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A Harnack type inequality for the Yamabe equation in low dimensions

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1 Introduction

Let (M, g) be an $n-$dimensional, smooth, compact Riemannian manifold without boundary. For $n = 2$, the uniformization theorem of Poincaré gives the existence of Riemannian metrics which are pointwise conformal to g and have constant Gauss curvature. For $n \geq 3$, the Yamabe conjecture states that there exist Riemannian metrics which are pointwise conformal to g and have constant scalar curvature. The Yamabe conjecture was proved through the works of Yamabe [41], Trudinger [40], Aubin [2], and Schoen [29]. For $n \geq 3$, let $\tilde{g} = u^{\frac{4}{n-2}} g$ for some positive function u, we have

$$R_{\tilde{g}} = u^{-\frac{n+2}{n-2}} \left(R_g u - \frac{4(n-1)}{n-2} \Delta_g u \right),$$

where R_g denotes the scalar curvature of g and Δ_g denotes the Laplace-Beltrami operator of g. The Yamabe conjecture is therefore equivalent to the solvability of

$$-L_g u = \bar{R} u^{\frac{n+2}{n-2}}, \quad u > 0, \qquad \text{in } M, \tag{1}$$

for $\bar{R} = -1, 0$ or 1, where $L_g = \Delta_g - c(n) R_g, c(n) = \frac{n-2}{4(n-1)}$, is the conformal Laplacian of g. It is known that the equation can not be solved for more than one of the $\bar{R}$. If the first eigenvalue of $-L_g$ is negative, there exists a solution of (1) for $\bar{R} = -1$, and the solution is unique. If the first eigenvalue of $-L_g$ is 0, solutions of (1) with $\bar{R} = 0$ are positive eigenfunctions associated with the first eigenvalue. On the other hand, when the first eigenvalue of $-L_g$ is positive, the structure of solutions of (1) with $\bar{R} = 1$ are in general more complicated (see, e.g., [29]).

We first assume that (M, g) is locally conformally flat and the first eigenvalue of $-L_g$ is positive, and we consider

$$-L_g u = u^{\frac{n+2}{n-2}}, \quad u > 0, \qquad \text{in } M, \tag{2}$$

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A Riemannian manifold is called locally conformally flat if near every point of M the metric g can be written in some local coordinates as $g = e^v(dx_1^2 + \cdots dx_n^2)$ for some function v.

For $Q \in M$ and $\lambda > 0$, let

$$\xi_{Q,\lambda}(P) = (n(n-2))^{\frac{n-2}{4}} \left(\frac{\lambda}{1 + \lambda^2 dist_g(P,Q)^2} \right)^{\frac{n-2}{2}}, \qquad P \in M.$$

Schoen proved in [31] that there exists some positive constant C, depending only on (M,g), such that for any smooth solution u of (2), there exist local maximum points $\mathcal{S} = \{P_1, \cdots, P_m\}$ of u such that

$$dist_g(P_i, P_j) \geq \frac{1}{C}, \qquad \forall\, i \neq j, \tag{3}$$

$$\frac{1}{C} u(P_i) \leq u(P_j) \leq C u(P_i), \qquad \forall\, i, j, \tag{4}$$

and

$$u(P) \leq C \sum_{l=1}^{m} \xi_{P_l, u(P_l)}(P), \qquad \forall\, P \in M. \tag{5}$$

By (3), $m = \#\mathcal{S}$ is bounded by some constant depending only on (M,g). A consequence of (3) and (5) is the following energy estimate: for any solution u of (2),

$$\int_M u^{\frac{2n}{n-2}} \leq C(M,g). \tag{6}$$

Another consequence of (3), (4) and (5) is

$$\sup_{dist_g(P,Q)<\epsilon} u \cdot \inf_{dist_g(P,Q)<4\epsilon} u \leq C(M,g)\epsilon^{2-n}, \qquad \forall\, \epsilon > 0 \text{ and } Q \in M. \tag{7}$$

Based on the above estimates and the positive mass theorem of Schoen and Yau [35], Schoen proved in [31] that

$$m = \#\mathcal{S} \ \text{ can be taken as 1 and } P_1 \text{ can be taken as a maximum point of } u, \tag{8}$$

and, if (M,g) is not conformally diffeomorphic to the standard $n-$sphere, that

$$\max_M u \leq C(M,g). \tag{9}$$

When (M,g) is conformally diffeomorphic to the standard $n-$sphere, estimate (9) is not valid. By standard elliptic estimates, bounds of derivatives of u follow from (9) and the equation satisfied by u. Another proof of (9) was given in [32].

Schoen [31] and Schoen and Zhang [36] also studied, for $n = 3$, the scalar curvature equation

$$-L_g u = K(x) u^5, \quad u > 0, \qquad \text{on } M, \tag{10}$$

where K is a positive smooth function on M. They established (3), (4), (5) and (8) for solutions u of (10), with C depending on K. Related compactness results on standard 2 and 3−spheres were established by Chang, Gursky and Yang [10] using different methods. The noncompactness of the conformal automorphism groups of the standard spheres makes certain aspects of the analysis more difficult on spheres. The standard spheres are the only compact Riemannian manifolds with noncompact conformal automorphism groups ([18], see also [33]).

The first author studied in [24] and [25], for $n \geq 3$, the scalar curvature equation

$$-L_g u = K(x) u^{\frac{n+2}{n-2}}, \quad u > 0, \qquad \text{on } M. \tag{11}$$

He introduced, for $\beta \geq 1$, a β−flatness condition $(*)_\beta$ (see Definition 0.4 in [25]) on K, and established (3), (4), (5), and, consequently, (6) and (7), for solutions u of (11), provided that K satisfies $(*)_{n-2}$. Positive smooth functions K satisfy $(*)_{n-2}$ for $n = 3, 4$. The condition $(*)_\beta$ is also monotone: if K satisfies $(*)_\beta$ then it satisfies $(*)_{\beta'}$ for $\beta' < \beta$. Estimate (8) is established in [24] for solutions of (11), provided that K satisfies $(*)_\beta$ for some $\beta > n-2$. It was shown in [25] that (8) does not hold in general under $(*)_{n-2}$, and multiple blow up points may occur to a sequence of solutions.

Condition $(*)_\beta$ for $\beta > n-2$ implies that

$$\nabla^\alpha K(x) = 0,\ 2 \leq |\alpha| \leq n-2, \text{ at points where } \nabla K(x) = 0, \tag{12}$$

a condition under which Escobar and Schoen [15] established the existence of solutions of (11) when (M, g), still locally conformally flat, is not conformally diffeomorphic to the standard sphere. Such an existence result is not valid on the real projective space $\mathbb{R}P^n$ for $n \geq 4$ under $(*)_{n-2}$, as shown by Bianchi and Egnell [6] and Bianchi [5]. This shows the relevance of the flatness order $n-2$. On the other hand, it is not known whether (12) is enough for any of the estimates (3), (4), (5), (6) and (7) to hold for solutions of (11), see the questions on page 552 of [25]. Estimates (3), (4) and (5) have played a central role in establishing the existence results in [36,24] and [25].

Schoen also established in [31] a local form of the estimate (7): let u be a smooth solution of

$$-\Delta u = u^{\frac{n+2}{n-2}},\ u > 0, \qquad \text{in } B_4, \tag{13}$$

then

$$\sup_{B_\epsilon} u \cdot \inf_{B_{4\epsilon}} u \leq C(n)\epsilon^{2-n}, \qquad \forall\, 0 < \epsilon < 1, \tag{14}$$

where Δ denotes the Laplacian and B_ϵ denotes the ball in $\mathbb{R}^n$ centered at the origin and of radius ϵ. For $n = 3$ and K being a positive function, estimate (14), with $C(K)$, was established in [31] for solutions of

$$-\Delta u = K(x)u^5,\ u > 0, \qquad \text{in } B_4.$$

Under the condition $(*)_{n-2}$ on K, such a local form of estimate (7) for

$$-\Delta u = K(x)u^{\frac{n+2}{n-2}},\ u > 0, \qquad \text{in } B_4$$

was established by Chen and Lin in [11]. The estimate does not hold in general under $(*)_\beta$ for $\beta < n-2$. Analogues of the Harnack type inequality (7) in dimension two were established by Brezis and Merle [7], Brezis, Li and Shafrir [8], Chen and Lin [13], and Li [26]. Such Harnack type inequalities on compact Kähler manifolds were established by Siu [38] and Tian [39].

When K satisfies $(*)_{n-2}$, any solution of (11) on the locally conformally flat manifold satisfies

$$\int_M u^{\frac{2n}{n-2}} \le C(M, g, K). \tag{15}$$

This is a consequence of the previously mentioned estimates (3) and (5). Estimate (15) also holds for a class of K satisfying $(*)_{\frac{n-2}{2}}$, a result in [12]. On the other hand, as partly conjectured by Korevaar and Schoen and shown by Chen and Lin [14], the energy estimate (15) fails in general for K with flatness order $\beta < \frac{n-2}{2}$.

Question 1.1. Does the energy estimate (15) hold under $(*)_\beta$ for $\frac{n-2}{2} \le \beta < n-2$?

Now we turn to general, i.e. not necessarily locally conformally flat, Riemannian manifolds (M, g) of dimension $n \ge 3$. It was conjectured by Schoen [32] that all solutions u of the Yamabe equation

$$-L_g u = u^{\frac{n+2}{n-2}}, \quad u > 0, \qquad \text{on } M, \tag{16}$$

satisfy

$$\max_M u \le C(M, g), \tag{17}$$

unless (M, g) is conformally diffeomorphic to the standard $n-$sphere. He proved this when (M, g) is locally conformally flat, as mentioned earlier. For the general case, he suggested an approach together with some estimates in [31] and [32]. Following these ideas, Li and Zhu [28] established, for $n = 3$, (3), (4), (5), (8), and, if (M, g) is not conformally diffeomorphic to the standard sphere, (17). In fact, the estimates were established for more general equations of the form $-\Delta_g u + k(x)u = K(x)u^5$. Equations $-\Delta_g u + k(x)u = u^{\frac{n+2}{n-2}}$, for $n \ge 3$, were studied by Bahri and Brezis [3].

In this paper we establish the Harnack type inequality on three and four dimensional Riemannian manifolds.

Let $B_1 \subset \mathbb{R}^n$, $n \ge 3$, be the unit ball centered at the origin, and let $(a_{ij}(x))$ be a smooth, $n \times n$ symmetric positive definite matrix function, defined on B_1, satisfying

$$\frac{1}{2}|\xi|^2 \le a_{ij}(x)\xi^i\xi^j \le 2|\xi|^2, \qquad \forall\, x \in B,\ \xi \in \mathbb{R}^n, \tag{18}$$

and, for some $\bar{a} > 0$,

$$\|a_{ij}\|_{C^3(B_1)} \le \bar{a}. \tag{19}$$

Consider the Riemannian metric

$$g = a_{ij}(x)dx^i dx^j \tag{20}$$

on B_1, and consider the Yamabe equation

$$-L_g u = u^{\frac{n+2}{n-2}}, \qquad u > 0, \quad \text{on} \quad B_1. \tag{21}$$

Our main result is

Theorem 1.1. *For $n = 3, 4$, let (B, g) be as above, then there exist some positive constants δ and C_0, depending only on $\bar{a}$, such that any smooth solution u of (21) satisfies*

$$\sup_{B(0,\epsilon)} u \cdot \inf_{B(0,4\epsilon)} u \le C_0 \epsilon^{2-n}, \qquad \forall\, 0 < \epsilon < \delta, \tag{22}$$

where $B(0, \epsilon)$ denotes the geodesic ball, with respect to g, centered at 0 with radius ϵ.

A consequence of the Harnack type inequality (7) is, as proved by Schoen in [31] (see also [11] for an alternative proof), the following energy estimate: for any solution u of (13),

$$\int_{B_1} u^{\frac{2n}{n-2}} \le C(n). \tag{23}$$

The same proof yields the following consequence of Theorem 1.1:

Corollary 1.1. *For $n = 3, 4$, let u be any smooth solution of (21), then*

$$\int_{B_{\frac{1}{2}}} u^{\frac{2n}{n-2}} \le C(\bar{a}).$$

Such Harnack type inequality for general conformally invariant fully nonlinear elliptic equations of second order have been established on locally conformally flat Riemannian manifolds by Li and Li in [19,20] and [23]. Theorem 1.1 can be viewed as a first step in an effort to establish the Harnack type inequality for conformally invariant fully nonlinear equations on general Riemannian manifolds. Such Harnack type inequality on general Riemannian manifolds would, as pointed out in [19], yield the energy estimate for solutions of a large class of conformally invariant fully nonlinear equations.

Now

Question 1.2. Does the conclusion of Theorem 1.1, with δ and C_0 depending also on n, hold for $n \ge 5$?

The above question is closely related to the previously mentioned conjecture of Schoen concerning the compactness of the moduli space of (16).

For reader′s convenience, we first give in Sect. 2 the proof of Theorem 1.1 for $n = 3$, and then, in Sect. 3, the proof for $n = 4$. Our proof, an application of the method of moving planes, uses the ansatz in Schoen′s proof of (14) in [31] (see also [8,11,13] and [26] where such ansatz was used). The main task in our proof of Theorem 1.1 is to produce suitable auxiliary functions so that the method of moving planes applies. The construction of the auxiliary functions for $n = 4$ is more delicate than that for $n = 3$. The method of moving planes has become a powerful tool in the study of nonlinear elliptic equations, see Alexandrov [1], Serrin [37], Gidas, Ni and Nirenberg [16], and Berestycki and Nirenberg [4], and others. The proof of (14) in [31] makes use of the Liouville type theorem of Caffarelli, Gidas and Spruck [9], while our proof of Theorem 1.1, as in our earlier paper [27], does not use the Liouville type theorem. This has played an important role in the proof of Li and Li ([19,20,23]) of the Harnack type inequality and the existence and compactness theorems for a fully nonlinear version of the Yamabe problem on locally conformally flat manifolds, under the circumstance that the associated Liouville type theorems were not available. Later they obtained such Liouville type theorems in [21] and [22], see also [23]. Our proof of Theorem 1.1 is by contradiction argument, as in the proof of (14) in [31], and therefore does not yield explicit constants δ and C_0. On the other hand, a direct proof has been given in [20] (see also [23]), and the argument can be applied here to obtain explicit constants δ and C_0.

2 Proof of Theorem 1.1 for $n = 3$:

In this section we establish Theorem 1.1 for $n = 3$. We argue by contradiction. Suppose that (22) does not hold, then for some $\bar{a} > 0$ there exist a sequence of Riemannian metrics $\{g_k\}$ of the form (20) and satisfying (18) and (19), but for some $\epsilon_k \to 0^+$ and some solutions u_k of (21) with g replaced by g_k, we have

$$\max_{B(0,\epsilon_k)} u_k \cdot \min_{B(0,4\epsilon_k)} u_k > k\epsilon_k^{2-n}, \tag{24}$$

where $B(0,\epsilon_k)$ denotes the geodesic ball with respect to g_k.

We will keep n in many formulas in this section, since they are valid in higher dimensions and will be used in Sect. 3 for $n = 4$.

By (18) and (19), there exists $\bar{\epsilon} = \bar{\epsilon}(n, \bar{a}) > 0$ such that the maximum principle holds for L_g on $B(0,r)$ for $0 < r \le \bar{\epsilon}$. Thus, since $L_g u_k \le 0$, we have

$$\min_{B(0,r)} u_k = \min_{\partial B(0,r)} u_k, \qquad \forall\, 0 < r \le \bar{\epsilon}. \tag{25}$$

For some $\bar{x}_k \in \overline{B(0,\epsilon_k)}$, $u_k(\bar{x}_k) = \max_{B(0,\epsilon_k)} u_k$, and, from the above,

$$u_k(\bar{x}_k)\epsilon_k^{\frac{n-2}{2}} \to \infty.$$

By a standard selection process (see, e.g., Lemma 5.1 and the proof of Theorem 5.1 in [27]), we can find $x_k \in B(\bar{x}_k, \epsilon_k/2)$ and $\sigma_k \in (0, \epsilon_k/4)$ satisfying

$$u_k(x_k)^{\frac{2}{n-2}}\sigma_k \to \infty, \tag{26}$$

$$u_k(x_k) \geq u_k(\bar{x}_k), \tag{27}$$

and

$$u_k(x) \leq C_1 u_k(x_k), \qquad \forall\, B(x_k, \sigma_k), \tag{28}$$

where C_1 is some universal constant.

It follows from (27), (25) and (24) that

$$u_k(x_k) \cdot \min_{\partial B(x_k, 2\epsilon_k)} u_k \cdot \epsilon_k^{n-2} \geq u_k(\bar{x}_k) \cdot \min_{B(0,4\epsilon_k)} u_k \cdot \epsilon_k^{n-2} \geq k \to \infty. \tag{29}$$

We use $\{z^1, \cdots, z^n\}$ to denote some geodesic normal coordinates centered at x_k (e.g., given by the exponential map). In the geodesic normal coordinates, $g = g_{ij}(z)dz^i dz^j$,

$$g_{ij}(z) = \delta_{ij} + O(r^2), \quad g := det(g_{ij}(z)) = 1 + O(r^2), \quad R_g(z) = O(1), \tag{30}$$

where $r = |z|$. Thus

$$\Delta_g u = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j u) = \Delta u + b_i\partial_i u + d_{ij}\partial_{ij}u,$$

where

$$b_j = O(r), \quad d_{ij} = O(r^2). \tag{31}$$

Here, and below, $\partial_i = \frac{\partial}{\partial z^i}$ and $\partial_{ij} = \frac{\partial^2}{\partial z^i \partial z^j}$.

The equation of u_k can be written as

$$L_g u_k + u_k^{\frac{n+2}{n-2}} = \Delta u_k + b_i\partial_i u_k + d_{ij}\partial_{ij}u_k - c(n)R_g u_k + u_k^{\frac{n+2}{n-2}} = 0 \quad \text{in } B(0, 3\epsilon_k). \tag{32}$$

We rescale u_k as

$$v_k(y) = M_k^{-1} u_k\left(M_k^{-\frac{2}{n-2}} y\right) \qquad \text{for} \quad |y| \leq 3\epsilon_k M_k^{\frac{2}{n-2}},$$

where $M_k = u_k(0)$. By (26) and the fact that $\epsilon_k \geq 4\sigma_k$,

$$\lim_{k\to\infty} \epsilon_k M_k^{\frac{2}{n-2}} = \lim_{k\to\infty} \sigma_k M_k^{\frac{2}{n-2}} = \infty. \tag{33}$$

The rescaled function v_k satisfies, using (32), (28) and (29),

$$\begin{cases} \Delta v_k + \bar{b}_i \partial_i v_k + \bar{d}_{ij}\partial_{ij} v_k - \bar{c} v_k + v_k^{\frac{n+2}{n-2}} = 0 & \text{for} \quad |y| < 3\epsilon_k M_k^{\frac{2}{n-2}}, \\ v_k(0) = 1, & \\ v_k(y) \le C_1, & |y| \le \sigma_k M_k^{\frac{2}{n-2}}, \\ \lim_{k\to\infty} \min_{|y|=2\epsilon_k M_k^{\frac{2}{n-2}}} \left(v_k(y)|y|^{n-2}\right) = \infty, & \end{cases} \tag{34}$$

where C_1 is the universal constant in (28),

$$\bar{b}_i(y) = M_k^{-\frac{2}{n-2}} b_i(M_k^{-\frac{2}{n-2}} y), \quad \bar{d}_{ij}(y) = d_{ij}(M_k^{-\frac{2}{n-2}} y), \tag{35}$$

and

$$\bar{c}(y) = c(n) R\left(M_k^{-\frac{2}{n-2}} y\right) M_k^{-\frac{4}{n-2}}. \tag{36}$$

Here, as we very often do later, we have omitted the k dependence in the notations of $\bar{b}_i$, $\bar{d}_{ij}$ and $\bar{c}$.

For $|y| \le 3\epsilon_k M_k^{\frac{2}{n-2}}$, we have, by (31),

$$|\bar{b}_i(y)| \le C M_k^{-\frac{4}{n-2}} |y|, \ |\bar{d}_{ij}(y)| \le C M_k^{-\frac{4}{n-2}} |y|^2, \ |\bar{c}(y)| \le C M_k^{-\frac{4}{n-2}}, \tag{37}$$

where C depends only on n and $\bar{a}$.

It follows from (33), (34) and (37), using standard elliptic estimates, that, along a subsequence, v_k converges in C^2 norm on any compact subset of $\mathbb{R}^n$ to a positive function U satisfying

$$\begin{cases} \Delta U + U^{\frac{n+2}{n-2}} = 0 & \text{in} \quad \mathbb{R}^n, \\ U(0) = 1, & 0 < U \le C_1. \end{cases} \tag{38}$$

For simplicity, we still use v_k to denote the subsequence.

Thus, for any $a > 0$, there exist constants $c(a) > 0$ and $\bar{k}_a > 0$, independent of k, such that

$$c(a) < v_k(y) \le v_k(y) + |\nabla v_k(y)| + |\nabla^2 v_k(y)| \le \frac{1}{c(a)}, \quad \forall\ |y| \le a \text{ and } k \ge \bar{k}_a. \tag{39}$$

For $x \in \mathbb{R}^n$ and $\lambda > 0$, let

$$v_k^{\lambda,x}(y) := \left(\frac{\lambda}{|y-x|}\right)^{n-2} v_k\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right)$$

denote the Kelvin transformation of v_k with respect to the ball centered at x and of radius λ.

We shall compare, for any fixed x, v_k and $v_k^{\lambda,x}$ and we shall always take, for simplicity, $x=0$. For $x\neq 0$, the arguments are similar. We use the notation v_k^λ instead of $v_k^{\lambda,0}$, i.e.

$$v_k^\lambda(y) = \left(\frac{\lambda}{|y|}\right)^{n-2} v_k(y^\lambda) \qquad \text{where} \quad y^\lambda = \frac{\lambda^2 y}{|y|^2}.$$

Set, for $\lambda > 0$,

$$\Sigma_\lambda = B\left(0, \epsilon_k M_k^{\frac{2}{n-2}}\right) \setminus \overline{B(0,\lambda)}.$$

From now on we restrict the domain to $B(0,\epsilon_k M_k^{\frac{2}{n-2}})$. This is needed for $x\neq 0$, since we would still have $B(x,\epsilon_k M_k^{\frac{2}{n-2}}) \subset B(0, 2\epsilon_k M_k^{\frac{2}{n-2}})$ for k large (depending on an upper bound of $|x|$), and the rest of the arguments are the same as that, given below, for $x=0$.

By (25) and (34),

$$\min_{|y|=\epsilon_k M_k^{\frac{2}{n-2}}} \left(v_k(y)|y|^{n-2}\right) \geq 2^{2-n} \min_{|y|=2\epsilon_k M_k^{\frac{2}{n-2}}} \left(v_k(y)|y|^{n-2}\right) \to \infty. \tag{40}$$

In the rest of this section, unless otherwise stated, we use the following notations: $\lambda_1 > 0$ denotes a fixed arbitrary large constant, $\lambda \in (0,\lambda_1]$, k is large (the largeness of k depends on λ_1), and C denotes various positive constants which are independent of k and λ (but allowed to depend on λ_1).

Since

$$\Delta v_k^\lambda(y) = \left(\frac{\lambda}{|y|}\right)^{n+2} \Delta v_k(y^\lambda),$$

we have, by (34)

$$\Delta v_k^\lambda(y) + v_k^\lambda(y)^{\frac{n+2}{n-2}} = E_1(y) \qquad y \in \Sigma_\lambda. \tag{41}$$

where

$$E_1(y) = -\left(\frac{\lambda}{|y|}\right)^{n+2} \left(\bar{b}_i(y^\lambda)\partial_i v_k(y^\lambda) + \bar{d}_{ij}(y^\lambda)\partial_{ij} v_k(y^\lambda) - \bar{c}(y^\lambda)v_k(y^\lambda)\right). \tag{42}$$

It follows from (37) that there exists $C_2 = C_2(\lambda_1)$ such that

$$|E_1(y)| \leq C_2 \lambda^{n+2} M_k^{-\frac{4}{n-2}} |y|^{-n-2}, \qquad \text{in } \Sigma_\lambda. \tag{43}$$

Let

$$w_\lambda = v_k - v_k^\lambda.$$

Here we have, for simplicity, omitted k in this notation. By (34) and (41),

$$\Delta w_\lambda + \bar{b}_i \partial_i w_\lambda + \bar{d}_{ij}\partial_{ij} w_\lambda - \bar{c} w_\lambda + \frac{n+2}{n-2}\xi^{\frac{4}{n-2}} w_\lambda = E_\lambda, \qquad \text{in } \Sigma_\lambda, \tag{44}$$

where ξ stays between v_k and v_k^λ, and

$$E_\lambda = -\bar{b}_i \partial_i v_k^\lambda - \bar{d}_{ij}\partial_{ij} v_k^\lambda + \bar{c} v_k^\lambda - E_1. \tag{45}$$

In the rest of this section, we take the specific value $n = 3$. A calculation yields, using (39), that

$$|\partial_i v_k^\lambda(y)| \le C\lambda |y|^{-2}, \qquad |\partial_{ij} v_k^\lambda(y)| \le C\lambda |y|^{-3}, \qquad \text{in } \Sigma_\lambda. \tag{46}$$

Using (37) and (46), we deduce from (45) the following

Lemma 2.1. *For some constant $C_3 = C_3(\lambda_1)$,*

$$|E_\lambda(y)| \le C_3 M_k^{-4} \lambda |y|^{-1}, \qquad \textit{in } \Sigma_\lambda. \tag{47}$$

Let

$$h_\lambda(y) = -C_3 \lambda M_k^{-4}(|y| - \lambda), \qquad \text{in } \Sigma_\lambda.$$

Lemma 2.2.

$$w_\lambda + h_\lambda \ge 0 \quad \textit{in } \Sigma_\lambda, \qquad \forall\, 0 < \lambda \le \lambda_1. \tag{48}$$

Proof of Lemma 2.2. We divide the proof into two steps.

Step 1. There exists $\lambda_{0,k} > 0$ such that (48) holds for all $0 < \lambda \le \lambda_{0,k}$.

To see this, we write

$$w_\lambda(y) = v_k(y) - v_k^\lambda(y) = \frac{1}{\sqrt{|y|}} \left(\sqrt{|y|} v_k(y) - \sqrt{|y^\lambda|} v_k(y^\lambda) \right).$$

Note that y and y^λ are on the same ray starting from the origin. Let, in polar coordinates,

$$f(r, \theta) = \sqrt{r} v_k(r, \theta).$$

By (39), there exists $r_0 > 0$ and $C > 0$ independent of k such that

$$\frac{\partial f}{\partial r}(r, \theta) > C r^{-\frac{1}{2}} \qquad \text{for} \quad 0 < r < r_0.$$

Consequently, for $0 < \lambda < |y| < r_0$, we have

$$\begin{aligned} & w_\lambda(y) + h_\lambda(y) = v_k(y) - v_k^\lambda(y) + h_\lambda(y) \\ > & \frac{1}{\sqrt{r_0}} C r_0^{-\frac{1}{2}} (|y| - |y^\lambda|) + h_\lambda(y) \\ > & \left(\frac{C}{r_0} - C_3 \lambda M_k^{-4}\right)(|y| - \lambda) \quad \text{since } |y| - |y^\lambda| > |y| - \lambda \\ > & 0. \end{aligned} \tag{49}$$

Since

$$|h_\lambda(y)| + v_k^\lambda(y) \le C(k, r_0)\lambda, \qquad r_0 \le |y| \le \epsilon_k M_k^{-2},$$

we can pick small $\lambda_{0,k} \in (0, r_0)$ (allowed to depend on k and r_0) such that for all $0 < \lambda < \lambda_{0,k}$ we have

$$w_\lambda(y) + h_\lambda(y) \ge \min_{|y| \le \epsilon_k M_k^{-2}} v_k(y) - C(k, r_0)\lambda_{0,k} > 0, \ \forall\, r_0 \le |y| \le \epsilon_k M_k^2.$$

Step 1 follows from this and (49).

Let

$$\bar\lambda^k = \sup\{0 < \lambda \le \lambda_1 \mid \quad w_\mu + h_\mu \ge 0 \quad \text{in} \quad \varSigma_\mu, \quad \text{for all } 0 < \mu \le \lambda\}. \tag{50}$$

Step 2. $\bar\lambda^k = \lambda_1$, i.e. (48) holds.

For this, the main estimate needed is

$$(\varDelta + \bar b_i \partial_i + \bar d_{ij}\partial_{ij} + \frac{n+2}{n-2}\xi^{\frac{4}{n-2}} - \bar c)(w_\lambda + h_\lambda) \le 0, \qquad \text{in} \quad \varSigma_\lambda, \tag{51}$$

i.e., in view of (44),

$$\varDelta h_\lambda + \bar b_i \partial_i h_\lambda + \bar d_{ij}\partial_{ij} h_\lambda + E_\lambda + (5\xi^4 - \bar c)h_\lambda \le 0 \qquad \text{in} \quad \varSigma_\lambda. \tag{52}$$

Since $h_\lambda < 0$ in $\varSigma_\lambda$,

$$5\xi^4 h_\lambda < 0 \qquad \text{in } \varSigma_\lambda.$$

The dominant term in (52) is

$$\varDelta h_\lambda(y) = -2C_3\lambda M_k^{-4}|y|^{-1}.$$

The rest of the terms are of higher orders. Indeed,

$$|\partial_i h_\lambda| \le C\lambda M_k^{-4}, \quad |\partial_{ij} h_\lambda| \le C\lambda M_k^{-4}|y|^{-1},$$

and, using also (37),

$$\begin{aligned} |\bar b_i(y)\partial_i h_\lambda| + |\bar d_{ij}(y)\partial_{ij} h_\lambda| + |\bar c h_\lambda| &\le C\lambda M_k^{-8}|y| \le C\lambda\epsilon_k^2 M_k^{-4}|y|^{-1} \\ &\le C_3\lambda M_k^{-4}|y|^{-1} \qquad\qquad \text{in} \quad \varSigma_\lambda. \end{aligned}$$

So, by (47) and the estimates above,

$$\begin{aligned} &\varDelta h_\lambda + \bar b_i \partial_i h_\lambda + \bar d_{ij}\partial_{ij} h_\lambda + E_\lambda + (5\xi^4 - \bar c)h_\lambda \\ &\le \varDelta h_\lambda + C_3\lambda M_k^{-4}|y|^{-1} + |E_\lambda| \\ &= -C_3\lambda M_k^{-4}|y|^{-1} + |E_\lambda| \le 0 \qquad \text{in} \quad \varSigma_\lambda. \end{aligned}$$

We see from (39) and the definitions of v_k^λ and h_λ that

$$|v_k^{\bar\lambda^k}(y)| + |h_{\bar\lambda^k}(y)| \le \frac{C(\lambda_1)}{|y|}, \qquad \forall\, |y| = \epsilon_k M_k^2.$$

Thus, by the boundary condition (40),

$$(w_{\bar\lambda^k} + h_{\bar\lambda^k})(y) > 0 \qquad \forall\, |y| = \epsilon_k M_k^2,$$

Since $w_{\bar\lambda^k} + h_{\bar\lambda^k}$ is non-negative and satisfies (52) with $\lambda = \bar\lambda^k$, we apply the strong maximum principle and the Hopf lemma to obtain

$$w_{\bar\lambda^k} + h_{\bar\lambda^k} > 0 \qquad \text{in } \Sigma_{\bar\lambda^k},$$

and

$$\frac{\partial}{\partial\nu}(w_{\bar\lambda^k} + h_{\bar\lambda^k}) > 0 \qquad \text{on } \partial B(0, \bar\lambda^k),$$

where $\frac{\partial}{\partial\nu}$ denotes the differentiation in the outer normal direction.

In view of the three estimates above, we must have $\bar\lambda^k = \lambda_1$. Step 2 is established. Lemma 2.2 is proved. □

Given any $\lambda_1 > 0$, since $\{v_k\}$ converges to U (along a subsequence) and h_λ converges to 0 on any compact subset of $\mathbb{R}^n$, we have, by sending k to ∞ in (48), that

$$U(y) \ge U^\lambda(y), \quad \text{for all } |y| \ge \lambda, \qquad 0 < \lambda < \lambda_1.$$

Since $\lambda_1 > 0$ is arbitrary, and since we can apply the same argument to compare v_k and $v_k^{\lambda,x}$, we have

$$U(y) \ge U^{\lambda,x}(y), \quad \text{for all } |y - x| \ge \lambda > 0.$$

This implies, by a calculus lemma (see, e.g., Lemma 11.2 in [27]), that U is a constant, contradicting to (38). Theorem 1.1 for $n = 3$ is proved. □

3 Proof of Theorem 1.1 for $n = 4$:

In this section we establish Theorem 1.1 for $n = 4$. The proof is along the same line of that for $n = 3$. The construction of the auxiliary function h_λ is more delicate, and we make use of coordinates with special properties to improve the estimate (37).

As for $n = 3$ we argue by contradiction. Suppose that (22) does not hold, then, as in Sect. 2, we can find $x_k \in B(0, 3\epsilon_k/2)$, $\epsilon_k \to 0$, and $\sigma_k \in (0, \epsilon_k/4)$ satisfying (26), (28) and (29). Let $\{z^1, \cdots, z^n\}$ be the conformal normal coordinates centered at x_k. For simplicity we write $R_{ij}(0)$, the Ricci curvature tensor at 0, as R_{ij}. The full curvature tensor R_{ijkl} is understood similarly. Let

$$A_{ij} = \frac{1}{2}R_{ij} - \frac{1}{4(n-1)}\left(\sum_l R_{ll}\right)\delta_{ij},$$

$$\varphi(z) = 1 + \frac{1}{2}A_{ij}z^i z^j,$$

$$\tilde g = \varphi^{\frac{4}{n-2}} g,$$

and

$$\tilde u = \varphi u.$$

Then, at 0, the Ricci curvature of $\tilde{g}$, $\tilde{R}_{ij}$, vanishes. By the conformal invariance of the equation satisfied by u, $\tilde{u}$ satisfies

$$-L_{\tilde{g}}\tilde{u} = \tilde{u}^{\frac{n+2}{n-2}}.$$

Since $\tilde{u}$ has essentially the same properties of u ((26), (28) and (29)), we will simply use u and g to denote $\tilde{u}$ and $\tilde{g}$ respectively. We will then have an additional property that $R_{ij}(0) = 0$. Consequently, $R(0) = 0$. We can also use the conformal normal coordinates of Lee and Parker [17], though the proof remains the same.

In the rest of this section, $n = 4$ unless otherwise stated. In such coordinates, we have

$$g = det(g_{ij}) = 1 - \frac{1}{3}R_{ij}z^i z^j + O(r^3), \tag{53}$$

$$g_{pq}(z) = \delta_{pq} + \frac{1}{3}R_{pijq}z^i z^j + O(r^3), \tag{54}$$

$$\Delta_g u = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j u\right) = \Delta u + b_i\partial_i u + d_{ij}\partial_{ij}u$$

where

$$b_j = \frac{1}{2g}\partial_i g g^{ij} + \partial_i g^{ij}, \qquad d_{ij} = g^{ij} - \delta^{ij}. \tag{55}$$

Since $R_{jp} = 0$, we have, by the expressions for g and g_{ij}, that

$$g^{ij} = \delta_{ij} - \frac{1}{3}R_{ipqj}z^p z^q + O(r^3),$$

$$\partial_i g^{ij} = -\frac{1}{3}R_{ipij}z^p - \frac{1}{3}R_{iiqj}z^q + O(r^2) = O(r^2)$$

and

$$\partial_i g = -\frac{2}{3}R_{ip}z^p + O(r^2) = O(r^2).$$

From the expressions above and $R(0) = 0$ we have

$$b_i = O(r^2), \quad d_{ij} = -\frac{1}{3}R_{ipqj}z^p z^q + O(r^3), \quad R = O(r). \tag{56}$$

Recall that $\bar{b}_i$, $\bar{d}_{ij}$ and $\bar{c}$ are defined in (35) and (36). By the estimates above we have

$$|\bar{b}_i(y)| \le CM_k^{-\frac{6}{n-2}}|y|^2, \quad |\bar{c}| \le CM_k^{-\frac{6}{n-2}}|y| \tag{57}$$

and

$$\bar{d}_{ij}(y) = -\frac{1}{3}M_k^{-\frac{4}{n-2}}R_{ipqj}y^p y^q + O(1)M_k^{-\frac{6}{n-2}}|y|^3, \tag{58}$$

We claim that there exists $C_4 > 0$ independent of k such that for large k,

$$v_k(y) \geq C_4|y|^{2-n}, \qquad \text{for} \quad 1 \leq |y| \leq 2\epsilon_k M_k^{\frac{2}{n-2}}. \tag{59}$$

Indeed for some $\epsilon > 0$, depending only on $\bar{a}$, the operator L_g satisfies the maximum principle in $B(x_k, 2\epsilon)$ and the Green's function G for $-L_g$ on $B(x_k, 2\epsilon)$ with respect to the Dirichlet boundary condition satisfies, for some $C > 0$ independent of k, $C^{-1}|x - x_k|^{2-n} \leq G(x, x_k) \leq C|x - x_k|^{2-n}$ for $0 < |x - x_k| < \epsilon/2$. By the boundary condition (29),

$$\lim_{k\to\infty} \min_{|x-x_k|=\epsilon_k} u_k(x) M_k \epsilon_k^{n-2} = \infty,$$

and therefore

$$u_k(x) \geq C^{-1}M_k^{-1}|x - x_k|^{2-n} \geq C^{-1}M_k^{-1}G(x, x_k), \qquad \forall\, |x - x_k| = \epsilon_k.$$

By (39),

$$u_k(x) = M_k v_k(M_k^{\frac{2}{n-2}}(x - x_k)) \geq C^{-1}M_k \geq C^{-1}M_k^{-1}G(x, x_k),$$
$$\forall\, |x - x_k| = M_k^{-\frac{2}{n-2}}.$$

Since $L_g u_k(x) \leq 0$ for $M_k^{-\frac{2}{n-2}} \leq |x - x_k| \leq \epsilon_k$, we have, by the maximum principle, that

$$u_k(x) \geq C^{-1}M_k^{-1}G(x, x_k) \geq C^{-1}M_k^{-1}|x - x_k|^{2-n},$$
$$\text{for} \quad M_k^{-\frac{2}{n-2}} \leq |x - x_k| \leq \epsilon_k.$$

This is, after scaling, the desired estimate (59).

Let $v_k, v_k^\lambda, w_\lambda, E_\lambda$ be defined as in Sect. 2. Recall that the equation of v_k^λ is (41) where E_1 satisfies (42) and (43). We will give an improved estimate for E_λ. In the rest of this section, unless otherwise stated, we use the following notations: $\lambda_1 > 0$ denotes a fixed arbitrarily large constant, $\lambda \in (0, \lambda_1]$, k is large (the largeness of k depends on λ_1, and also on λ_0 and r_0 which will appear later), and C denotes various positive constants which are independent of k and λ (but allowed to depend on λ_1).

Lemma 3.1. *For some positive constant $C_3 = C_3(\lambda_1)$,*

$$|E_\lambda(y)| \leq C_3\lambda^2 M_k^{-2}|y|^{-3} + C_3\lambda^2 M_k^{-3}|y|^{-1}, \quad \text{in } \Sigma_\lambda. \tag{60}$$

Proof of Lemma 3.1. In Σ_λ we have, by (39),

$$\partial_i v_k^\lambda(y) = (2 - n)\frac{\lambda^{n-2}}{|y|^n} y^i v_k(y^\lambda) + O(1)\lambda^n|y|^{-n} = O(1)\lambda^{n-2}|y|^{1-n}, \tag{61}$$

and

$$\partial_{ij} v_k^\lambda(y) = (n-2)(n\frac{\lambda^{n-2}}{|y|^{n+2}}y^i y^j - \frac{\lambda^{n-2}}{|y|^n}\delta_{ij})v_k(y^\lambda) + O(1)\lambda^n|y|^{-1-n}, \quad (62)$$

where $O(1)$ depends on λ_1 but is independent of k.

It follows from (45),(43), (57),(58), (61) and (62) that

$$\begin{aligned} E_\lambda(y) = & O(1)M_k^{-\frac{6}{n-2}}|y|^2(\lambda^{n-2}|y|^{1-n}) \\ & + \left(\frac{1}{3}M_k^{-\frac{4}{n-2}}R_{ipqj}y^p y^q + O(1)M_k^{-\frac{6}{n-2}}|y|^3\right)\cdot \\ & \left((n-2)\left(n\frac{\lambda^{n-2}}{|y|^{n+2}}y^i y^j - \frac{\lambda^{n-2}}{|y|^n}\delta_{ij}\right)v_k\left(y^\lambda\right) + O(1)\lambda^n|y|^{-n-1}\right) \\ & + O(1)M_k^{-\frac{6}{n-2}}|y|\left(\frac{\lambda}{|y|}\right)^{n-2} v_k(y^\lambda) + O(1)M_k^{-\frac{4}{n-2}}\frac{\lambda^{n+2}}{|y|^{n+2}}. \end{aligned}$$

Using the antisymmetry property of R_{ipqj} and the fact that $R_{pq} = 0$, we have $R_{ipqj}y^p y^q \delta_{ij} = -R_{pq}y^p y^q = 0$ and $R_{ipqj}y^p y^q y^i y^j = 0$. Estimate (60) follows from the above. □

For $\alpha < 4$ and $\alpha \neq 2$, let

$$f_\alpha(z) = -\frac{1}{(4-\alpha)(2-\alpha)}[|z|^{2-\alpha} - 1] - \frac{1}{2(4-\alpha)}[|z|^{-2} - 1], \qquad |z| \geq 1.$$

Then

$$f_\alpha(z) = 0, \qquad |z| = 1,$$

and, for $|z| \geq 1$,

$$\Delta f_\alpha(z) = -|z|^{-\alpha},$$

$$f_\alpha(z) \leq 0, \ \ |f_\alpha(z)| \leq C(\alpha)|z|^{\max\{0,2-\alpha\}}, \quad (63)$$

$$|\nabla f_\alpha(z)| \leq C(\alpha)(|z|^{1-\alpha} + |z|^{-3}) \leq C(\alpha)|z|^{1-\alpha},$$

and

$$|\nabla^2 f_\alpha(z)| \leq C(\alpha)(|z|^{-\alpha} + |z|^{-4}) \leq C(\alpha)|z|^{-\alpha}. \quad (64)$$

Define

$$h_\lambda(y) = 2C_3\lambda M_k^{-2} f_3(\frac{y}{\lambda}) + 2C_3\lambda^3 M_k^{-3} f_1(\frac{y}{\lambda}), \qquad y \in \Sigma_\lambda.$$

Then

$$h_\lambda(y) \leq 0 \qquad \text{in } \Sigma_\lambda, \quad (65)$$

and

$$\Delta h_\lambda(y) = -2C_3\lambda^2 M_k^{-2}|y|^{-3} - 2C_3\lambda^2 M_k^{-3}|y|^{-1}, \qquad y \in \Sigma_\lambda. \quad (66)$$

Lemma 3.2.

$$w_\lambda + h_\lambda > 0 \qquad in \quad \Sigma_\lambda. \tag{67}$$

Proof of Lemma 3.2. As in the proof of Lemma 2.2, we divide the proof into two steps.

Step 1. There exists $\lambda_0 > 0$ independent of k such that (67) holds for all $0 < \lambda < \lambda_0$.

To see this, we write

$$w_\lambda(y) = v_k(y) - v_k^\lambda(y) = |y|^{-1}(|y|v_k(y) - |y^\lambda|v_k(y^\lambda)).$$

Let, in polar coordinates,

$$f(r,\theta) = rv_k(r,\theta).$$

By (39), there exist $r_0 > 0$ and $C > 0$ independent of k such that

$$\frac{\partial f}{\partial r}(r,\theta) > C > 0, \qquad \text{for } 0 < r < r_0.$$

Consequently,

$$w_\lambda(y) \ge C^{-1}|y|^{-1}|y - y^\lambda| \ge \frac{1}{Cr_0}(|y| - \lambda), \qquad \text{for } 0 < \lambda < |y| < r_0.$$

On the other hand, for $y \in \Sigma_\lambda$,

$$\begin{aligned} |h_\lambda(y)| &\le C\lambda M_k^{-2}|f_3(\frac{y}{\lambda})| + C\lambda^3 M_k^{-3}|f_1(\frac{y}{\lambda})| \\ &\le C\lambda M_k^{-2}||\frac{y}{\lambda}|^{-2} - 1| + C\lambda^3 M_k^{-3}||\frac{y}{\lambda}| - 1| \le CM_k^{-2}(|y| - \lambda). \end{aligned}$$

It follows that

$$w_\lambda + h_\lambda \ge (\frac{1}{Cr_0} - \frac{C}{M_k^2})(|y| - \lambda) > 0 \quad 0 < \lambda < |y| < r_0. \tag{68}$$

Now for $\epsilon_k M_k \ge |y| \ge r_0$, we have, by (63), (39) and (59), that $|h_\lambda(y)| \le CM_k^{-2} \le C\epsilon_k^2|y|^{-2} < \frac{1}{2}v_k(y)$. Therefore, for $\epsilon_k M_k \ge |y| \ge r_0$,

$$v_k(y) - v_k^\lambda(y) + h_\lambda(y) > \frac{1}{2}v_k(y) - v_k^\lambda(y) > \frac{1}{2}v_k(y) - \left(\frac{\lambda}{|y|}\right)^2 \max_{B(0,r_0)} v_k. \tag{69}$$

Because of (39), (59), (68) and (69), we can choose $\lambda_0 > 0$ independent of k such that (67) holds for $0 < \lambda < \lambda_0$.

Now we define $\bar\lambda^k$ as in (50).

Step 2. $\bar\lambda^k = \lambda_1$, i.e. (67) holds.

We know from Step 1 that $\lambda_0 \le \bar\lambda^k \le \lambda_1$. We want to show, for $\lambda_0 \le \lambda \le \lambda_k \le \lambda_1$, that

$$(\Delta + \bar b_i \partial_i + \bar d_{ij}\partial_{ij} + 3\xi^2 - \bar c)(w_\lambda + h_\lambda) \le 0, \qquad \text{in } \Sigma_\lambda. \tag{70}$$

In view of (44), this is equivalent to

$$(\Delta + \bar b_i \partial_i + \bar d_{ij}\partial_{ij} + 3\xi^2 - \bar c)h_\lambda + E_\lambda \le 0 \qquad \text{in} \quad \Sigma_\lambda, \tag{71}$$

for $\lambda_0 \le \lambda \le \bar\lambda^k \le \lambda_1$.

In the following, we assume that $\lambda_0 \le \lambda \le \bar\lambda^k \le \lambda_1$. We recall that $\lambda_0 > 0$ is independent of k, a fact which we use below. Using (57), (58) and (63), we have, for $y \in \Sigma_\lambda$, that

$$\begin{aligned} |\bar b_i||\partial_i h_\lambda(y)| &\le C M_k^{-3}|y|^2[M_k^{-2}|\nabla f_3\left(\frac{y}{\lambda}\right)| + M_k^{-3}|\nabla f_1\left(\frac{y}{\lambda}\right)|] \\ &\le CM_k^{-5} + CM_k^{-6}|y|^2 \le C\epsilon_k^3 M_k^{-2}|y|^{-3} + C\epsilon_k^3 M_k^{-3}|y|^{-1}, \\ |\bar c(y)h_\lambda(y)| &\le CM_k^{-5}|y| \le C\epsilon_k M_k^{-3}|y|^{-1}, \end{aligned}$$

and, by (58),

$$\begin{aligned} |\bar d_{ij}(y)||\partial_{ij}h_\lambda(y)| &\le CM_k^{-2}|y|^2|\nabla^2 h_\lambda(y)| \\ &\le C\epsilon_k^2\left[M_k^{-2}\left|\nabla^2 f_3\left(\frac{y}{\lambda}\right)\right| + M_k^{-3}\left|\nabla^2 f_1\left(\frac{y}{\lambda}\right)\right|\right] \\ &\le C\epsilon_k^2 M_k^{-2}|y|^{-3} + C\epsilon_k^2 M_k^{-3}|y|^{-1}. \end{aligned}$$

Putting together the above estimates and using (66) and Lemma 3.1, we have in Σ_λ that

$$\begin{aligned} &\left(\Delta + \bar b_i\partial + \bar d_{ij}\partial_{ij} - \bar c\right) h_\lambda(y) \\ &\le -2C_3\lambda^2 M_k^{-2}|y|^{-3} - 2C_3\lambda^2 M_k^{-3}|y|^{-1} + C\epsilon_k\left(M_k^{-2}|y|^{-3} + M_k^{-3}|y|^{-1}\right) \\ &\le -C_3\lambda^2 M_k^{-2}|y|^{-3} - C_3\lambda^2 M_k^{-3}|y|^{-1} \le -|E_\lambda(y)|. \end{aligned}$$

Estimate (71), and therefore (70), follows from this since $h_\lambda \le 0$ in Σ_λ. By (39) and the explicit expression of h_λ,

$$|v_k^{\bar\lambda^k}(y)| + |h_{\bar\lambda^k}(y)| \le \frac{C}{|y|^2}, \qquad \forall\, |y| = \epsilon_k M_k.$$

Thus, by the boundary condition (40),

$$(w_{\bar\lambda^k} + h_{\bar\lambda^k})(y) > 0, \qquad \forall\, |y| = \epsilon_k M_k.$$

Step 2 follows from the corresponding arguments in the proof of Lemma 2.2. The rest of the proof of Lemma 3.2 is the same as that in the proof of Lemma 2.2. □

Proof of Theorem 1.1 for $n = 4$. Given Lemma 3.2, the proof is the same as that of Theorem 1.1 for $n = 3$.

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