Solutions to Select Problems of Assignment 11

6.3.3 (a) $V=\mathbb{C}^2$, $T\left(\begin{bmatrix} z_1\\ z_2\end{bmatrix}\right) = \begin{bmatrix}2z_1+iz_2\\ (1-i)z_1\end{bmatrix}$, $x=\begin{bmatrix}3-i\\ 1+2i\end{bmatrix}$.

Method 1: $\langle x, T\left(\begin{bmatrix} z_1\\ z_2\end{bmatrix}\right)\rangle = \langle T^*(x), \begin{bmatrix} z_1\\ z_2\end{bmatrix}\rangle \Rightarrow (3-i)\overline{(2z_1+iz_2)} + (1+2i)\overline{((1-i)z_1)}$

$= a\bar z_1 + b\bar z_2$, if $T^*(x)=\begin{bmatrix} a\\ b\end{bmatrix}$. Working out the L.H.S, we get

$(3-i)(2\bar z_1 + \bar i\, \bar z_2) + (1+2i)((1-\bar i)\bar z_1) = a\bar z_1+b\bar z_2 \Rightarrow$

$6\bar z_1 - 2i\bar z_1 - 3i\bar z_2 - \bar z_2 + (1+i+2i-2)\bar z_1 = a\bar z_1+b\bar z_2 \Rightarrow$

$(5+i)\bar z_1 + (-3i-1)\bar z_2 = a\bar z_1 + b\bar z_2$. Since this relation is to hold for all z_1, z_2, we conclude $a=5+i$, $b=-3i-1$. i.e $T^*\left(\begin{bmatrix}3-i\\ 1+2i\end{bmatrix}\right) = \begin{bmatrix}5+i\\ -3i-1\end{bmatrix}$.

Method 2: $\beta=\left\{\begin{bmatrix}1\\ 0\end{bmatrix},\begin{bmatrix}0\\ 1\end{bmatrix}\right\}$ is an O.N.B for $\mathbb{C}^2$, and $T\left(\begin{bmatrix}1\\ 0\end{bmatrix}\right) = \begin{bmatrix}2\\ 1-i\end{bmatrix} = 2\begin{bmatrix}1\\ 0\end{bmatrix} + (1-i)\begin{bmatrix}0\\ 1\end{bmatrix}$,

$T\left(\begin{bmatrix}0\\ 1\end{bmatrix}\right) = \begin{bmatrix}i\\ 0\end{bmatrix} = i\begin{bmatrix}1\\ 0\end{bmatrix} + 0\cdot\begin{bmatrix}0\\ 1\end{bmatrix}$. Thus $[T]_\beta = \begin{bmatrix}2 & i\\ 1-i&0\end{bmatrix}$.

By Thm 6.10, $[T^*]_\beta = [T]_\beta^* = \begin{bmatrix}2&1+i\\ -i&0\end{bmatrix}$.

So $[T^*(x)]_\beta = \begin{bmatrix}2&1+i\\ -i&0\end{bmatrix}\begin{bmatrix}3-i\\ 1+2i\end{bmatrix} = \begin{bmatrix}5+i\\ -1-3i\end{bmatrix} \Rightarrow T^*\left(\begin{bmatrix}3-i\\ 1+2i\end{bmatrix}\right) = (5+i)\begin{bmatrix}1\\ 0\end{bmatrix} + (-1-3i)\begin{bmatrix}0\\ 1\end{bmatrix}$

6.5.6. Since $T(f)=hf$, we find $T^*(f)$ by $\langle f, T(g)\rangle = \langle T^*(f), g\rangle$, ie.

$\int_0^1 f(t)\overline{h(t)g(t)}\,dt = \int_0^1 T^*(f)(t)\overline{g(t)}\,dt \Leftrightarrow \int_0^1 f(t)\bar h(t)\bar g(t)\,dt = \int_0^1 T^*(f)(t)\bar g(t)\,dt$.

Since this equality holds for all $g\in C([0,1])$, $\Rightarrow T^*(f) = \bar h(t) f(t)$.

$\Rightarrow T^*T(f) = T^*(hf) = \bar h(t)h(t)f(t) = |h(t)|^2 f(t)$.

T is unitary iff $T^*T = Id \Leftrightarrow |h(t)|^2 \equiv 1$.

6.6.5 (a) $\forall x\in V$, $\exists!\, y\in R(T)$, $z\in N(T)$, s.t. $x=y+z$, and $y=T(x)$. Since T is an orthogonal projection, $y\perp z$. Thus by Pythagorean thm, $\|x\|^2 = \|y\|^2+\|z\|^2 \ge \|y\|^2 = \|T(x)\|^2 \Rightarrow$ $\|x\|\ge\|T(x)\|$. $\tilde T:\mathbb{R}^2\to\mathbb{R}^2$ defined by $\tilde T\left(\begin{bmatrix}x\\ y\end{bmatrix}\right) = \begin{bmatrix}x\\ x\end{bmatrix}$ is a projection of $\mathbb{R}^2$ onto $W_1=\left\{\begin{bmatrix}x\\ y\end{bmatrix}: y=x\right\}$ along $W_2=\left\{\begin{bmatrix}0\\ y\end{bmatrix}\right\}$. For $\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}$, $\tilde T\left(\begin{bmatrix}1\\ 0\end{bmatrix}\right) = \begin{bmatrix}1\\ 1\end{bmatrix} \Rightarrow \|\tilde T\left(\begin{bmatrix}1\\ 0\end{bmatrix}\right)\| > \|\begin{bmatrix}1\\ 0\end{bmatrix}\|$.

If $T:V\to V$ is a projection onto W_1 along W_2 s.t. $\|T(x)\|=\|x\|$ for all $x\in V$, then T is an orthogonal operator from V to V. Since V is finite dimensional, T must be an isomorphism. Furthermore, $W_2=\{0\}$, for otherwise, if W_2 contains some $\vec w_2\ne\vec 0$, then $T(\vec w_2)=\vec 0$ violating the assumption $\|T(\vec w_2)\|=\|\vec w_2\|$. Thus $V=W_1$ and $T(x)\equiv x$ for all $x\in V$.

6.6.5.(b) Suppose T is a projection s.t. $\|T(x)\| \le \|x\|$ for all $x \in V$.

Let W_1, W_2 be the subspaces defining T: $V = W_1 \oplus W_2$, $W_1 = R(T)$. It suffices to prove $W_1 \perp W_2$. Suppose not, then $\exists$ some $\vec{w_2} \in W_2$, and some $\vec{w_1} \in W_1$ s.t. $\langle \vec{w_1}, \vec{w_2} \rangle \neq 0$. By modifying $\vec{w_1}$ by some scalar multiple of $\vec{w_1}$, if necessary, we may $\vec{w_1} \in W_1$, and $\vec{w_2} \in W_2$ are chosen s.t. $\langle \vec{w_1}, \vec{w_2} \rangle$ is a non-zero real number.

Consider $x = t\vec{w_1} + \vec{w_2}$ for real scalars t. By assumption,

$$\|T(t\vec{w_1} + \vec{w_2})\|^2 \le \|t\vec{w_1} + \vec{w_2}\|^2, \text{ for all } t \in \mathbb{R}.$$

But, $T(t\vec{w_1} + \vec{w_2}) = t\vec{w_1}$ by def'n of T, so $\|T(t\vec{w_1} + \vec{w_2})\|^2 = t^2\|\vec{w_1}\|^2$.

On the other hand, $\|t\vec{w_1} + \vec{w_2}\|^2 = t^2\|\vec{w_1}\|^2 + 2\,\mathrm{Re}\{t\langle \vec{w_1}, \vec{w_2} \rangle\} + \|\vec{w_2}\|^2$

$$\Rightarrow \quad t^2\|\vec{w_1}\|^2 + 2t\langle \vec{w_1}, \vec{w_2} \rangle + \|\vec{w_2}\|^2 \ge t^2\|\vec{w_1}\|^2,$$

$$\Rightarrow \quad 2t\langle \vec{w_1}, \vec{w_2} \rangle + \|\vec{w_2}\|^2 \ge 0, \text{ for all } t \in \mathbb{R}.$$

When $\langle \vec{w_1}, \vec{w_2} \rangle \neq 0$, real, we can obviously choose t st.

$2t\langle \vec{w_1}, \vec{w_2} \rangle + \|\vec{w_2}\|^2 < 0$! This forces $\langle \vec{w_1}, \vec{w_2} \rangle = 0$, for any $\vec{w_1} \in W_1$, and $\vec{w_2} \in W_2$, proving T is an orthogonal projection.