640:350:01 Homework 9

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2.4.7

Let A be an $n \times n$ matrix.

(a) Suppose that $A^2 = O$. Prove that A is not invertible.

(b) Suppose that $AB = O$ for some nonzero $n \times n$ matrix B. Could A be invertible? Explain.

(a) Proof:

Assume $A^2 = O$. Suppose A is invertible. Then $A = AI = A(AA^{-1}) = (AA)A^{-1} = A^2A^{-1} = OA^{-1} = O$. But $A = O$ clearly contradicts A being invertible. Thus A is not invertible. ■

(b) Proof:

Assume $AB = O$. Suppose A is invertible. Then $B = IB = (A^{-1}A)B = A^{-1}(AB) = A^{-1}O = O$. This contradicts B being nonzero. Thus A is not invertible. ■

2.4.10(b)

Let A and B be $n \times n$ matrices such that $AB = I_n$. Prove $A = B^{-1}$ (and hence $B = A^{-1}$).

Proof:

Assume $AB = I_n$. By part (a), A and B are invertible. Thus $A = AI_n = A(BB^{-1}) = (AB)B^{-1} = I_nB^{-1} = B^{-1}$ and $B = I_nB = (A^{-1}A)B = A^{-1}(AB) = A^{-1}I_n = A^{-1}$. ■

2.5.2(c)

Find the change of coordinate matrix that changes β'-coordinates into β-coordinates, where $\beta = \{(2,5),(-1,-3)\}$ and $\beta' = \{e_1, e_2\}$ are ordered bases for $\mathbb{R}^2$.

The change of coordinate matrix we want to find is $[I_{\mathbb{R}^2}]_{\beta'}^{\beta}$. We see that

$$I_{\mathbb{R}^2}(e_1) = e_1 = (1,0) = 3(2,5) + 5(-1,-3)$$
$$I_{\mathbb{R}^2}(e_2) = e_2 = (0,1) = -1(2,5) - 2(-1,-3)$$

Thus

$$[I_{\mathbb{R}^2}]_{\beta'}^{\beta} = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$$

and, for any $(x, y) \in \mathbb{R}^2$,

$$[x]_\beta = [I_{\mathbb{R}^2}(x)]_\beta = [I_{\mathbb{R}^2}]_{\beta'}^{\beta}[x]_{\beta'} = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x - y \\ 5x - 2y \end{pmatrix}$$

2.5.5

Let T be the linear operator on $P_1(\mathbb{R})$ defined by $T\big(p(x)\big) = p^{'}(x)$, the derivative of $p(x)$. Let $\beta = \{1, x\}$ and $\beta^{'} = \{1 + x, 1 - x\}$. Use Theorem 2.23 and the fact that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

to find $[T]_{\beta'}$.

We first find $[T]_\beta$. We see that

$$T(1) = 0 = (1 \quad x)\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$T(x) = 1 = (1 \quad x)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus

$$[T]_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

We now find $Q = \left[I_{P_1(\mathbb{R})}\right]_{\beta'}^{\beta}$. We see that

$$I_{P_1(\mathbb{R})}(1 + x) = 1 + x = (1 \quad x)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$I_{P_1(\mathbb{R})}(1 - x) = 1 - x = (1 \quad x)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Thus

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

By Theorem 2.23, we now have

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

2.5.7

In $\mathbb{R}^2$, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $T(x, y)$, where

(a) T is the reflection of $\mathbb{R}^2$ about L.

(b) T is the projection on L along the line perpendicular to L.

(a) Let $w_1 = (1, m)$, a vector parallel to L, and $w_2 = (-m, 1)$, a vector perpendicular to L, be vectors in $\mathbb{R}^2$. Clearly $\beta = \{w_1, w_2\}$ and $\beta' = \{e_1, e_2\}$ are ordered bases for $\mathbb{R}^2$. We now find $[T]_\beta$. We see that

$$T(w_1) = w_1 = (w_1 \quad w_2)\begin{pmatrix}1\\0\end{pmatrix}$$

$$T(w_2) = -w_2 = (w_1 \quad w_2)\begin{pmatrix}0\\-1\end{pmatrix}$$

Thus

$$[T]_\beta = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

We now find $Q = [I_{\mathbb{R}^2}]_{\beta'}^{\beta}$. We see that

$$I_{\mathbb{R}^2}(e_1) = e_1 = (1,0) = \frac{1}{m^2+1}(1,m) + \frac{-m}{m^2+1}(-m,1)$$

$$I_{\mathbb{R}^2}(e_2) = e_2 = (0,1) = \frac{m}{m^2+1}(1,m) + \frac{1}{m^2+1}(-m,1)$$

Thus

$$Q = \frac{1}{m^2+1}\begin{pmatrix}1 & m\\-m & 1\end{pmatrix}$$

and

$$Q^{-1} = \begin{pmatrix}1 & -m\\m & 1\end{pmatrix}$$

By Theorem 2.23, we now have

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q = \begin{pmatrix}1 & -m\\m & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\frac{1}{m^2+1}\begin{pmatrix}1 & m\\-m & 1\end{pmatrix}$$

$$= \frac{1}{m^2+1}\begin{pmatrix}1-m^2 & 2m\\2m & m^2-1\end{pmatrix}$$

Thus

$$T\begin{pmatrix}x\\y\end{pmatrix} = \left[T\begin{pmatrix}x\\y\end{pmatrix}\right]_{\beta'} = [T]_{\beta'}\left[\begin{pmatrix}x\\y\end{pmatrix}\right]_{\beta'} = [T]_{\beta'}\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{m^2+1}\begin{pmatrix}1-m^2 & 2m\\2m & m^2-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$= \frac{1}{m^2+1}\begin{pmatrix}(1-m^2)x + 2my\\2mx + (m^2-1)y\end{pmatrix}$$

(b) Let w_1, w_2, β, and β' be defined as in part (a). We now find $[T]_\beta$. We see that

$$T(w_1) = w_1 = (w_1 \quad w_2)\begin{pmatrix}1\\0\end{pmatrix}$$

$$T(w_2) = 0 = (w_1 \quad w_2)\begin{pmatrix}0\\0\end{pmatrix}$$

Thus

$$[T]_\beta = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

The change of coordinate matrix $Q = [I_{\mathbb{R}^2}]_{\beta'}^{\beta}$ and its inverse Q^{-1} are the same as in part (a).

By Theorem 2.23, we now have

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q = \begin{pmatrix}1 & -m\\m & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\frac{1}{m^2+1}\begin{pmatrix}1 & m\\-m & 1\end{pmatrix} = \frac{1}{m^2+1}\begin{pmatrix}1 & m\\m & m^2\end{pmatrix}$$

Thus

$$T\begin{pmatrix}x\\y\end{pmatrix} = \left[T\begin{pmatrix}x\\y\end{pmatrix}\right]_{\beta'} = [T]_{\beta'}\left[\begin{pmatrix}x\\y\end{pmatrix}\right]_{\beta'} = [T]_{\beta'}\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{m^2+1}\begin{pmatrix}1 & m\\m & m^2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$= \frac{1}{m^2+1}\begin{pmatrix}x+my\\mx+m^2y\end{pmatrix}$$