

640:350:01 Homework 14

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3.2.19

Let A be an $m \times n$ matrix with rank m and B be an $n \times p$ matrix with rank n . Determine the rank of AB .

We see that

$$\begin{aligned}\text{rank}(AB) &= \dim(\text{col}(AB)) = \dim\left\{\sum_{i=1}^p x_i \text{col}_i(AB) \mid x \in F^p\right\} = \dim\left\{\sum_{i=1}^p x_i A \text{col}_i(B) \mid x \in F^p\right\} \\ &= \dim\left\{A \sum_{i=1}^p x_i \text{col}_i(B) \mid x \in F^p\right\} = \dim\{Ax \mid x \in \text{col}(B)\}\end{aligned}$$

We know that the columns of B are in F^n , that is, $\text{col}(B) \subseteq F^n$. But the rank of B is n , so $\dim(\text{col}(B)) = n$ and thus $\text{col}(B) = F^n$. Hence

$$\text{rank}(AB) = \dim\{Ax \mid x \in F^n\} = \dim\left\{\sum_{i=1}^n x_i \text{col}_i(A) \mid x \in F^n\right\} = \dim(\text{col}(A)) = \text{rank}(A) = m$$

3.3.8

Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined by $T(a, b, c) = (a + b, b - 2c, a + 2c)$. For each vector v in \mathbb{R}^3 , determine whether $v \in R(T)$.

(a) $v = (1, 3, -2)$

(b) $v = (2, 1, 1)$

We see that

$$[T]_{\beta} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$$

where β is the standard basis for \mathbb{R}^3 . We will attempt to solve the equation $[T]_{\beta}[x]_{\beta} = [v]_{\beta}$ to see if the given vectors are in $R(T)$.

(a) The vector $v = (1, 3, -2)$ is in $R(T)$ since

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 3 \\ 1 & 0 & 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & -1 & 2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and thus, for example, letting $c = 0$, we have $T(-2, 3, 0) = (1, 3, -2)$.

(b) The vector $v = (2, 1, 1)$ is in $R(T)$ since

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and thus, for example, letting $c = 0$, we have $T(1, 1, 0) = (2, 1, 1)$.

4.3.11

A matrix $M \in M_{n \times n}(\mathbb{C})$ is called skew-symmetric if $M^t = -M$. Prove that if M is skew-symmetric and n is odd, then M is not invertible. What happens if n is even?

Proof:

Let M be an $n \times n$ skew-symmetric matrix, where n is odd. Suppose M is invertible. Then we must have $\det(M) \neq 0$. Since $M^t = -M$ and thus $M^t M^{-1} = -MM^{-1} = -I$, we must have

$$\begin{aligned} 1 &= \det(I) = \det(MM^{-1}) = \det(M) \cdot \det(M^{-1}) = \det(M^t) \cdot \det(M^{-1}) = \det(M^t M^{-1}) \\ &= \det(-I) = \prod_{i=1}^n (-I)_{ii} = (-1)^n = -1 \end{aligned}$$

which is a contradiction. Thus M is not invertible. ■

If n is instead even, no conclusion can be made about whether M is invertible or not.

For example, $M = O$ is skew-symmetric and not invertible, while

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is skew-symmetric yet clearly invertible.

4.3.12

A matrix $Q \in M_{n \times n}(\mathbb{R})$ is called orthogonal if $QQ^t = I$. Prove that if Q is orthogonal, then $\det(Q) = \pm 1$.

Proof:

Let Q be an $n \times n$ orthogonal matrix. Then $(\det(Q))^2 = \det(Q) \cdot \det(Q) = \det(Q) \cdot \det(Q^t) = \det(QQ^t) = \det(I) = 1$, which implies $\det(Q) = \pm 1$. ■

4.3.14

Let $\beta = \{u_1, u_2, \dots, u_n\}$ be a subset of F^n containing n distinct vectors, and let B be the matrix in $M_{n \times n}(F)$ having u_j as column j . Prove that β is a basis for F^n if and only if $\det(B) \neq 0$.

Proof:

β is a basis for F^n

iff $\text{span}(\beta) = \text{col}(B) = F^n$

iff $\text{rank}(B) = n$

iff B is invertible

iff $\det(B) \neq 0$. ■