640:350:01 Homework 6

Timothy J. Shields

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2.1.5

Let $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be defined by $T\big(f(x)\big) = xf(x) + f'(x)$. Prove that T is linear, find bases for both $N(T)$ and $R(T)$, compute the nullity and rank of T, verify the dimension theorem, and determine whether T is one-to-one or onto.

Let $c \in \mathbb{R}$ and $f, g \in P_2(\mathbb{R})$. Then

$$T\big((f+g)(x)\big) = x\big((f+g)(x)\big) + (f+g)'(x) = x\big(f(x)+g(x)\big) + \big(f'(x)+g'(x)\big)$$
$$= \big(xf(x)+f'(x)\big) + \big(xg(x)+g'(x)\big) = T\big(f(x)\big) + T\big(g(x)\big)$$

and

$$T\big((cf)(x)\big) = x\big((cf)(x)\big) + (cf)'(x) = xc\big(f(x)\big) + c\big(f'(x)\big) = c\big(xf(x)+f'(x)\big)$$
$$= cT\big(f(x)\big),$$

so T is linear.

Since T is linear, $0 \in N(T)$, that is, $\{0\} \subseteq N(T)$. Let $h(x) = a_0 + a_1x + a_2x^2 \in N(T)$. Then $T\big(h(x)\big) = xh(x) + h'(x) = a_0x + a_1x^2 + a_2x^3 + a_1 + 2a_2x = a_11 + (a_0 + 2a_2)x + a_1x^2 + a_2x^3 = 0$. Since $\{1, x, x^2, x^3\}$ is a basis for $P_3(\mathbb{R})$ and is thus linearly independent, the only solution to this equation is the trivial solution. Clearly this implies $a_0 = a_1 = a_2 = 0$. Thus $h(x) = 0 \in \{0\}$. So $N(T) \subseteq \{0\}$. Hence $\{0\} = N(T)$ and thus $\emptyset$ is a basis for $N(T)$.

Clearly $\beta = \{1, x, x^2\}$ is a basis for $P_2(\mathbb{R})$. Then by Theorem 2.2, $R(T) = \operatorname{span}\{T(1), T(x), T(x^2)\} = \operatorname{span}\{x, x^2+1, x^3+2x\}$. For $a_1, a_2, a_3 \in \mathbb{R}$, examine solutions to $a_1x + a_2(x^2+1) + a_3(x^3+2x) = a_21 + (a_1+2a_3)x + a_2x^2 + a_3x^3 = 0$. Since $\{1, x, x^2, x^3\}$ is a basis for $P_3(\mathbb{R})$ and is thus linearly independent, the only solution to this equation is the trivial solution. Clearly this implies $a_1 = a_2 = a_3 = 0$. So $\gamma = \{x, x^2+1, x^3+2x\}$ is linearly independent and is thus a basis for $R(T)$.

The nullity and rank of T are simply $\operatorname{nullity}(T) = \dim\big(N(T)\big) = |\emptyset| = 0$ and $\operatorname{rank}(T) = \dim\big(R(T)\big) = |\gamma| = 3$. Thus $\operatorname{nullity}(T) + \operatorname{rank}(T) = 3 = |\beta| = \dim\big(P_2(\mathbb{R})\big)$, verifying the dimension theorem.

Since $N(T) = \{0\}$, by Theorem 2.4, T is one-to-one. Since $\dim\big(R(T)\big) < \dim\big(P_3(\mathbb{R})\big)$, we have $R(T) \neq P_3(\mathbb{R})$. Thus T is not onto.

2.2.4

Define

$$T: M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R}) \text{ by } T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2.$$

Let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \text{ and } \gamma = \{1, x, x^2\}.$$

Compute $[T]_\beta^\gamma$.

For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$, we have

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Also,

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (1+0) + (2\cdot 0)x + 0x^2 = 1 = \begin{pmatrix} 1 & x & x^2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (0+1) + (2\cdot 0)x + 1x^2 = 1 + x^2 = \begin{pmatrix} 1 & x & x^2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = (0+0) + (2\cdot 0)x + 0x^2 = 0 = \begin{pmatrix} 1 & x & x^2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = (0+0) + (2\cdot 1)x + 0x^2 = 2x = \begin{pmatrix} 1 & x & x^2 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

Thus we have

$$[T(A)]_\gamma = [T]_\beta^\gamma [A]_\beta = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+b \\ 2d \\ b \end{pmatrix}.$$

2.2.9

Let V be the vector space of complex numbers over the field $\mathbb{R}$. Define $T: V \to V$ by $T(z) = \overline{z}$, where $\overline{z}$ is the complex conjugate of z. Prove that T is linear, and compute $[T]_\beta$, where $\beta = \{1, i\}$.

Let $k \in \mathbb{R}$, and $u, v \in V$, where for some $a, b, c, d \in \mathbb{R}$ we have $u = a + bi$ and $v = c + di$. Then

$$T(u+v) = \overline{u+v} = \overline{(a+bi)+(c+di)} = \overline{(a+c)+(b+d)i} = (a+c)-(b+d)i$$
$$= (a-bi)+(c-di) = \overline{(a+bi)} + \overline{(c+di)} = \overline{u} + \overline{v} = T(u) + T(v)$$

and

$$T(ku) = \overline{ku} = \overline{k(a+bi)} = \overline{ka+kbi} = ka - kbi = k(a-bi) = k\overline{(a+bi)} = k\overline{u} = kT(u).$$

Thus T is linear. Now, for any $z = x + yi \in V$, where $x, y \in \mathbb{R}$, we have

$$z = x + yi = \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Also,

$$T(1) = T(1+0i) = 1 - 0i = \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$T(i) = T(0+i) = 0 - i = \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Thus we have

$$[T(z)]_\beta = [T]_\beta [z]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}.$$

2.2.10

Let V be a vector space with the ordered basis $\beta = \{v_1, v_2, \dots, v_n\}$. Define $v_0 = 0$. By Theorem 2.6, there exists a linear transformation $T: V \to V$ such that $T(v_j) = v_j + v_{j-1}$ for $j = 1, 2, \dots, n$. Compute $[T]_\beta$.

Let $u \in V$. Then there exist unique $a_1, a_2, \dots, a_n \in F$ such that

$$u = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

For $i = 1, 2, \dots, n$, let $e_i \in F^n$ be the $n \times 1$ column vector which has 1 as its ith component and 0 elsewhere. Then we have

$$T(v_1) = v_1 + v_0 = v_1 + 0 = v_1 = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} e_1,$$

and for $j = 2, 3, \dots, n$,

$$T(v_j) = v_j + v_{j-1} = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} (e_j + e_{j-1}).$$

Thus we have

$$[T(u)]_\beta = [T]_\beta [u]_\beta = \begin{pmatrix} 1 & 1 & 0 & & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ a_2 + a_3 \\ a_3 + a_4 \\ \vdots \\ a_{n-1} + a_n \\ a_n \end{pmatrix}.$$