640:350:01 Homework 23-24

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6.1.9

Let β be a basis for a finite-dimensional inner product space V.

(a) Prove that if $\langle x, z\rangle = 0$ for all $z \in \beta$, then $x = 0$.

(b) Prove that if $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in \beta$, then $x = y$.

(a) Proof:

Let $x \in V$. Suppose $\beta = \{v_1, v_2, \dots, v_n\}$. Then, for some scalars $c_1, c_2, \dots, c_n$, we have $x = \sum_{i=1}^{n} c_i v_i$. Assume $\langle x, z\rangle = 0$ for all $z \in \beta$. Then

$$\langle x, x\rangle = \left\langle x, \sum_{i=1}^{n} c_i v_i \right\rangle = \sum_{i=1}^{n} \overline{c_i}\langle x, v_i\rangle = \sum_{i=1}^{n} \overline{c_i} 0 = 0$$

Thus, by Theorem 6.1(d), we have $x = 0$. ∎

(b) Proof:

Let $x, y \in V$. Assume $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in \beta$. Then $\langle x - y, z\rangle = \langle x, z\rangle - \langle y, z\rangle = 0$ for all $z \in \beta$, which by part (a) implies that $x - y = 0$. Thus $x = y$. ∎

6.1.11

Prove the *parallelogram law* on an inner product space V; that is, show for all $x, y \in V$ that

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

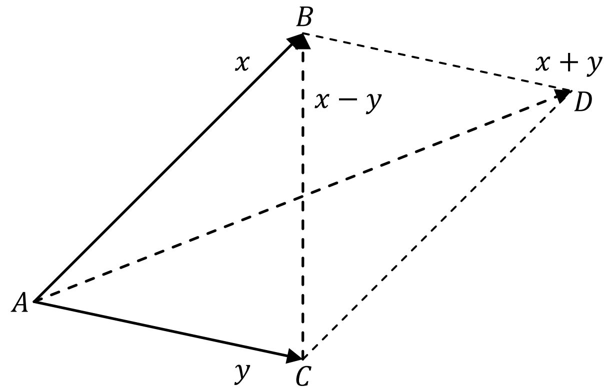
What does this equation state about parallelograms in $\mathbb{R}^2$?

Proof:

Let $x, y \in V$. Then

$$\begin{aligned}\|x + y\|^2 + \|x - y\|^2 &= \langle x + y, x + y\rangle + \langle x - y, x - y\rangle \\ &= \langle x, x + y\rangle + \langle y, x + y\rangle + \langle x, x - y\rangle - \langle y, x - y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle + \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\ &= 2\langle x, x\rangle + 2\langle y, y\rangle = 2\|x\|^2 + 2\|y\|^2 \quad \blacksquare\end{aligned}$$

We now examine two vectors in $\mathbb{R}^2$ and how the parallelogram law applies to them. Schematically, the relationship between $x, y, x-y$, and $x+y$ is



Let P_1P_2 denote the distance between points P_1 and P_2. Then the parallelogram law relates the lengths of the sides of the parallelogram to the lengths of its diagonals by

$$(AD)^2 + (BC)^2 = (AB)^2 + (AC)^2 + (BD)^2 + (CD)^2$$

6.1.15(a)

Prove that if V is an inner product space, then $|\langle x, y\rangle| = \|x\| \cdot \|y\|$ if and only if one of the vectors x or y is a multiple of the other.

Proof:

First, note that if either x or y is the zero vector, then the statement is trivially true. So let x and y be nonzero vectors in V.

(i) Assume $|\langle x, y\rangle| = \|x\| \cdot \|y\|$. Let $a = \langle x, y\rangle / \|y\|^2$ and let $z = x - ay$. Then $y \perp z$, since

$$\langle z, y\rangle = \langle x - ay, y\rangle = \langle x, y\rangle - a\langle y, y\rangle = \langle x, y\rangle - \frac{\langle x, y\rangle}{\|y\|^2}\|y\|^2 = 0$$

It follows that $ay \perp z$. Also, by assumption,

$$|a| = \left|\frac{\langle x, y\rangle}{\|y\|^2}\right| = \frac{|\langle x, y\rangle|}{\|y\|^2} = \frac{\|x\| \cdot \|y\|}{\|y\|^2} = \frac{\|x\|}{\|y\|}$$

implying $\|x\| = |a| \cdot \|y\|$. Now, since $x = ay + z$, we must have

$$\|x\|^2 = \|ay + z\|^2$$

But, since $ay \perp z$, by Exercise 10 we must have

$$\|ay + z\|^2 = \|ay\|^2 + \|z\|^2 = |a|^2 \cdot \|y\|^2 + \|z\|^2$$

Furthermore, since $\|x\| = |a| \cdot \|y\|$, we have

$$\|ay + z\|^2 = \|x\|^2 + \|z\|^2$$

Thus $\|x\|^2 = \|ay + z\|^2 = \|x\|^2 + \|z\|^2$, implying $\|z\|^2 = 0$, that is, $z = 0$. Returning to the definition of z, we see that $x = ay$, that is, x is a multiple of y.

(ii) Now assume x is a multiple of y, that is, for some scalar c, $x = cy$. Then

$$|\langle x, y\rangle| = |\langle cy, y\rangle| = |c\langle y, y\rangle| = |c| \cdot |\langle y, y\rangle| = |c| \cdot \|y\|^2 = |c| \cdot \|y\| \cdot \|y\| = \|cy\| \cdot \|y\| = \|x\| \cdot \|y\|$$

The case of y being a multiple of x proceeds similarly. ■

6.2.6

Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $x \in V - W$, prove that there exists $y \in W^\perp$ such that $\langle x, y\rangle \neq 0$.

Proof:

Let $x \in V$. Assume $x \notin W$. By Theorem 6.6, there exist unique vectors $u \in W$ and $y \in W^\perp$ such that $x = u + y$. Suppose $y = 0$. Then $x = u \in W$, a contradiction. Thus $y \neq 0$. Also note that $u \perp y$ since $u \in W$ and $y \in W^\perp$. We now see that, since $y \neq 0$,

$$\langle x, y\rangle = \langle u + y, y\rangle = \langle u, y\rangle + \langle y, y\rangle = \langle y, y\rangle \neq 0 \quad ■$$

6.2.10

Let W be a finite-dimensional subspace of an inner product space V. Prove that there exists a projection T on W along $W^\perp$ that satisfies $N(T) = W^\perp$. In addition, prove that $\|T(x)\| \leq \|x\|$ for all $x \in V$.

Proof:

We know by Exercise 13(d) that $V = W \oplus W^\perp$. Let $n = \dim(V)$. Let $\beta = \{v_1, v_2, \dots, v_s\}$ and $\gamma = \{w_1, w_2, \dots, w_t\}$ be bases for W and $W^\perp$, respectively, where $s + t = n$. Then $\beta \cup \gamma$ is a basis for V. Let $T: V \to V$ be the linear transformation defined by $T(v_i) = v_i$ for $1 \leq i \leq s$ and $T(w_i) = 0$ for $1 \leq i \leq t$.

We prove $\forall x \in W, T(x) = x$ and $\forall y \in W^{\perp}, T(y) = 0$:

Let $x \in W$. Then there exist scalars $c_1, c_2, \ldots, c_s$ such that

$$x = \sum_{i=1}^{s} c_i v_i$$

Thus

$$T(x) = T\left(\sum_{i=1}^{s} c_i v_i\right) = \sum_{i=1}^{s} c_i T(v_i) = \sum_{i=1}^{s} c_i v_i = x$$

Now let $y \in W^{\perp}$. Then there exist scalars $d_1, d_2, \ldots, d_t$ such that

$$y = \sum_{i=1}^{t} d_i w_i$$

Thus

$$T(y) = T\left(\sum_{i=1}^{t} d_i w_i\right) = \sum_{i=1}^{t} d_i T(w_i) = \sum_{i=1}^{t} d_i 0 = 0$$

We prove $N(T) = W^{\perp}$:

Let $x \in V$. By Theorem 6.6, there exist unique vectors $u \in W$ and $z \in W^{\perp}$ such that $x = u + z$. Thus $T(x) = T(u + z) = T(u) + T(z) = u$. Thus $x \in N(T)$ if and only if $u = 0$, which occurs if and only if $x = z \in W^{\perp}$. Therefore $N(T) = W^{\perp}$.

We prove $\|T(x)\| \leq \|x\|$:

Let $x \in V$. By Theorem 6.6, there exist unique vectors $u \in W$ and $z \in W^{\perp}$ such that $x = u + z$. As before, we have $T(x) = u$. Thus, since $\|z\| \geq 0$,

$$\|T(x)\| = \|u\| \leq \|u\| + \|z\|$$

Now, since $u \in W$ and $z \in W^{\perp}$, we must have $u \perp z$. Thus, by Exercise 10 of Section 6.1,

$$\|u\| + \|z\| = \|u + z\| = \|x\|$$

Therefore we conclude that

$$\|T(x)\| \leq \|x\| \quad \blacksquare$$