640:350:01 Homework 4

Timothy J. Shields

February 12, 2009

1.6.7

The vectors $u_1 = (2, -3, 1)$, $u_2 = (1, 4, -2)$, $u_3 = (-8, 12, -4)$, $u_4 = (1, 37, -17)$, and $u_5 = (-3, -5, 8)$ generate $\mathbb{R}^3$. Find a subset of the set $\{u_1, u_2, u_3, u_4, u_5\}$ that is a basis for $\mathbb{R}^3$.

We examine the reduced row echelon form of the matrix having $u_i^t, i = 1, 2, \ldots, 5$ as its columns.

$$\begin{pmatrix} 2 & 1 & -8 & 1 & -3 \\ -3 & 4 & 12 & 37 & -5 \\ 1 & -2 & -4 & -17 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -4 & -17 & 8 \\ 0 & -2 & 0 & -14 & 19 \\ 0 & 5 & 0 & 35 & -19 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -4 & -17 & 8 \\ 0 & 3 & 0 & 21 & 0 \\ 0 & 5 & 0 & 35 & -19 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & -4 & -17 & 8 \\ 0 & 1 & 0 & 7 & 0 \\ 0 & 5 & 0 & 35 & -19 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -4 & -3 & 8 \\ 0 & 1 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & -19 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -4 & -3 & 0 \\ 0 & 1 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

From this we see that, since $\beta = \{u_1, u_2, u_5\}$ generates $\mathbb{R}^3$ and is linearly independent, β is a basis for $\mathbb{R}^3$. The reason for this is as follows. Let A be the matrix having u_1^t, u_2^t, and u_5^t as its columns. Then, for all $b \in \mathbb{R}^3$, $Ax = b$ has a unique solution x, because the reduced row echelon form of A is I, that is, A is invertible and $x = A^{-1}b$.

1.6.11

Let u and v be distinct vectors of a vector space V. Show that if $\{u, v\}$ is a basis for V and a and b are nonzero scalars, then both $\{u + v, au\}$ and $\{au, bv\}$ are also bases for V.

Proof:

Assume $\{u, v\}$ is a basis for V. Then $\dim(V) = 2$, and, by Theorem 1.10, Corollary 2, to show that a set S having two elements is a basis for V, it suffices to show that either S generates V or S is linearly independent. Let a and b be nonzero scalars. We will show that $\{u + v, au\}$ and $\{au, bv\}$ are linearly independents sets and thus bases for V.

First we examine solutions to $x(u+v)+y(au)=0$, where x and y are scalars. We have $(x+ay)u+xv=0$. But, by assumption $\{u,v\}$ is linearly independent, so the only solution to this is the trivial solution. Thus $x=0$, and, since a is nonzero, $y=0$. Hence $\{u+v,au\}$ is linearly independent and, since it has two elements, is thus a basis for V.

Next we examine solutions to $x(au)+y(bv)=0$. We have $(ax)u+(by)v=0$. But, by assumption $\{u,v\}$ is linearly independent, so the only solution to this is the trivial solution. Thus $ax=0$ and $by=0$. Since a and b are nonzero, we have $x=y=0$. Hence $\{au,bv\}$ is linearly independent and, since it has two elements, is thus a basis for V. ■

1.6.14

Find bases for the following subspaces of F^5:

$$W_1=\{(a_1,a_2,a_3,a_4,a_5)\in F^5: a_1-a_3-a_4=0\}$$

and

$$W_2=\{(a_1,a_2,a_3,a_4,a_5)\in F^5: a_2=a_3=a_4 \text{ and } a_1+a_5=0\}.$$

What are the dimensions of W_1 and W_2?

(i) Let $(a_1,a_2,a_3,a_4,a_5)\in W_1$, where $a_1,a_2,a_3,a_4,a_5\in F$ and $a_1-a_3-a_4=0$. Then $a_1=a_3+a_4$. Thus $(a_1,a_2,a_3,a_4,a_5)=(a_3+a_4,a_2,a_3,a_4,a_5)=a_2(0,1,0,0,0)+a_3(1,0,1,0,0)+a_4(1,0,0,1,0)+a_5(0,0,0,0,1)$. Hence we see that

$$\beta=\{(0,1,0,0,0),(1,0,1,0,0),(1,0,0,1,0),(0,0,0,0,1)\}\subseteq W_1$$

spans W_1. We now examine solutions to $x_1(0,1,0,0,0)+x_2(1,0,1,0,0)+x_3(1,0,0,1,0)+x_4(0,0,0,0,1)=0$. Clearly the only solution is $x_1=x_2=x_3=x_4=0$, so β is linearly independent. Thus, since β is a subset of W_1 that spans W_1 and is linearly independent, β is a basis for W_1 and $\dim(W_1)=|\beta|=4$.

(ii) Let $(a_1,a_2,a_3,a_4,a_5)\in W_2$, where $a_1,a_2,a_3,a_4,a_5\in F$ and $a_2=a_3=a_4$ and $a_1+a_5=0$. Then $a_5=-a_1$. Thus $(a_1,a_2,a_3,a_4,a_5)=(a_1,a_2,a_2,a_2,-a_1)=a_1(1,0,0,0,-1)+a_2(0,1,1,1,0)$. Hence we see that

$$\beta=\{(1,0,0,0,-1),(0,1,1,1,0)\}\subseteq W_2$$

spans W_2. We now examine solutions to $x_1(1,0,0,0,-1)+x_2(0,1,1,1,0)=0$. Clearly the only solution is $x_1=x_2=0$, so β is linearly independent. Thus, since β is a subset of W_2 that spans W_2 and is linearly independent, β is a basis for W_2 and $\dim(W_2)=|\beta|=2$.