640:350:01 Homework 5

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2.1.14(b)

Let V and W be vector spaces and $T: V \to W$ be linear. Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.

Proof:

(i) Assume S is linearly independent. Suppose $T(S)$ is linearly dependent. Then there exist a finite number of vectors $w_1, w_2, \dots, w_n \in T(S)$ and scalars $a_1, a_2, \dots, a_n \in F$ not all zero such that $a_1 w_1 + a_2 w_2 + \cdots + a_n w_n = 0_W$. But, since $w_1, w_2, \dots, w_n \in T(S)$, for each $i = 1, 2, \dots, n$, there exists $v_i \in S$ such that $T(v_i) = w_i$. Thus $a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n) = 0_W$. Since T is linear, we have $T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) = 0_W$. Suppose $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$ is distinct from 0_V. Then $T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) = T(0_V) = 0_W$, that is, both $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$ and 0_V map to 0_W, a contradiction to T being one-to-one. Thus $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0_V$. But some $a_i, i = 1, 2, \dots, n$ is nonzero, which implies S is linearly dependent, a contradiction. Thus $T(S)$ is linearly independent.

(ii) Assume $T(S)$ is linearly independent. Suppose S is linearly dependent. Then there exist a finite number of vectors $v_1, v_2, \dots, v_n \in S$ and scalars $a_1, a_2, \dots, a_n \in F$ not all zero such that $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0_V$. It follows that $T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) = T(0_V) = 0_W$. Since T is linear, we have $a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n) = 0_W$. Clearly $T(v_i) \in T(S), i = 1, 2, \dots, n$, so, since some $a_i, i = 1, 2, \dots, n$ is nonzero, $T(S)$ is linearly dependent, a contradiction. Thus S is linearly independent. ■

2.1.17(b)

Let V and W be finite-dimensional vector spaces and $T: V \to W$ be linear. Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

Proof:

Assume $\dim(V) > \dim(W)$. Suppose T is one-to-one. Then, by Theorem 2.4, $N(T) = \{0\}$, that is, $\text{nullity}(T) = 0$. By Theorem 2.3, it follows that $\text{nullity}(T) + \text{rank}(T) = \text{rank}(T) = \dim(V)$. But $\text{rank}(T)$ is the dimension of $R(T)$, which by Theorem 2.1 is a subspace of W, so $\dim(V) = \dim\big(R(T)\big) \leq \dim(W)$ — a contradiction. Thus T is not one-to-one. ■