.

5.2.3(b)

Test T for diagonalizability, and if T is diagonalizable, find a basis β for $P_2(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix, where $T\colon P_2(\mathbb{R}) \to P_2(\mathbb{R}), T(ax^2 + bx + c) = cx^2 + bx + a$.

Let γ be the standard ordered basis for $P_2(\mathbb{R})$. Then clearly

$$[T]_\gamma = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial of T is

$$\det([T]_\gamma - \lambda I) = \det \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & 1 - \lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & -\lambda \\ 0 & 1 - \lambda & 0 \\ -\lambda & 0 & 1 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & 0 & -\lambda \\ 0 & 1 - \lambda & 0 \\ 0 & 0 & 1 - \lambda^2 \end{bmatrix} = (-1)(1)(1 - \lambda)(1 - \lambda^2)$$

$$= (-1)(\lambda - 1)^2(\lambda + 1)$$

Thus the characteristic polynomial of T splits. The eigenvalues of T are $\lambda_1 = 1$ and $\lambda_2 = -1$. We now find the eigenspaces corresponding to each eigenvalue.

[‡] See footnote of previous page.

$$\lambda_1\colon \begin{bmatrix}-1 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & -1\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \Longrightarrow E_{\lambda_1} = \left\{ \begin{bmatrix}1 & x & x^2\end{bmatrix} \left(s\begin{bmatrix}1\\0\\1\end{bmatrix} + t\begin{bmatrix}0\\1\\0\end{bmatrix} \right) \middle| s, t \in \mathbb{R} \right\}$$

$$\lambda_2\colon \begin{bmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} \Longrightarrow E_{\lambda_2} = \left\{ \begin{bmatrix}1 & x & x^2\end{bmatrix} \left(t\begin{bmatrix}-1\\0\\1\end{bmatrix} \right) \middle| t \in \mathbb{R} \right\}$$

We clearly see that $\beta_1 = \{x^2+1, x\}$ and $\beta_2 = \{x^2-1\}$ are bases for E_{λ_1} and E_{λ_2}, respectively. Thus, since the multiplicities of λ_1 and λ_2 equal the dimensions of E_{λ_1} and E_{λ_2}, respectively, by Theorem 5.9, T is diagonalizable, and $\beta = \beta_1 \cup \beta_2 = \{x^2+1, x, x^2-1\}$ is a basis for $P_2(\mathbb{R})$ consisting of eigenvectors of T. We see that $[T]_\beta$ is diagonal:

$$T(x) = x, \qquad T(x^2+1) = x^2+1, \qquad T(x^2-1) = -x^2+1$$

$$[T]_\beta = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

5.2.3(c)

Test T for diagonalizability, and if T is diagonalizable, find a basis β for $\mathbb{R}^3$ such that $[T]_\beta$ is a diagonal matrix, where $T\colon \mathbb{R}^3 \to \mathbb{R}^3, T\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = \begin{bmatrix}a_2\\-a_1\\2a_3\end{bmatrix}$.

Let γ be the standard ordered basis for $\mathbb{R}^3$. Then

$$[T]_\gamma = \begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 2\end{bmatrix}$$

The characteristic polynomial of T is

$$\det([T]_\gamma - \lambda I) = \det\begin{bmatrix}-\lambda & 1 & 0\\ -1 & -\lambda & 0\\ 0 & 0 & 2-\lambda\end{bmatrix} = \det\begin{bmatrix}1 & \lambda & 0\\ -\lambda & 1 & 0\\ 0 & 0 & 2-\lambda\end{bmatrix} = \det\begin{bmatrix}1 & \lambda & 0\\ 0 & \lambda^2+1 & 0\\ 0 & 0 & 2-\lambda\end{bmatrix}$$

$$= (\lambda^2+1)(2-\lambda) = (-1)(\lambda-i)(\lambda+i)(\lambda-2)$$

Thus the characteristic polynomial of T splits over $\mathbb{C}$, but not over $\mathbb{R}$. Therefore, by the converse of Theorem 5.6, T is not diagonalizable over $\mathbb{R}$.

5.2.7

For $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$, find an expression for A^n, where n is an arbitrary positive integer.

The characteristic polynomial of A is

$$\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$$

Thus the characteristic polynomial of A splits. The eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = -1$. We now find the eigenspaces corresponding to each eigenvalue.

$$\lambda_1\colon \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Longrightarrow E_{\lambda_1} = \left\{ t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$$

$$\lambda_2\colon \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Longrightarrow E_{\lambda_2} = \left\{ t\begin{bmatrix} 2 \\ -1 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$$

We clearly see that $\beta_1 = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ and $\beta_2 = \left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$ are bases for E_{λ_1} and E_{λ_2}, respectively. Since the multiplicities of λ_1 and λ_2 equal the dimensions of E_{λ_1} and E_{λ_2}, respectively, by Theorem 5.9, A is diagonalizable, and $\beta = \beta_1 \cup \beta_2 = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$. Let $D = [T]_\beta = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$ and $Q = [I_{\mathbb{R}^2}]_\beta^\gamma = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$, where γ is the standard ordered basis for $\mathbb{R}^2$. Then $Q^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ and $A = QDQ^{-1}$. Let $n \geq 1$. Then

$$A^n = \underbrace{(QDQ^{-1})(QDQ^{-1})\cdots(QDQ^{-1})}_{n} = Q[D(Q^{-1}Q)D(Q^{-1}Q)\cdots(Q^{-1}Q)D]Q^{-1}$$

$$= Q[DIDI\cdots ID]Q^{-1} = Q\,\underbrace{DD\cdots D}_{n}\,Q^{-1} = QD^nQ^{-1}$$

But $D^n = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}^n = \begin{bmatrix} 5^n & 0 \\ 0 & (-1)^n \end{bmatrix}$, so

$$A^n = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5^n & 0 \\ 0 & (-1)^n \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5^n & 2(5^n) \\ (-1)^n & -(-1)^n \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 5^n + 2(-1)^n & 2(5^n) - 2(-1)^n \\ 5^n - (-1)^n & 2(5^n) + (-1)^n \end{bmatrix}$$

5.2.12(b)

Let T be an invertible linear operator on a finite dimensional vector space V. Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

Proof:

Assume T is diagonalizable. Then, by Theorem 5.1, there exists an ordered basis $\beta = \{v_1, v_2, \dots, v_n\}$ for V consisting of eigenvectors of T. Let λ_j be the eigenvalue corresponding to v_j for $1 \leq j \leq n$. By part (a) and the assumption that T is invertible, λ_j^{-1} is an eigenvalue of T^{-1} for $1 \leq j \leq n$. Thus, for $1 \leq j \leq n$,

$$T^{-1}(v_j) = T^{-1}(\lambda_j^{-1}\lambda_j v_j) = \lambda_j^{-1}T^{-1}(\lambda_j v_j) = \lambda_j^{-1}T^{-1}\left(T(v_j)\right) = \lambda_j^{-1}v_j$$

It follows that

$$\begin{aligned}[T^{-1}]_\beta &= \left[[T^{-1}(v_1)]_\beta \quad [T^{-1}(v_2)]_\beta \quad \cdots \quad [T^{-1}(v_n)]_\beta\right] \\ &= \left[[\lambda_1^{-1}v_1]_\beta \quad [\lambda_2^{-1}v_2]_\beta \quad \cdots \quad [\lambda_n^{-1}v_n]_\beta\right] = [\lambda_1^{-1}e_1 \quad \lambda_2^{-1}e_2 \quad \cdots \quad \lambda_n^{-1}e_n]\end{aligned}$$

which is diagonal. Thus T^{-1} is diagonalizable. ■