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February 19, 2009

2.3.9

Find linear transformations $U, T: F^2 \to F^2$ such that $UT = T_0$ but $TU \neq T_0$. Use your answer to find matrices A and B such that $AB = O$ but $BA \neq O$.

Define $U(x,y) = (y,0)$ and $T(x,y) = (x,0)$ for all $x, y \in F$. Then clearly U and T are linear, and

$$\forall x, y \in F, \qquad (UT)(x,y) = U\big(T(x,y)\big) = U(x,0) = (0,0)$$

but, for example,

$$(TU)(1,1) = T\big(U(1,1)\big) = T(1,0) = (1,0) \neq (0,0).$$

Therefore $UT = T_0$ but $TU \neq T_0$. Let β be the standard ordered basis for F^2. Then

$$[U]_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad [T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and $A = [U]_\beta$ and $B = [T]_\beta$ are matrices such that $[U]_\beta [T]_\beta = O$ but

$$[T]_\beta [U]_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq O.$$

2.3.11

Let V be a vector space, and let $T: V \to V$ be linear. Prove that $T^2 = T_0$ if and only if $R(T) \subseteq N(T)$.

Proof:

(i) Assume $T^2 = T_0$. Let $v \in R(T)$. Then for some $u \in V$ we have $T(u) = v$. But by assumption $T^2(u) = T_0(u) = 0$ and thus $T^2(u) = T\big(T(u)\big) = T(v) = 0$. Thus $v \in N(T)$.

(ii) Assume $R(T) \subseteq N(T)$. Let $u \in V$. Then $T(u) \in R(T)$ and thus by assumption $T(u) \in N(T)$. But this means that $T\big(T(u)\big) = 0$, so $T^2(u) = T\big(T(u)\big) = 0 = T_0(u)$. Since this is true for all $u \in V$, $T^2 = T_0$. ∎