640:350:01 Homework 3

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1.4.9

Show that the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ generate $M_{2\times 2}(F)$.

Proof:

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_{2\times 2}(F)$.

Then $A = a_{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{12}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_{21}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. ■

1.4.13

Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$, then $\mathrm{span}(S_1) \subseteq \mathrm{span}(S_2)$. In particular, if $S_1 \subseteq S_2$ and $\mathrm{span}(S_1) = V$, deduce that $\mathrm{span}(S_2) = V$.

Proof:

Let S_1 and S_2 be subsets of a vector space V such that $S_1 \subseteq S_2$. Let $u \in \mathrm{span}(S_1)$. Then, for some finite number of vectors $x_1, x_2, \dots, x_n$ in S_1 and scalars $a_1, a_2, \dots, a_n$, we have

$$u = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

But since $S_1 \subseteq S_2$, we have $x_1, x_2, \dots, x_n$ in S_2, so u is a linear combination of vectors in S_2. Thus $u \in \mathrm{span}(S_2)$.

Now assume $\mathrm{span}(S_1) = V$. Then, by Theorem 1.5, $\mathrm{span}(S_2) \subseteq V$ and, by assumption, $V \subseteq \mathrm{span}(S_1) \subseteq \mathrm{span}(S_2)$. ■

1.5.5

Show that the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $P_n(F)$.

Proof:

We examine the solutions to the equation $a_0 1 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$. Since this equality must hold for all $x \in \mathbb{R}$, it must hold for $x = 0$, so $a_0 1 + a_1 0 + a_2 0 + \cdots + a_n 0 = a_0 = 0$. Thus $a_0 = 0$. Taking derivatives, $a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1} = 0$. As before, when $x = 0$, we have $a_1 + 2a_2 0 + 3a_3 0 + \cdots + na_n 0 = a_1 = 0$. Thus $a_1 = 0$. Repeating this process, we obtain

$$a_0 = a_1 = a_2 = \cdots = a_n = 0.$$

Thus the only solution is the trivial solution and the set is linearly independent. ■

1.5.9

Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

Proof:

(i) Assume $\{u, v\}$ is linearly dependent. Then, for some scalars a, b not both zero, we have $au + bv = 0$. If $a \neq 0$, then $u = \frac{-b}{a} v$, so u is a multiple of v. If $b \neq 0$, then $v = \frac{-a}{b} u$, so v is a multiple of u.

(ii) Suppose u is a multiple of v. Then, for some scalar c, $u = cv$. Thus $1u + (-c)v = 0$. Since 1 is nonzero, $\{u, v\}$ is linearly dependent. Now suppose v is a multiple of u. Then, for some scalar d, $v = du$. Thus $(-d)u + 1v = 0$. Since 1 is nonzero, $\{u, v\}$ is linearly dependent. ■

1.5.14

Prove that a set S is linearly dependent if and only if $S = \{0\}$ or there exist distinct vectors $v, u_1, u_2, \dots, u_n$ in S such that v is a linear combination of $u_1, u_2, \dots, u_n$.

Proof:

(i) Assume S is linearly dependent. Then, for some finite number of distinct vectors $u_1, u_2, \dots, u_n$ in S and scalars $a_1, a_2, \dots, a_n$ not all zero,

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0.$$

If $n = 1$, then $a_1u_1 = 0$ and $a_1 \neq 0$, so $u_1 = 0$ and thus $S = \{0\}$. If $n > 1$, then, since $a_1, a_2, \dots, a_n$ are not all zero, for some $i = 1, 2, \dots, n$, $a_i \neq 0$. Hence we have

$$u_i = \frac{-a_1}{a_i}u_1 + \frac{-a_2}{a_i}u_2 + \cdots + \frac{-a_{i-1}}{a_i}u_{i-1} + \frac{-a_{i+1}}{a_i}u_{i+1} + \cdots + \frac{-a_n}{a_i}u_n.$$

(ii) In the case that $S = \{0\}$, S is linearly dependent because $1 \cdot 0 = 0$. In the case $S \neq \{0\}$, we assume that there exist distinct vectors $v, u_1, u_2, \dots, u_n$ in S such that v is a linear combination of $u_1, u_2, \dots, u_n$. Hence, for some scalars $a_1, a_2, \dots, a_n$,

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n.$$

Adding $(-1)v$ to both sides gives

$$(-1)v + a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0.$$

Since not all of the coefficients are zero, S is linearly dependent. ■