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3.2.17

Prove that if B is a 3×1 matrix and C is a 1×3 matrix, then the 3×3 matrix BC has rank at most 1. Conversely, show that if A is any 3×3 matrix having rank 1, then there exist a 3×1 matrix B and a 1×3 matrix C such that $A = BC$.

Proof:

(i) Let B be a 3×1 matrix and C be a 1×3 matrix. Then clearly the dimension of the subspace generated by the columns of B is at most 1 and thus $\text{rank}(B) \leq 1$. But, by Theorem 3.7, $\text{rank}(BC) \leq \text{rank}(B)$. Hence $\text{rank}(BC) \leq 1$.

(ii) Let A be a 3×3 matrix having rank 1. Then the dimension of the subspace generated by the columns of A is 1. Let $\beta = \{B\}$ be a basis for this subspace, where B is a 3×1 matrix. Then there exists unique scalars x_1, x_2, and x_3 such that $\text{col}_1(A) = x_1 B$, $\text{col}_2(A) = x_2 B$, and $\text{col}_3(A) = x_3 B$. Let $C = (x_1 \quad x_2 \quad x_3)$ be a 1×3 matrix. Then

$$BC = B(x_1 \quad x_2 \quad x_3) = (x_1 B \quad x_2 B \quad x_3 B) = (\text{col}_1(A) \quad \text{col}_2(A) \quad \text{col}_3(A)) = A \ \blacksquare$$

3.2.21

Let A be an $m \times n$ matrix with rank m. Prove that there exists an $n \times m$ matrix B such that $AB = I_m$.

Proof:

Let A be an $m \times n$ matrix with rank m. Then the dimension of the subspace S generated by the columns of A is m. But S is a subspace of F^m, which also has dimension m. Thus $S = F^m$. Since clearly $n \geq m$ and the columns of A generate F^m, we can reduce the set of columns of A to a basis for F^m.

Let $r_1, r_2, \dots, r_m \in \{1, 2, \dots, n\}$ such that $r_1 < r_2 < \cdots < r_m$, and let $\beta = \{\mathrm{col}_{r_1}(A), \mathrm{col}_{r_2}(A), \dots, \mathrm{col}_{r_m}(A)\}$ be a basis for F^m constructed by reducing the set of columns of A. For reference, let $\Gamma = \{1, 2, \dots, m\}$ and $\Delta = \{1, 2, \dots, n\} - \{r_1, r_2, \dots, r_m\}$. Let C be the $m \times m$ matrix such that, for all $k \in \Gamma$, $\mathrm{col}_k(C) = \mathrm{col}_{r_k}(A)$. We see that $\mathrm{rank}(C) = m$ and thus C is invertible. Now let B be the $n \times m$ matrix such that, for all $k \in \Gamma$, $\mathrm{row}_{r_k}(B) = \mathrm{row}_k(C^{-1})$, and, for all $k \in \Delta$, $\mathrm{row}_k(B) = 0$.

We now show that $AB = I_m$. Let $i, j \in \Gamma$. Then

$$(AB)_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj} = \sum_{k \in \Gamma} A_{ir_k} B_{r_k j} + \sum_{k \in \Delta} A_{ik} B_{kj} = \sum_{k \in \Gamma} C_{ik} C_{kj}^{-1} + \sum_{k \in \Delta} A_{ik} 0 = (CC^{-1})_{ij} + 0$$

$$= (I_m)_{ij}$$

Thus $AB = I_m$. ■