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Compactness of solutions to the Yamabe problem. II

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1. Introduction

Let (M^n, g) be a compact, smooth, connected Riemannian manifold (without boundary) of dimension $n \geq 3$. The Yamabe conjecture has been proved through the works of Yamabe [92], Trudinger [91], Aubin [4] and Schoen [79]: There exist constant scalar curvature metrics on M which are pointwise conformal to g.

Consider the Yamabe equation and its sub-critical approximations:

$$-L_g u = n(n-2)u^p, \qquad u > 0, \quad \text{on} \quad M, \tag{1}$$

where $1 < p \leq \frac{n+2}{n-2}$, $L_g = \Delta_g - c(n)R_g$, Δ_g is the Laplace-Beltrami operator associated with g, R_g is the scalar curvature of g, and $c(n) = \frac{(n-2)}{4(n-1)}$.

Let

$$\mathcal{M}_p = \{u \in C^2(M) \mid u \text{ satisfies } (1)\}.$$

Schoen initiated the investigation of the compactness of $\mathcal{M}_p$ and proved the following remarkable result in 1991, see [82], under the assumption that (M, g) is locally conformally flat and is not conformally diffeomorphic to standard spheres: For any $1 < 1 + \epsilon \leq p \leq \frac{n+2}{n-2}$ and for any non-negative integer k,

$$\|u\|_{C^k(M,g)} \leq C, \qquad \forall\, u \in \mathcal{M}_p, \tag{2}$$

where C is some constant depending only on (M, g), ϵ and k. He also announced in the same paper the same result for general manifolds, without the locally conformally flat assumption. The proof of this claim has not been made available. For general manifolds of dimension $n = 3$, a proof was given by Li and Zhu in [73]; while for $n = 4$, a combination of the results of Li and Zhang [70] and Druet [44] yields a proof, with the H^1 bound given in [70] and the L^∞ bound under the assumption of an H^1 bound given in [44].

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Let W_g denote the Weyl tensor (see for instance [6] for the definition), we consider the following two cases:

1°. $3 \le n \le 7$,

2°. $n \ge 8$ and $|W_g| + |\nabla W_g| > 0$ on M.

Theorem 1.1. *Let (M^n, g) be a compact, smooth, connected Riemannian manifold which is not locally conformally flat. Assume either 1° or 2°. Then, for any $1 < 1 + \epsilon \le p \le \frac{n+2}{n-2}$ and for any non-negative integer k, (2) holds for some constant C depending only on (M^n, g), ϵ and k.*

Remark 1.1. The proof of Theorem 1.1 in Case 1° makes use of the deep positive mass theorem of Schoen and Yau in [83]. If we assume the positive mass theorem for $n = 8, 9$, then our proof yields the conclusion of Theorem 1.1 on any non-locally conformally flat Riemannian manifolds of dimension $n = 8, 9$.

Remark 1.2. Theorem 1.1 in the case $n \le 4$ was, as mentioned earlier, already known. Theorem 1.1 in the case $n = 5, 6, 7$ as well as in the case $n \ge 8$ under $|W_g| > 0$ on M was announced in November 2003 by the first author in his talk at the Joint Analysis Seminar in Princeton University and also announced in our note [71]. These were also independently proved by Marques [75]. The case $n = 5$ was proved independently by Druet [45]. Theorem 1.1 was announced by the first author at the international conference in honor of Haim Brezis′s sixtyth birthday in Paris, June 9-13, 2004. Some further results for dimensions $n \ge 10$ will be given in a forthcoming paper [72].

Since the first eigenvalue of $-L_g$ is positive, multiplying (1) by u and integrating by parts on M lead to $\max_M u \ge C^{-1}$ for some positive constant C depending only (M, g) and ϵ. On the other hand, once we know that $u \le C$ on M for some C depending only on (M, g) and ϵ, an application of the Harnack inequality yields $u \ge C^{-1} \max_M u$ on M for some C depending only on (M, g) and ϵ. A consequence of Theorem 1.1 is, by some arguments in [82] and [66], the following

Corollary 1.1. *Under the hypothesis of Theorem 1.1, there exists some constant $\bar{C}$, depending only on (M, g) and ϵ, such that for all $1 < 1 + \epsilon \le p \le \frac{n+2}{n-2}$ and for all $C > \bar{C}$,*

$$deg\left(v - n(n-2)(L_g)^{-1}(v^p), \mathcal{O}_C, 0\right) = -1,$$

where $\mathcal{O}_C := \{v \in C^{2,\alpha}(M) \ : \ 1/C < v < C, \|v\|_{C^{2,\alpha}(M)} < C\}$, $0 < \alpha < 1$, and deg denotes the Leray-Schauder degree.

For the Leray-Schauder degree theory, see for instance [77].

Much of this paper is devoted to the fine analysis of blow up solutions of (1). In fact we establish the following local version of such estimates. Let $\Omega \subset M$ be an open connected subset of M, and let $\Omega_\epsilon := \{P \in \Omega \mid dist_g(P, \partial\Omega) > \epsilon)$. For $Q \in \Omega$ and $\mu > 0$, let

$$\xi_{Q,\mu}(P) = \left(\frac{\mu}{1 + \mu^2 \mathrm{dist}_g(P, Q)^2}\right)^{\frac{n-2}{2}}, \quad P \in \Omega.$$

We are interested in solutions of

$$-L_g u = n(n-2)u^p, \qquad u > 0, \quad \text{in } \Omega. \tag{3}$$

Theorem 1.2. *Let (M^n, g) be a compact, smooth, connected Riemannian manifold of dimension $n \geq 10$, and let $\Omega \subset M$ be an open connected subset, $1 < 1 + \epsilon \leq p \leq \frac{n+2}{n-2}$. Suppose that u is a smooth solution of (3) satisfying, for some $\bar{P} \in \Omega$ and some constant $\bar{b} \geq 1$, that*

$$\nabla u(\bar{P}) = 0, \qquad 1 \leq \sup_\Omega u \leq \bar{b} u(\bar{P}). \tag{4}$$

Then, for any $\epsilon' > 0$,

$$|W_g(\bar{P})|_g + |\nabla_g W_g(\bar{P})|_g u(\bar{P})^{-\frac{2}{n-2}} \leq C u(\bar{P})^{-\frac{4}{n-2}+\epsilon'}. \tag{5}$$

where C is some positive constant depending only on ϵ, ϵ', $dist_g(\bar{P}, \partial\Omega)$, $\bar{b}$, a positive lower bound for the injectivity radius of (M, g), a positive lower bound of the first eigenvalue of $-L_g$ on Ω with zero Dirichlet boundary condition, and a positive upper bound of the norm of the curvature tensor of (M, g) together with its covariant derivatives up to the eighth order.

Remark 1.3. Theorem 1.1 in the case $n \geq 10$ follows immediately from Theorem 1.2.

The estimates we establish for dimensions $n \leq 9$ in the next theorem is much stronger than that for $n \geq 10$. This is the reason that Theorem 1.1 for $n \leq 7$, as well as for dimensions $n = 8, 9$ as mentioned in Remark 1.1, hold without any assumption on the Weyl tensor.

Theorem 1.3. *Let (M^n, g) be a compact, smooth, connected Riemannian manifold of dimension $3 \leq n \leq 9$, and let $\Omega \subset M$ be an open connected subset, $1 < 1+\epsilon \leq p \leq \frac{n+2}{n-2}$. Suppose that u is a smooth solution of (3) satisfying (4) for some $\bar{P} \in \Omega$ and some constant $\bar{b} \geq 1$. Then, for any $\delta > 0$, there exist some positive constant C and some positive integer m, which depend only on ϵ, $dist_g(\bar{P}, \partial\Omega)$, $\bar{b}$, δ, a positive lower bound for the injectivity radius of (M, g), a positive lower bound for the first eigenvalue of $-L_g$ on Ω with zero Dirichlet boundary condition, and a positive upper bound of the norm of the curvature tensor of (M, g) together with its covariant derivatives up to the eighth order, and there exist local maximum points $\mathcal{S} := \{P_1, \cdots, P_m\} \subset \Omega_{2\delta}$ of u, such that*

$$dist_g(P_i, P_j) \geq \frac{1}{C}, \qquad \forall i \neq j, \tag{6}$$

$$\frac{1}{C} u(P_i) \leq u(P_j) \leq C u(P_i), \qquad \forall i, j, \tag{7}$$

$$|W_g(P_i)|_g \leq \begin{cases} \dfrac{C}{\sqrt{\log u(P_i)}}, \text{ if } n = 6, \\ C u(P_i)^{-\frac{n-6}{n-2}}, \text{ if } 7 \leq n \leq 9, \end{cases} \qquad \forall\, i \tag{8}$$

$$|\nabla W_g(P_i)|_g \le \begin{cases} \dfrac{C}{\sqrt{\log u(P_i)}}, & \text{if } n=8, \\ Cu(P_i)^{-\frac{n-8}{n-2}}, & \text{if } n=9. \end{cases} \qquad \forall\, i \tag{9}$$

$$\frac{1}{C}\sum_{l=1}^{m} \xi_{P_l, u(P_l)^{\frac{2}{n-2}}}(P) \le u(P) \le C\sum_{l=1}^{m} \xi_{P_l, u(P_l)^{\frac{2}{n-2}}}(P), \quad \forall P \in \Omega_{4\delta} \tag{10}$$

and, for $|\alpha| = 0, 1, 2$ *and* $P \in \Omega_{4\delta}$,

$$|\nabla_g^\alpha (u - \sum_{l=1}^{m} \xi_{P_l, u(P_l)^{\frac{2}{n-2}}})(P)|$$
$$\le \begin{cases} Cu(\bar{P})^{-1+\frac{2|\alpha|}{n-2}}(1+u(\bar{P})^{\frac{2}{n-2}} dist(P,\mathcal{S}))^{-|\alpha|}, & \text{if } n=3,4,5, \\ C(\epsilon')u(\bar{P})^{-1+\frac{2|\alpha|+2\epsilon'}{n-2}}(1+u(\bar{P})^{\frac{2}{n-2}} dist(P,\mathcal{S}))^{-\epsilon'-|\alpha|}, & \text{if } n=6, \\ Cu(\bar{P})^{1+\frac{2|\alpha|-8}{n-2}}(1+u(\bar{P})^{\frac{2}{n-2}} dist(P,\mathcal{S}))^{6-n-|\alpha|}, & \text{if } n=7,8,9. \end{cases} \tag{11}$$

Estimates (6), (7) and (10) on locally conformally flat manifolds, without the assumption (4), were established by Schoen in [81]. Analogues of such local results do not hold for many other problems, including Harmonic maps and Ginzburg-Landau vortices, with similar loss of compactness – similar in the sense that the energy is quantized when a sequence of solutions blows up. It is interesting to note that the analogue of (6) in such a local setting (under the assumption (4)) fails even for the equation $-\Delta u = Ve^u$ in dimension $n = 2$, as demonstrated by Chen in [41]. On the other hand, it was proved by the first author in [68] that analogues of (6), (7) and (10) hold for solutions of such equations defined globally on compact Riemannian surfaces.

The proof of the Yamabe conjecture ([92]), [91,4] and [79]), a milestone in the studies of nonlinear elliptic equations, concerns the existence of a minimizer of some functional with lack of compactness. Many further studies have been devoted to related critical exponent equations which address important issues including those concerning non-minimal solutions or approximate solutions to such equations, see for instance Brezis and Nirenberg ([23]) and Bahri and Coron ([11] and [12]). These studies have led to different proofs of the Yamabe conjecture in the case $n \le 5$ and in the case (M, g) is locally conformally flat, see Bahri and Brezis [10] and Bahri [9]. For the Nirenberg problem and the Yamabe problem on manifolds with boundary, which are related to the Yamabe problem, compactness of solutions has been studied by Schoen ([81]), Schoen and Zhang ([84]), Chang et al. ([32]), Li ([66]) and [67]), Han and Li ([57]), Chen and Lin ([35]), Felli and Ould Ahmedou ([51]), and Escobar and Garcia [49]. Much of the analysis in these works can be made purely local. One way to achieve this is a Harnack type inequality of Schoen ([81]): For $n \ge 3$, let u be a smooth positive solution of

$$-\Delta u = u^{\frac{n+2}{n-2}}, \qquad \text{in } B_4 \subset \mathbb{R}^n,$$

then

$$\mathrm{sup}_{\mathrm{B}_1}\mathrm{u}\cdot\mathrm{inf}_{\mathrm{B}_2}\mathrm{u}\le \mathrm{C}(\mathrm{n}). \tag{12}$$

A consequence of this is, as proved in [81], $\int_{B_1} u^{\frac{2n}{n-2}} \le C(n)$. Analogous results, extensions, as well as different proofs of (12) can be found in [87,90,81,22,21, 74,34,33,68,16,17,36,69,64,14,78,37,65,15,88,89], and others. In particular, it was proved in our paper [70] that such Harnack type inequality holds on non-locally conformally flat Riemannian manifolds of dimension $n = 3, 4$. Therefore the conclusion of Theorem 1.3 holds in dimension $n = 3, 4$ without the assumption (4).

The main step in the proof of Theorem 1.2 and Theorem 1.3 is to establish Theorem 2.1. The proof of Theorem 2.1 is based on the method of moving planes, using the ansatz of Schoen in his proof of (12) in [81] (see also [21,34,35,68] and [70] where such ansatz was used). The method of moving planes has become a powerful tool in the study of nonlinear elliptic equations, see Alexandrov [1], Serrin [85], Gidas et al. [54], Berestycki and Nirenberg [18], and others. The main task in our proof of Theorem 2.1 is to construct suitable auxiliary functions so that the method of moving planes can be applied. To do this we make use of numerous results and methods from previous works, some of which are described below. We need to make use of the Liouville type theorem of Caffarelli et al. in [25] which identifies the limit of the rescaled blow-up sequence of solutions, and we need to establish strong enough convergence rate of the difference of the rescaled blow-up sequence of solutions and its limit. Certain rate of convergence of the difference of the rescaled blow-up sequence of solutions and its limit was established by Chen and Lin in [35] for the scalar curvature equations in the Euclidean space, and we adapt their methods to establish iterated estimates on such convergence rate on larger and larger balls with improved estimates after each iteration. For dimensions $n \ge 8$, the iterated estimates also yield stronger and stronger decay estimates on the Weyl tensor and its first covariant derivatives at points where the sequence of solutions blow up. Here we also make use of a Pohozaev type identity as well as some properties of the conformal normal coordinates as established by Lee and Parker [63], Cao [27], Günther [53], and Hebey and Vaugon [60]. To construct auxiliary functions we also adapt the way of using the spherical harmonics by Caffarelli et al. in [26]. The proof of Theorem 1.1 is based on Theorem 1.2-1.3 and the positive mass theorem of Schoen and Yau in [83]. Theorem 1.2-1.3 provide strong enough pointwise estimates for blow-up solutions as well as, for higher dimensions, strong enough decay estimates for the Weyl tensor and its first covariant derivatives at the blow up points. These estimates allow us to use the positive mass theorem through the Pohozaev type identity as in Schoen [80]. We note that the Pohozaev identity has been used by Arkinson and Peletier [5] and Brezis and Peletier [24] to obtain pointwise estimates for blow up solutions of related critical exponent equations, see also [81,56,84,66,67], and others, for the extensive use of the Pohozaev identity in establishing pointwise estimates to blow up solutions of critical exponent equations.

Most of the works mentioned above concern the compactness of solutions or the fine pointwise analysis of blow up solutions to the Yamabe equation and some related ones, which often yield the existence of solutions through the use of degree theories. There have been many works on the existence of solutions to the Yamabe problem, the Nirenberg problem, and the Yamabe problem on manifolds with boundary, see for instance [76,62,42,43,61,50,29,30,39,38,55,59,31,46,47,19, 28,48,7,8,40,2,58] and [3]. There have also been works on parabolic flows associated with the Yamabe problem and the Yamabe problem on manifolds with boundary, see for instance [93,86] and [20].

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2. Main estimates

Let $B_1 \subset \mathbb{R}^n$, $n \geq 3$, be the unit ball centered at the origin, and let $(a_{ij}(x))$ be a smooth, $n \times n$ symmetric positive definite matrix function, defined on B_1, satisfying

$$\frac{1}{2}|\xi|^2 \leq a_{ij}(x)\xi^i\xi^j \leq 2|\xi|^2, \qquad \forall\, x \in B_1,\ \xi \in \mathbb{R}^n, \tag{13}$$

and, for some $\bar{a} > 0$,

$$\|a_{ij}\|_{C^8(B_1)} \leq \bar{a}. \tag{14}$$

Consider the Riemannian metric

$$g = a_{ij}(x)dx^i dx^j \tag{15}$$

on B_1, and consider

$$-L_g u = n(n-2)u^p, \qquad u > 0, \quad \text{on} \quad B_1. \tag{16}$$

Theorem 2.1. *Let (B_1, g) be as above and let u be a solution of (16), with $1 < 1+\epsilon \leq p \leq \frac{n+2}{n-2}$, satisfying, for some $\bar{b} \geq 1$,*

$$\nabla u(0) = 0, \qquad 1 \leq \sup_{B_1} u \leq \bar{b} u(0). \tag{17}$$

Then there exist some positive constants δ and C_0, depending only on n, $\bar{b}$, ϵ and $\bar{a}$, such that

$$u(0)u(x)|x|^{n-2} \leq C_0, \qquad \forall\, 0 < |x| \leq \delta, \qquad \textit{if} \quad 3 \leq n \leq 9, \tag{18}$$

$$|W_g(0)|_g \leq \begin{cases} \dfrac{C_0}{\sqrt{\log u(0)}}, \textit{ if } n = 6, \\ C_0 u(0)^{-\frac{n-6}{n-2}}, \textit{ if } 7 \leq n \leq 9, \end{cases} \tag{19}$$

$$|\nabla_g W_g(0)|_g \leq \begin{cases} \dfrac{C_0}{\sqrt{\log u(0)}}, \textit{ if } n = 8, \\ C_0 u(0)^{-\frac{n-8}{n-2}}, \textit{ if } n = 9. \end{cases} \tag{20}$$

If $n \geq 10$*, then for all* $\epsilon_1 > 0$*, there exists* $C(\epsilon_1) > 0$ *such that*

$$|W_g(0)|_g + |\nabla_g W_g(0)|_g u(0)^{-\frac{2}{(n-2)}} \leq C(\epsilon_1) u(0)^{-\frac{4}{(n-2)}+\epsilon_1}. \tag{21}$$

Remark 2.1. Theorem 1.2 follows from (21).

We first prove Theorem 2.1 for $p = \frac{n+2}{n-2}$. We point out the changes needed for $p < \frac{n+2}{n-2}$ in Sect. 5. Suppose that the conclusion of Theorem 2.1 for $p = \frac{n+2}{n-2}$ does not hold, then for some $\bar{a} > 0$, $\bar{b} \geq 1$, there exist a sequence of Riemannian metrics $\{\tilde{g}_k\}$ of the form (15) that satisfy (13) and (14), and some solutions u_k of (16), with $p = \frac{n+2}{n-2}$ and with g replaced by $\tilde{g}_k$, satisfying (17), such that one of the following happens:

$$\max_{|x|<\frac{1}{k}} \left(u_k(0) u_k(x) |x|^{n-2} \right) \geq k, \tag{22}$$

$$|W_{\tilde{g}_k}(0)|_{\tilde{g}_k} > \begin{cases} \dfrac{k}{\sqrt{\log u_k(0)}}, \text{ if } n = 6, \\ k u_k(0)^{-\frac{n-6}{n-2}}, \text{ if } 7 \leq n \leq 9, \end{cases} \tag{23}$$

$$|\nabla_{\tilde{g}_k} W_{\tilde{g}_k}(0)|_{\tilde{g}_k} > \begin{cases} \dfrac{k}{\sqrt{\log u_k(0)}}, \text{ if } n = 8, \\ k u_k(0)^{-\frac{n-8}{n-2}}, \text{ if } n = 9, \end{cases} \tag{24}$$

or, for some $\epsilon_5 > 0$ independent of k,

$$|W_{\tilde{g}_k}(0)|_{\tilde{g}_k} + |\nabla_{\tilde{g}_k} W_{\tilde{g}_k}(0)|_{\tilde{g}_k} u_k(0)^{-\frac{2}{n-2}} > k u_k(0)^{-\frac{4}{(n-2)}+\epsilon_5}, \qquad \text{if } n \geq 10. \tag{25}$$

We will simply use g to denote $\tilde{g}_k$.

Let $\bar{P}$ be a point on (M, g), it was proved in [63], together with some improvement in [27] and [53], that there exists some function φ (with control) near $\bar{P}$ such that the conformal metric $\tilde{g} = e^{\varphi} g$ satisfies, in $\tilde{g}-$normal coordinates $\{x^1, \cdots, x^n\}$ centered at $\bar{P}$

$$\det(\tilde{g}_{ij}) = 1.$$

Such coordinates are called conformal normal coordinates. As well known, we may assume that we work in conformal normal coordinates. In conformal normal coordinates (we write g_{ij} instead of $\tilde{g}_{ij}$), we have, at $x = 0$,

$$R_{ij} = 0,\ R_{,i} = 0,\ Sym_{ijk} R_{ij,k} = 0,\ \Delta_g R = -\frac{1}{6}|W|^2, \tag{26}$$

where R_{ijkl} denotes the curvature tensor evaluated at 0, R_{ij} denotes the Ricci curvature tensor at 0, $R_{ijkl,p}$ denotes covariant derivative of the curvature tensor at 0, etc., repeated indices denote summation over the indices, and

$$Sym_{p_1 \cdots p_m} A_{p_1 \cdots p_m} := \sum_{\sigma} A_{p_{\sigma(1)} \cdots p_{\sigma(m)}}$$

where σ runs through all permutations of $1, 2, \cdots, m$.

We make a conformal change of the metric $\hat{g} = e^{\varphi} g_k$ and let $\{z^1, \cdots, z^n\}$ be the conformal normal coordinates centered at the origin. After the conformal change, u_k becomes $\hat{u}_k = e^{\frac{n-2}{4}\varphi} u_k$. As well known all relevant properties of u_k hold for $\hat{u}_k$, and we simply assume that $g_{ij}(z)dz^i dz^j$ is already in conformal normal coordinates. In local coordinates,

$$g_{pq}(x) = \delta_{pq} + \frac{1}{3} R_{pijq} x^i x^j + \frac{1}{6} R_{pijq,k} x^i x^j x^k + \left(\frac{1}{20} R_{pijq,kl} + \frac{2}{45} R_{pijm} R_{qklm} \right) x^i x^j x^k x^l + O(r^5).$$

In conformal normal coordinates, write

$$\Delta_g = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j) = \Delta + b_i \partial_i + d_{ij} \partial_{ij},$$

where (g^{ij}) denotes the inverse matrix of (g_{ij}), $\partial_i = \frac{\partial}{\partial z^i}$, $\partial_{ij} = \frac{\partial^2}{\partial z^i \partial z^j}$, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial z^i \partial z^i}$,

$$\begin{aligned} b_i(x) &= \partial_j g^{ij}(x) \\ &= -\frac{1}{6} R_{ia,b} x^a x^b - \frac{1}{6} R_{iabp,p} x^a x^b - \left(\frac{1}{20} R_{ia,bc} - \frac{1}{15} R_{ipad} R_{pbcd} \right. \\ &\quad \left. - \frac{1}{15} R_{iapd} R_{pbcd} + \frac{1}{20} R_{iabp,pc} + \frac{1}{20} R_{iabp,cp} \right) x^a x^b x^c + O(r^4), \end{aligned}$$

and

$$\begin{aligned} d_{ij}(x) = g^{ij} - \delta_{ij} &= -\frac{1}{3} R_{ipqj} x^p x^q - \frac{1}{6} R_{ipqj,k} x^p x^q x^k \\ &\quad - \left(\frac{1}{20} R_{ipqj,kl} - \frac{1}{15} R_{ipqm} R_{jklm} \right) x^p x^q x^k x^l + O(r^5). \end{aligned}$$

By (17) and (22), $M_k := u_k(0) \to \infty$. Write $(g_k)_{ij}(y) = g_{ij}(M_k^{-\frac{2}{n-2}} y) dy^i dy^j$, then

$$\Delta_{g_k} = \Delta + \bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij},$$

where

$$\bar{b}_i(y) = M_k^{-\frac{2}{n-2}} b_i \left(M_k^{-\frac{2}{n-2}} y \right), \quad \bar{d}_{ij}(y) = d_{ij} \left(M_k^{-\frac{2}{n-2}} y \right).$$

Let

$$v_k(y) := M_k^{-1} u_k \left(M_k^{-\frac{2}{n-2}} y \right), \tag{27}$$

$$c(x) = c(n) R_g(x), \quad \text{and} \quad \bar{c}(y) = c(n) R_g \left(M_k^{-\frac{2}{n-2}} y \right) M_k^{-\frac{4}{n-2}}.$$

Then

$$|\bar{b}_i(y)| = O(1)M_k^{-\frac{6}{n-2}}|y|^2, \quad |\bar{d}_{ij}(y)| = O(1)M_k^{-\frac{4}{n-2}}|y|^2, \qquad (28)$$
$$\bar{c}(y) = O(1)M_k^{-\frac{8}{n-2}}|y|^2.$$

The rescaled function v_k satisfies

$$\begin{cases} \Delta_{g_k} v_k(y) - \bar{c} v_k(y) + n(n-2)v_k(y)^{\frac{n+2}{n-2}} = 0, & |y| \le \frac{1}{2}M_k^{\frac{2}{n-2}}, \\ 1 = v_k(0) \ge (\bar{b}^{-1} + \circ(1))v_k(y), \quad |y| \le \frac{1}{2}M_k^{\frac{2}{n-2}}, & \nabla v_k(0) = 0. \end{cases} \qquad (29)$$

Note that (22) is the same as

$$\max_{|y| \le \frac{1}{k}M_k^{\frac{2}{n-2}}} (v_k(y)|y|^{n-2}) \ge k, \qquad (30)$$

where $|y| := \sqrt{(y^1)^2 + \cdots + (y^n)^2}$. Since we eventually draw contradiction for large k, so throughout the paper k is large unless otherwise stated.

By standard elliptic estimates, solutions v_k of (29), after passing to a subsequence (still denoted as v_k, etc.), converge in $C^2_{loc}(\mathbb{R}^n)$ to some positive function U satisfying $U(0) = 1, \nabla U(0) = 0$ and

$$-\Delta U = n(n-2)U^{\frac{n+2}{n-2}} \qquad \text{in } \mathbb{R}^n.$$

By the Liouville type theorem in [25],

$$U(y) = (1 + |y|^2)^{-\frac{n-2}{2}} \qquad \text{in } \mathbb{R}^n.$$

For some universal constant $\delta_1 > 0$, the Green′s function $G(0,x)$ of $-L_g$ on $B(0, 3\delta_1)$, with respect to zero Dirichlet boundary condition, is positive and satisfies

$$\frac{1}{C} dist_g(0,x)^{2-n} \le G(0,x) \le C dist_g(0,x)^{2-n}, \qquad x \in B(0, 2\delta_1) \setminus \{0\},$$
$$\lim_{x \to 0} G(0,x) dist_g(0,x)^{n-2} = \frac{1}{(n-2)\omega_n}, \qquad (31)$$

where ω_n denotes the volume of the standard $(n-1)-$sphere and $C > 0$ is universal. By the convergence of v_k to U and the above behavior of the Green′s function,

$$u_k(x) \ge \frac{1}{C} M_k^{-1} G(0,x), \qquad \text{on } \partial \left(B(0, 3\delta_1) \setminus B\left(0, M_k^{-\frac{2}{n-2}}\right)\right).$$

Since u_k is a supersolution we obtain, using the maximum principle,

$$u_k(x) \ge C^{-1} M_k^{-1} G(0,x), \qquad \text{on } B(0, \delta_1) \setminus B\left(0, M_k^{-\frac{2}{n-2}}\right),$$

i.e.

$$v_k(y) \geq \frac{1}{C(1+|y|^{n-2})}, \qquad \forall\, 0<|y|\leq \delta_1 M_k^{\frac{2}{n-2}}. \tag{32}$$

For $\lambda > 0$ and for any function v, let

$$v^\lambda(y) := \left(\frac{\lambda}{|y|}\right)^{n-2} v(y^\lambda), \qquad y^\lambda := \frac{\lambda^2 y}{|y|^2},$$

denote the Kelvin transformation of v, and let

$$\begin{aligned} \Sigma_\lambda &:= B\left(0, \frac{1}{2}M_k^{\frac{2}{n-2}}\right) \setminus \overline{B(0,\lambda)} = \left\{ y \mid \lambda < |y| < \frac{1}{2}M_k^{\frac{2}{n-2}} \right\}, \\ w_\lambda(y) &:= v_k(y) - v_k^\lambda(y), \qquad y \in \Sigma_\lambda. \end{aligned}$$

A calculation yields (see [70])

$$\Delta w_\lambda + \bar{b}_i \partial_i w_\lambda + \bar{d}_{ij}\partial_{ij} w_\lambda - \bar{c} w_\lambda + n(n+2)\xi^{\frac{4}{n-2}} w_\lambda = E_\lambda, \qquad \text{in } \Sigma_\lambda, \tag{33}$$

where $\xi > 0$ is given by

$$n(n+2)\xi^{\frac{4}{n-2}} = \begin{cases} n(n-2)\dfrac{v_k^{\frac{n+2}{n-2}} - (v_k^\lambda)^{\frac{n+2}{n-2}}}{v_k - v_k^\lambda}, & v_k \neq v_k^\lambda, \\ n(n+2) v_k^{\frac{4}{n-2}}, \quad v_k = v_k^\lambda, \end{cases} \tag{34}$$

and

$$\begin{aligned} E_\lambda = &\left(\bar{c}(y) v_k^\lambda(y) - \left(\frac{\lambda}{|y|}\right)^{n+2} \bar{c}(y^\lambda) v_k(y^\lambda) \right) - (\bar{b}_i \partial_i v_k^\lambda + \bar{d}_{ij}\partial_{ij} v_k^\lambda) \\ &+ \left(\frac{\lambda}{|y|}\right)^{n+2} \left(\bar{b}_i(y^\lambda)\partial_i v_k(y^\lambda) + \bar{d}_{ij}(y^\lambda)\partial_{ij} v_k(y^\lambda) \right). \end{aligned} \tag{35}$$

By the convergence of v_k to U,

$$\sigma_k := \|v_k - U\|_{C^2(B_2)} \to 0 \quad \text{as } k \to \infty.$$

Proposition 2.1. *For $n \geq 3$, let v_k satisfy (29). Then for $\lambda \in (0,2]$ and $y \in \Sigma_\lambda$,*

$$E_\lambda = \bar{c}(y) U^\lambda(y) - \left(\frac{\lambda}{|y|}\right)^{n+2} \bar{c}(y^\lambda) U(y^\lambda) + O(1)\sigma_k M_k^{-\frac{4}{n-2}} |y|^{-n}. \tag{36}$$

where $|O(1)| \leq C_0$ for some positive constant C_0 independent of λ, y and k.

Proof. For any radially symmetric function $w(y)$, we have, in conformal normal coordinates,

$$(\Delta_{g_k} - \Delta)w = (\bar{b}_i\partial_i + \bar{d}_{ij}\partial_{ij})w \equiv 0. \tag{37}$$

Thus

$$I := (\bar{b}_i\partial_i v_k^\lambda + \bar{d}_{ij}\partial_{ij}v_k^\lambda) = (\bar{b}_i\partial_i + \bar{d}_{ij}\partial_{ij})[(v_k - U)^\lambda].$$

A calculation yields

$$\begin{aligned}
&\partial_i \left\{ \left(\frac{\lambda}{|y|}\right)^{n-2} (v_k - U)(y^\lambda) \right\} \\
&= \partial_i \left\{ \left(\frac{\lambda}{|y|}\right)^{n-2} \right\} (v_k - U)(y^\lambda) + \left(\frac{\lambda}{|y|}\right)^{n-2} \partial_i \left\{ (v_k - U)(y^\lambda) \right\}, \\
&\partial_{ij} \left\{ \left(\frac{\lambda}{|y|}\right)^{n-2} (v_k - U)(y^\lambda) \right\}. \\
&= \partial_{ij} \left\{ \left(\frac{\lambda}{|y|}\right)^{n-2} \right\} (v_k - U)(y^\lambda) + \partial_i \left\{ \left(\frac{\lambda}{|y|}\right)^{n-2} \right\} \partial_j \left\{ (v_k - U)(y^\lambda) \right\} \\
&\quad + \partial_j \left\{ \left(\frac{\lambda}{|y|}\right)^{n-2} \right\} \partial_i \left\{ (v_k - U)(y^\lambda) \right\} + \left(\frac{\lambda}{|y|}\right)^{n-2} \partial_{ij} \left\{ (v_k - U)(y^\lambda) \right\}.
\end{aligned}$$

It follows, using (37) and $\bar{d}_{ij} \equiv \bar{d}_{ji}$, that

$$\begin{aligned}
I = &\left(\frac{\lambda}{|y|}\right)^{n-2} \bar{b}_i\partial_i \left\{ (v_k - U)(y^\lambda) \right\} + 2\bar{d}_{ij}\partial_i \left\{ \left(\frac{\lambda}{|y|}\right)^{n-2} \right\} \partial_j \left\{ (v_k - U)(y^\lambda) \right\} \\
&+ \left(\frac{\lambda}{|y|}\right)^{n-2} \bar{d}_{ij}\partial_{ij} \left\{ (v_k - U)(y^\lambda) \right\}.
\end{aligned}$$

Since $(v_k - U)(0) = 0$ and $\nabla(v_k - U)(0) = 0$, we have

$$(v_k - U)(y^\lambda) = O(1)\sigma_k|y^\lambda|^2, \qquad |\nabla(v_k - U)(y^\lambda)| = O(1)\sigma_k|y^\lambda|. \tag{38}$$

Using (38) and (29), we obtain

$$I = O(1)\sigma_k M_k^{-\frac{4}{n-2}}|y|^{-n}.$$

Similarly,

$$\left(\frac{\lambda}{|y|}\right)^{n+2} \left(\bar{b}_i(y^\lambda)\partial_i v_k(y^\lambda) + \bar{d}_{ij}(y^\lambda)\partial_{ij}v_k(y^\lambda)\right) = O(1)\sigma_k M_k^{-\frac{4}{n-2}}|y|^{-n}.$$

and

$$|\bar{c}(y)||v_k^\lambda(y) - U^\lambda(y)| + \left(\frac{\lambda}{|y|}\right)^{n+2} |\bar{c}(y^\lambda)||v_k(y^\lambda) - U(y^\lambda)|$$

$$= O(1)\sigma_k M_k^{-\frac{4}{n-2}} |y|^{-n}.$$

Proposition 2.1 is established. □

For $\bar{l} \geq 2$, write the Taylor expansion of $R(x)$ at 0:

$$R(x) = \sum_{l=2}^{\bar{l}} \sum_{|\alpha|=l} \frac{\partial_\alpha R}{\alpha!} x^\alpha + O(|x|^{\bar{l}+1}). \tag{39}$$

Thus, with $r = |y|, y = r\theta$,

$$\bar{c}(y)U^\lambda(y) - \left(\frac{\lambda}{|y|}\right)^{n+2} \bar{c}(y^\lambda)U(y^\lambda) \tag{40}$$

$$= \sum_{l=2}^{\bar{l}} M_k^{-\frac{4+2l}{n-2}} H_{l,\lambda}(r) \sum_{|\alpha|=l} \frac{\partial_\alpha R}{\alpha!} \theta^\alpha + O\left(M_k^{-\frac{4}{n-2}}\right) |M_k^{-\frac{2}{n-2}} y|^{\bar{l}+1} \cdot \left(\frac{\lambda}{|y|}\right)^{n-2},$$

where

$$H_{l,\lambda}(r) = c(n)\lambda^{n-2} r^{2+l-n} \left[1 - \left(\frac{\lambda}{r}\right)^{4+2l}\right] U\left(\frac{\lambda^2}{r}\right). \tag{41}$$

Let

$$\bar{R}^{(l)} := \frac{1}{|\mathbb{S}^{n-1}|} \int_{\theta \in \mathbb{S}^{n-1}} \sum_{|\alpha|=l} \frac{\partial_\alpha R}{\alpha!} \theta^\alpha, \tag{42}$$

and

$$\tilde{R}^{(l)}(\theta) := -\bar{R}^{(l)} + \sum_{|\alpha|=l} \frac{\partial_\alpha R}{\alpha!} \theta^\alpha, \qquad \theta \in \mathbb{S}^{n-1}. \tag{43}$$

By (26),

$$\bar{R}^{(2)} = \frac{1}{2n} \Delta R = -\frac{1}{12n} |W|^2, \qquad \text{and } \bar{R}^{(3)} = 0. \tag{44}$$

We assume that for some constants $\gamma \geq 0$ and $C \geq 0$,

$$M_k^{\frac{\gamma}{(n-2)^2}} = \circ\left(M_k^{\frac{2}{n-2}}\right), \qquad \|v_k - U\|_{C^2\left(B\left(0, 2M_k^{\frac{\gamma}{(n-2)^2}}\right)\right)} \leq C M_k^{-\frac{\gamma}{n-2}}. \tag{45}$$

We deduce from Proposition 2.1, using (41), the following

Corollary 2.1. *For $n \geq 3$, let v_k satisfy (29). We assume (45). Then, for $\lambda \in (0, 2]$ and for $y \in \Sigma_\lambda$, we have, for some positive constant C_0 independent of k, λ and y,*

$$E_\lambda(y) \leq \sum_{l=2}^{3} M_k^{-\frac{4+2l}{n-2}} H_{l,\lambda}(r) \tilde{R}^{(l)}(\theta) + C_0 M_k^{-\frac{4+\gamma}{n-2}} r^{-n} + C_0 M_k^{-\frac{12}{n-2}} r^{6-n}, \tag{46}$$

In the proof of the following iterated estimates on the rate of convergence of $v_k - U$, we make use of some ideas in Chen and Lin [35] and Caffarelli et al. [26], in addition to the way of Schoen [81] in using the method of moving planes to prove the Harnack type inequality (12).

Proposition 2.2. *For $n \geq 3$, we assume that (45) holds for some constants $0 \leq \gamma < 2(n-2)$, and $C \geq 0$. Let v_k satisfy (29). Then there exist some positive constants $\delta' > 0$ and $C_2 > 0$, independent of k, such that*

$$\|v_k - U\|_{C^2(B(0,\delta' R_k))} \leq C_2 (R_k)^{2-n},$$

for any $\{R_k\}$ satisfying, for some $\bar{\epsilon} \in (0,1)$ independent of k,

$$2 \leq R_k = \circ\left(M_k^{\frac{2}{n-2}}\right), \tag{47}$$

$$R_k = \circ(1) M_k^{\frac{4+\gamma}{(n-2)(n-2+\bar{\epsilon})}}, \tag{48}$$

$$R_k = \circ(1) M_k^{\frac{12}{(n-2)(\max\{6, n-2+\bar{\epsilon}\})}}, \tag{49}$$

$$R_k = O(1) M_k^{\frac{8}{(n-2)\max\{4+\bar{\epsilon}, n-2\}}}. \tag{50}$$

A consequence of Proposition 2.2 is

Corollary 2.2. *Let v_k satisfy (29). For any $\epsilon > 0$, let*

$$R_k = \begin{cases} M_k^{\frac{2-\epsilon}{n-2}}, \text{ if } 3 \leq n \leq 6, \\ M_k^{\frac{8}{(n-2)^2}}, \text{ if } n \geq 7. \end{cases}$$

Then

$$\limsup_{k\to\infty} (R_k)^{n-2} \cdot \|v_k - U\|_{C^2(B(0,R_k))} < \infty.$$

Proof of Corollary 2.2 by using Proposition 2.2. Taking first $\gamma = 0$ in (45) and applying Proposition 2.2 with $R_k = M_k^{\frac{a}{(n-2)^2}}$ for $0 < a < \min\{4, 2(n-2)\}$, we see that (45) holds now for any $0 < \gamma < \min\{4, 2(n-2)\}$. Corollary 2.2 for $n = 3, 4$ is established. For $n \geq 5$, since we now deduce that (45) holds for any $0 < \gamma < \min\{4, 2(n-2)\}$, we can apply Proposition 2.2 with $R_k = M_k^{\frac{a}{(n-2)^2}}$, $0 < a < \min\{8, 2(n-2)\}$, and γ very close to $\min\{4, 2(n-2)\}$, and know that (45) holds for any $0 < \gamma < \min\{8, 2(n-2)\}$. Corollary 2.2 for $n = 5, 6$ is established. For $n \geq 7$, we already know that (45) holds for any $0 < \gamma < 8$. Take γ very close to 8, we can apply Proposition 2.2 with $R_k = M_k^{\frac{8}{(n-2)^2}}$ to conclude the proof of Corollary 2.2 for $n \geq 7$. $\square$

We prove Proposition 2.2 by the method of moving spheres, and we need to construct appropriate auxiliary functions to handle the error term E_λ. For $n \geq 3$ and $\alpha < 2$, let

$$f_{n,\alpha}(r) = -\frac{1}{(n-\alpha)(2-\alpha)}[r^{2-\alpha} - 1] - \frac{1}{(n-\alpha)(n-2)}[r^{2-n} - 1], \quad r \geq 1.$$

Clearly,

$$f_{n,\alpha}(1) = f'_{n,\alpha}(1) = 0, \qquad 0 \leq -f_{n,\alpha}(r) \leq C(n,\alpha) r^{2-\alpha}, \qquad r \geq 1. \quad (51)$$

Thinking of $f_{n,\alpha}(r)$ as a radially symmetric function in $\mathbb{R}^n$, and let Δ denote the Laplacian in $\mathbb{R}^n$, we have

$$\Delta f_{n,\alpha}(r) = f''_{n,\alpha}(r) + \frac{n-1}{r} f'_{n,\alpha}(r) = -r^{-\alpha}, \quad r \geq 1, \quad (52)$$

$$\left|\frac{d^i}{dr^i} f_{n,\alpha}(r)\right| \leq C(n,\alpha) |\Delta f_{n,\alpha}(r)| r^{2-i}, \qquad r \geq 1, \ i = 0, 1, 2. \quad (53)$$

Since

$$\int_{\theta \in \mathbb{S}^{n-1}} \tilde{R}^{(l)}(\theta) = 0,$$

we can write

$$\tilde{R}^{(l)}(\theta) = \sum_{j=1}^{l} \sum_{i=1}^{I_j} a^l_{ji} Y^{(i)}_j(\theta), \quad (54)$$

where $Y^{(i)}_j(\theta)$ are spherical harmonics of degree j satisfying, for some $\mu_j \geq n-1$,

$$-\Delta_{\mathbb{S}^{n-1}} Y^{(i)}_j(\theta) = \mu_j Y^{(i)}_j(\theta). \quad (55)$$

Consider, for $\frac{1}{2} < \lambda < 2$,

$$\begin{cases} \Delta h_{l,j,\lambda}(r) + \left(V_\lambda(r) - \frac{\mu_j}{r^2}\right) h_{l,j,\lambda}(r) = -H_{l,\lambda}(r), & \lambda < r < M_k^{\frac{2}{n-2}}, \\ h_{l,j,\lambda}(r) \geq 0, & \lambda < r < M_k^{\frac{2}{n-2}}, \\ h_{l,j,\lambda}(\lambda) = 0, \quad h_{l,j,\lambda}\left(M_k^{\frac{2}{n-2}}\right) = 0, & \end{cases} \quad (56)$$

where

$$V_\lambda(r) := \begin{cases} n(n-2)\dfrac{U(r)^{\frac{n+2}{n-2}} - U^\lambda(r)^{\frac{n+2}{n-2}}}{U(r) - U^\lambda(r)}, \ \lambda \neq 1, \\ n(n+2) U(r)^{\frac{4}{n-2}}, \ \lambda = 1. \end{cases}$$

By Proposition 6.1 in Appendix A, there exists some small $\epsilon_4 = \epsilon_4(n) \in (0, \frac{1}{2})$ such that for $\lambda \in [1 - \epsilon_4, 1 + \epsilon_4]$, equation (56) has a unique classical solution satisfying

$$\sum_{i=0}^{2} \left|\frac{d^i}{dr^i} h_{l,j,\lambda}(r)\right| r^{n-l-4+i} \leq C, \ \ \lambda < r < M_k^{\frac{2}{n-2}}, \quad (57)$$

where $C > 0$ depends only on n and l. From now on we only consider λ in this range.

Let, with our notation $r = |y|$ and $y = r\theta$,

$$\tilde{h}^{(i)}_{l,j,\lambda}(y) := M_k^{-\frac{4+2l}{n-2}} h_{l,j,\lambda}(r) a^l_{ji} Y^{(i)}_j(\theta), \qquad \lambda \le r \le 4\delta_1 M_k^{\frac{2}{n-2}}.$$

Then, by (56), we have, for $y \in \Sigma_\lambda$,

$$(\Delta + V_\lambda)\left(\sum_{l=2}^{3}\sum_{j=1}^{l}\sum_{i=1}^{I_j} \tilde{h}^{(i)}_{l,j,\lambda}(y)\right) = -\sum_{l=2}^{3} M_k^{-\frac{4+2l}{n-2}} H_{l,\lambda}(r)\tilde{R}^{(l)}(\theta). \tag{58}$$

Now we construct the auxiliary functions which will be used in the proof of Proposition 2.2. Let, for $y \in \Sigma_\lambda$,

$$\tilde{h}_{1,\lambda}(y) = \sum_{l=2}^{3}\sum_{j=1}^{l}\sum_{i=1}^{I_j} \tilde{h}^{(i)}_{l,j,\lambda}(y), \tag{59}$$

$$\tilde{h}_{3,\lambda}(y) = QM_k^{-\frac{4+\gamma}{n-2}} f_{n,2-\hat{\epsilon}}\left(\frac{r}{\lambda}\right), \tag{60}$$

$$\tilde{h}_{4,\lambda}(y) = QM_k^{-\frac{12}{n-2}} f_{n,\min\{n-6,2-\hat{\epsilon}\}}\left(\frac{r}{\lambda}\right), \tag{61}$$

and

$$h_\lambda(y) = \tilde{h}_{1,\lambda}(y) + \tilde{h}_{3,\lambda}(y) + \tilde{h}_{4,\lambda}(y), \tag{62}$$

where $\hat{\epsilon} = \bar{\epsilon}/9$ and $Q > C_0$, independent of k, is some large constant to be fixed later.

Since $M_k^{-\frac{2}{n-2}}|y| = O(1)$ for $y \in \Sigma_\lambda$, we have, by (57),

$$|\tilde{h}_{1,\lambda}(y)| \le C\sum_{l=2}^{3} M_k^{-\frac{4+2l}{n-2}} |h_{l,j,\lambda}(y)| \le CM_k^{-\frac{8}{n-2}}|y|^{6-n}, \quad y \in \Sigma_\lambda. \tag{63}$$

Similarly,

$$|\nabla\tilde{h}_{1,\lambda}(y)| \le CM_k^{-\frac{8}{n-2}}|y|^{5-n}, \quad |\nabla^2\tilde{h}_{1,\lambda}(y)| \le CM_k^{-\frac{8}{n-2}}|y|^{4-n}, \qquad y \in \Sigma_\lambda. \tag{64}$$

By (51) , we have, for some $C > 0$ independent of k and Q,

$$|\tilde{h}_{3,\lambda}(y)| \le CQM_k^{-\frac{4+\gamma}{n-2}} r^{\hat{\epsilon}}, \qquad y \in \Sigma_\lambda, \tag{65}$$

$$|\tilde{h}_{4,\lambda}(y)| \le CQM_k^{-\frac{12}{n-2}} r^{\max\{8-n,\hat{\epsilon}\}}, \qquad \forall\, \lambda < |y| < \frac{1}{2}M_k^{\frac{2}{n-2}}. \tag{66}$$

We also know from (52) and (53) that, for all $y \in \Sigma_\lambda$,

$$\Delta\tilde{h}_{3,\lambda}(y) = -Q\lambda^{-\hat{\epsilon}} M_k^{-\frac{4+\gamma}{n-2}} \cdot r^{\hat{\epsilon}-2}, \tag{67}$$

$$\Delta\tilde{h}_{4,\lambda}(y) = -Q\lambda^{\min\{n-8,\hat{\epsilon}\}} M_k^{-\frac{12}{n-2}} r^{-\min\{n-6,2-\hat{\epsilon}\}}, \tag{68}$$

$$|\nabla^i\tilde{h}_{m,\lambda}(y)| \le C|y|^{2-i}|\Delta\tilde{h}_{m,\lambda}(y)|, \qquad i = 0,1,2, \;\; m = 3,4. \tag{69}$$

Lemma 2.1. *For $n \geq 3$, we assume that (45) holds for some constants $0 \leq \gamma < 2(n-2)$ and $C \geq 0$. Let v_k satisfy (29), and let $\{R_k\}$ satisfy (47), (48) and (49). Then for any $\epsilon > 0$, there exists $k_0 > 1$ (can depend on ϵ and Q), such that for all $k \geq k_0$,*

$$\min_{|y|=r} v_k(y) \leq (1+\epsilon)U(r), \qquad \forall\, 0 < r \leq R_k. \tag{70}$$

Proof of Lemma 2.1. We prove it by a contradiction argument. Suppose (70) is not true, then there exists $\epsilon_0 > 0$, such that

$$\min_{|y|=r_k} v_k(y) > (1+\epsilon_0)U(r_k),$$

for a sequence of $r_k \in (0, R_k]$. We have written $U(r)$ for $U(y)$, $|y| = r$.

By the convergence of v_k to U, we know $r_k \to \infty$. Thus

$$\min_{|y|=r_k} v_k(y) \geq (1+\epsilon_0/2) r_k^{2-n}. \tag{71}$$

Fixing a small $\epsilon_4' \in (0, \epsilon_4(n))$, independent of k, such that

$$U^\lambda(y) \leq (1+\frac{\epsilon_0}{8})|y|^{2-n} \qquad \forall\, 0 < \lambda \leq 1+\epsilon_4',\ |y| = r_k.$$

It follows that

$$v_k^\lambda(y) \leq (1+\frac{\epsilon_0}{4})|y|^{2-n}, \qquad \forall\, 0 < \lambda \leq 1+\epsilon_4',\ |y| = r_k. \tag{72}$$

We will derive a contradiction by applying the method of moving spheres to $w_\lambda + h_\lambda$ with $1-\epsilon_4' \leq \lambda \leq 1+\epsilon_4'$.

Let

$$\hat{\Sigma}_\lambda := \{y\,;\,|\ \lambda < |y| < r_k\}.$$

We know that $h_\lambda = 0$ on ∂B_λ and, in view of (47), (48), (49), (66) and (63),

$$h_\lambda(y) = \circ(1)|y|^{2-n}, \qquad y \in \hat{\Sigma}_\lambda, \tag{73}$$

where $\circ(1)$ denotes some quantity going to zero as $k \to \infty$, uniform in y.

Step 1. For $\lambda_0 = 1-\epsilon_4'$,

$$w_{\lambda_0}(y) + h_{\lambda_0}(y) \geq 0, \qquad \forall\ y \in \hat{\Sigma}_{\lambda_0}. \tag{74}$$

Since $\lambda_0 < 1$, there exist some small positive constant $\epsilon_5 \leq \bar{\epsilon}_0/10$ and some large constant $R_1 > 10$ such that

$$U(y) - U^{\lambda_0}(y) \geq \epsilon_5(|y|-\lambda_0)|y|^{1-n}, \qquad |y| > \lambda_0, \tag{75}$$

$$U(y) > \left(1-\frac{\epsilon_5}{2}\right)|y|^{2-n}, \qquad |y| = R_1, \tag{76}$$

$$U^{\lambda_0}(y) < (1-4\epsilon_5)|y|^{2-n}, \qquad |y| \geq R_1. \tag{77}$$

Since v_k converges in C^1 to U in the region $\lambda_0 \le |y| \le R_1$, $h_{\lambda_0}(y) = 0$ for $|y| = \lambda_0$, and since $|h_{\lambda_0}(y)| + |\nabla h_{\lambda_0}(y)| = \circ(1)$ in the same region and uniform in y, we deduce from (75), (76) and (77) that, for large k as always,

$$w_{\lambda_0}(y) + h_{\lambda_0}(y) > 0, \qquad \lambda_0 < |y| \le R_1, \tag{78}$$

$$v_k(y) > (1-\epsilon_5)|y|^{2-n}, \qquad |y| = R_1, \tag{79}$$

$$v_k^{\lambda_0}(y) \le (1-3\epsilon_5)|y|^{2-n}, \qquad |y| \ge R_1. \tag{80}$$

Let $G(0,x)$ be the Greens function of $-L_g$ on $B(0,3\delta_1)$ as at the beginning of this section. Using the maximum principle, we compare u_k and $(1-\epsilon_5)(n-2)\sigma_n M_k^{-1} G(0,x)$ as at the beginning of this section to obtain, by (79) and (31), for some $\delta_2 > 0$ independent of k,

$$v_k(y) \ge (1-2\epsilon_5)|y|^{2-n}, \qquad R_1 \le |y| \le \delta_2 M_k^{\frac{2}{n-2}}. \tag{81}$$

Using (81), (80) and (73), we obtain

$$w_{\lambda_0}(y) + h_{\lambda_0}(y) > 0, \qquad R_1 \le |y| \le r_k.$$

Step 1 follows from this and (78).

For $\lambda_1 = 1 + \epsilon_4'$, let

$$\bar\lambda_k = \sup\{\lambda_0 \le \lambda \le \lambda_1 \mid \mathrm{w}_\mu + \mathrm{h}_\mu \ge 0 \text{ in } \hat\Sigma_\mu \text{ for all } \lambda_0 \le \mu \le \lambda\}.$$

Step 2. $\bar\lambda^k = \lambda_1$.

Let

$$\hat O_\lambda := \{y \in \hat\Sigma_\lambda \mid v_k(y) < 2v_k^\lambda(y)\}.$$

It follows from (73) that, for large k (the largeness of k may depend on Q),

$$w_\lambda + h_\lambda(y) = v_k(y) - v_k^\lambda(y) + h_\lambda(y) \ge v_k^\lambda(y) + h_\lambda(y) > 0 \text{ in } \hat\Sigma_\lambda \setminus \hat O_\lambda, \tag{82}$$

We also know from (71), (72) and (73) that

$$w_{\bar\lambda^k}(y) + h_{\bar\lambda^k}(y) > 0, \qquad |y| = r_k. \tag{83}$$

Recall that v_k satisfies (29), and $w_\lambda := v_k - v_k^\lambda$ satisfies (33), with ξ given by (34) and E_λ, defined by (35), satisfying (46). To complete the proof of Lemma 2.1 we need the following

Lemma 2.2. *For $n \ge 3$, we assume that (45) holds for some constants $0 \le \gamma < 2(n-2)$ and $C \ge 0$. Let v_k satisfy (29) and let $\{r_k\} = \circ(M_k^{\frac{2}{n-2}})$. Then we have, for $1 - \epsilon_4' \le \lambda \le 1 + \epsilon_4'$, and for a large constant Q,*

$$(\Delta_{g_k} - \bar c + n(n+2)\xi^{\frac{4}{n-2}})h_\lambda + E_\lambda \le 0 \quad \text{in} \quad \hat O_\lambda. \tag{84}$$

By (51) and (37),

$$\tilde{h}_{m,\lambda} \le 0, \quad \Delta_{g_k}\tilde{h}_{m,\lambda} \equiv \Delta\tilde{h}_{m,\lambda}, \qquad m = 3, 4.$$

It follows, using (67), (68) and the smallness of $|\lambda - 1| \le \epsilon_4'$, that, for $y \in \Sigma_\lambda$,

$$(\Delta_{g_k} + n(n+2)\xi^{\frac{4}{n-2}})\tilde{h}_{3,\lambda} \le -\frac{Q}{2}M_k^{-\frac{4+\gamma}{n-2}} \cdot r^{\hat{\epsilon}-2}, \tag{85}$$

$$(\Delta_{g_k} + n(n+2)\xi^{\frac{4}{n-2}})\tilde{h}_{4,\lambda} \le -\frac{Q}{2}M_k^{-\frac{12}{n-2}} r^{-\min\{n-6, 2-\hat{\epsilon}\}}. \tag{86}$$

Using (29), (47) and (69), we have, for $y \in \hat{\Sigma}_\lambda$,

$$|\bar{c}\tilde{h}_{m,\lambda}(y)| \le CM_k^{-\frac{8}{n-2}}|y|^2 \cdot |y|^2|\Delta\tilde{h}_{m,\lambda}(y)| = \circ(1)|\Delta\tilde{h}_{m,\lambda}(y)|, \ \ m = 3, 4. \tag{87}$$

Putting together the above four estimates, we have

$$\sum_{m=3}^{4}\left(\Delta_{g_k} - \bar{c} + n(n+2)\xi^{\frac{4}{n-2}}\right)\tilde{h}_{m,\lambda} \le -\frac{Q}{4}D_{k,\hat{\epsilon}}(r), \qquad y \in \hat{\Sigma}_\lambda. \tag{88}$$

where

$$D_{k,\hat{\epsilon}}(r) := M_k^{-\frac{4+\gamma}{n-2}} \cdot r^{\hat{\epsilon}-2} + M_k^{-\frac{12}{n-2}} r^{-\min\{n-6, 2-\hat{\epsilon}\}}.$$

The right hand side of (88) is of good sign and will be used to absorb other terms.

Using (29), (63) and (64), we have, for $y \in \hat{\Sigma}_\lambda$,

$$|\bar{c}||\tilde{h}_{1,\lambda}(y)| + |\bar{b}_i\partial_i\tilde{h}_{1,\lambda}(y)| + |\bar{d}_{ij}\partial_{ij}\tilde{h}_{1,\lambda}(y)| \le CM_k^{-\frac{12}{n-2}}|y|^{6-n} \le CD_{k,\hat{\epsilon}}(r). \tag{89}$$

We give an estimate of ξ, given by (34), in the following

Lemma 2.3. *For $n \ge 3$, we assume (45) for some $0 \le \gamma \le 2(n-2)$. Let v_k satisfy (29). Then, there exists C, independent of k, such that*

$$|n(n+2)\xi^{\frac{4}{n-2}}(y) - V_\lambda(|y|)| \le CM_k^{-\frac{\gamma}{n-2}}|y|^{n-6}, \qquad \lambda \le |y| \le 2M_k^{\frac{\gamma}{(n-2)^2}}, \tag{90}$$

and

$$|n(n+2)\xi^{\frac{4}{n-2}} - V_\lambda| \le C|y|^{-4}, \quad y \in \hat{O}_\lambda. \tag{91}$$

Proof of Lemma 2.3. By (45),

$$v_k(y) = U(y) + a(y), \qquad v_k^\lambda(y) = U^\lambda(y) + b(y), \qquad \lambda \le |y| \le 2M_k^{\frac{\gamma}{(n-2)^2}},$$

where $a(y)$ and $b(y)$ satisfy

$$|a(y)| + |b(y)| \le CM_k^{-\frac{\gamma}{n-2}}, \qquad \lambda \le |y| \le 2M_k^{\frac{\gamma}{(n-2)^2}}.$$

Then, with $n^*=\frac{n+2}{n-2}$ and for $\lambda\le|y|\le 2M_k^{\frac{\gamma}{(n-2)^2}}$,

$$\begin{aligned}
\frac{v_k^{n^*}-v_k^\lambda(y)^{n^*}}{v_k-v_k^\lambda} &= \frac{(U+a)^{n^*}-(U^\lambda+b)^{n^*}}{(U+a)-(U^\lambda+b)}\\
&= \frac{\int_0^1\frac{d}{dt}\{(t(U+a)+(1-t)(U^\lambda+b))^{n^*}\}dt}{(U+a)-(U^\lambda+b)}\\
&= n^*\int_0^1(t(U+a)+(1-t)(U^\lambda+b))^{\frac{4}{n-2}}dt\\
&= n^*\int_0^1(tU+(1-t)U^\lambda)^{\frac{4}{n-2}}dt+O(1)(|a(y)|+|b(y)|)|y|^{n-6}\\
&= \frac{1}{n(n-2)}V_\lambda+O\left(M_k^{-\frac{\gamma}{n-2}}|y|^{n-6}\right).
\end{aligned}$$

Estimate (90) is established. Estimate (91) is obvious. □

By (90) and (63), we have, for $\lambda\le|y|\le 2M_k^{\frac{\gamma}{(n-2)^2}}$,

$$|n(n+2)\xi^{\frac{4}{n-2}}-V_\lambda||\tilde h_{1,\lambda}|\le CM_k^{-\frac{8+\gamma}{n-2}}\le CM_k^{-\frac{4+\gamma}{n-2}}r^{-2}=O(1)D_{k,\hat\epsilon}(r). \qquad (92)$$

By (91), (63) and the fact $\gamma\le 2(n-2)$, we have, for $y\in\hat O_\lambda$ and $|y|\ge 2M_k^{\frac{\gamma}{(n-2)^2}}$,

$$|n(n+2)\xi^{\frac{4}{n-2}}-V_\lambda||\tilde h_{1,\lambda}|\le CM_k^{-\frac{8}{n-2}}|y|^{2-n}\le CM_k^{-\frac{4+\gamma}{n-2}}|y|^{-2}=O(1)D_{k,\hat\epsilon}(r). \qquad (93)$$

By (59), (58), (92) and (93),

$$\begin{aligned}
&(\Delta+n(n+2)\xi^{\frac{4}{n-2}})\tilde h_{1,\lambda}\\
&\le(\Delta+V_\lambda)\tilde h_{1,\lambda}+|n(n+2)\xi^{\frac{4}{n-2}}-V_\lambda||\tilde h_{1,\lambda}|\\
&\le-\sum_{l=2}^3 M_k^{-\frac{4+2l}{n-2}}H_{l,\lambda}(r)\tilde R^{(l)}(\theta)+O(1)D_{k,\hat\epsilon}(r),\qquad \text{in } \hat O_\lambda.
\end{aligned}$$

Thus, in view of (89),

$$\begin{aligned}
&\left(\Delta_{g_k}-\bar c+n(n+2)\xi^{\frac{4}{n-2}}\right)\tilde h_{1,\lambda}\\
&\le-\sum_{l=2}^3 M_k^{-\frac{4+2l}{n-2}}H_{l,\lambda}(r)\tilde R^{(l)}(\theta)+CD_{k,\hat\epsilon}(r),\qquad \text{in } \hat O_\lambda. \qquad (94)
\end{aligned}$$

Then we fix a large constant Q, estimate (84) follows from (46), (94) and (88). Lemma 2.2 is established. □

Now we establish Step 2: We know that $w_{\bar\lambda^k}+h_{\bar\lambda^k}$ is non-negative in $\hat\Sigma_{\bar\lambda^k}$, and, by (84) , satisfies

$$\left(\Delta_{g_k}-\bar c+n(n+2)\xi^{\frac{4}{n-2}}\right)(w_{\bar\lambda^k}+h_{\bar\lambda^k})\le 0,\qquad \text{in } \hat O_{\bar\lambda^k}. \qquad (95)$$

Since $w_{\bar\lambda^k}+h_{\bar\lambda^k}$ satisfies (83) and (82) with $\lambda=\bar\lambda^k$, we apply the strong maximum principle and the Hopf lemma to obtain

$$w_{\bar\lambda^k}+h_{\bar\lambda^k}>0 \qquad \text{in } \hat\Sigma_{\bar\lambda^k},$$

and

$$\frac{\partial}{\partial\nu}(w_{\bar\lambda^k}+h_{\bar\lambda^k})>0 \qquad \text{on } \partial B(0,\bar\lambda^k),$$

where $\frac{\partial}{\partial\nu}$ denotes the differentiation in the outer normal direction. In view of (83) and the above two estimates above, we must have $\bar\lambda^k=\lambda_1$. Step 2 is established.

By Step 2, $w_{\lambda_1}+h_{\lambda_1}\ge 0$ in $\hat\Sigma_{\lambda_1}$. Sending k to infinity, we obtain

$$U^{\lambda_1}(y)\le U(y), \qquad \forall\, |y|\ge\lambda_1. \tag{96}$$

But the above is not satisfied by U, a fact easily checked using $\lambda_1>1$. This leads to contradiction. Lemma 2.1 is established. □

Lemma 2.4. *Under the hypotheses of Lemma 2.1, there exist $\delta\in(0,1)$ and $C>0$, independent of k, such that for all large k,*

$$v_k(y)\le CU(y), \qquad |y|\le\delta R_k.$$

A consequence of Proposition 2.2 and Lemma 2.4 is

Corollary 2.3. *For $n\ge 8$ and for any $\epsilon>0$, there exists some constant $C>1$, independent of k, such that*

$$v_k(y)\le CU(y), \qquad \forall\,|y|\le M_k^{\frac{12-\epsilon}{(n-2)^2}}.$$

Proof of Corollary 2.3. By Corollary 2.2, (45) is satisfied with $\gamma=8$. For any $\epsilon>0$, let $R_k=M_k^{\frac{12-\epsilon}{(n-2)^2}}$, and let $\bar\epsilon>0$ be sufficiently small (depending on ϵ). Using $n\ge 8$, we easily check that hypotheses of Lemma 2.4 are satisfied, and Corollary 2.3 follows from the lemma. □

Proof of Lemma 2.4. The proof is very similar to the proof of lemma 3.2 in [35]. Let G_k (will be denoted as G) be the Green's function of $-L_{g_k}$ on $B(0,\delta_1 M_k^{\frac{2}{n-2}})$ with respect to zero Dirichlet boundary data, where δ_1 is the constant above (31), and let y_1 be a minimum point of v_k on $|y|=R_k$. For $\epsilon>0$, there exists some constant $\delta\in(0,1)$, independent of k, such that, for large k, the following estimates go through:

$$\begin{aligned} v_k(y_1) &\ge \int_{B\left(0,\delta_1 M_k^{\frac{2}{n-2}}\right)} G(y_1,\eta)n(n-2)v_k(\eta)^{\frac{n+2}{n-2}}dV_{g_k}\\ &\ge \int_{B(0,\delta R_k)} G(y_1,\eta)n(n-2)v_k(\eta)^{\frac{n+2}{n-2}}dV_{g_k}, \end{aligned}$$

and, using (47),

$$G(y_1,\eta)\ge \frac{(1-\epsilon/2)}{(n-2)\omega_n}|y_1-\eta|^{2-n}\ge \frac{(1-3\epsilon/4)}{(n-2)\omega_n}|y_1|^{2-n},\qquad |\eta|=\delta R_k,$$

where ω_n denotes the volume of the standard $(n-1)-$sphere. Since $dV_{g_k}=(1+\circ(1))d\eta$, we have

$$v_k(y_1)\ge \frac{(1-\epsilon)n}{\omega_n}|y_1|^{2-n}\int_{B(0,\delta R_k)} v_k^{\frac{n+2}{n-2}}(\eta)d\eta$$

On the other hand, by Lemma 2.1,

$$v_k(y_1)\le (1+\epsilon)U(y_1)\le (1+2\epsilon)|y_1|^{2-n}.$$

So

$$\int_{B(0,\delta R_k)} v_k^{\frac{n+2}{n-2}}(\eta)d\eta\le (1+4\epsilon)\omega_n/n.$$

A direct computation gives,

$$\int_{\mathbb{R}^n} U^{\frac{n+2}{n-2}}=\frac{\omega_n}{n}.$$

By the convergence of v_k to U, there exists some R_1, depending only on n and ϵ, such that, for large k,

$$\int_{R_1\le|\eta|\le\delta R_k} v_k^{\frac{n+2}{n-2}}d\eta\le 5\epsilon.$$

Using the second line of (29),

$$\int_{R_1\le|\eta|\le\delta R_k} v_k^{\frac{2n}{n-2}}d\eta\le (\bar b+1)\int_{R_1\le|\eta|\le\delta R_k} v_k^{\frac{n+2}{n-2}}d\eta\le 5(\bar b+1)\epsilon.$$

For each $2R_1<r<\delta R_k/2$, we consider $\tilde v_k(z)=r^{\frac{n-2}{2}}v_k(rz)$ for $1/2<|z|<2$. Then $\tilde v_k$ satisfies

$$\begin{aligned}&\frac{1}{\sqrt{g(rz)}}\partial_{z_i}(\sqrt{g(rz)}g^{ij}(rz)\partial_{z_j}\tilde v_k(z))-\bar c r^2\tilde v_k(z)\\&+n(n-2)\tilde v_k(z)^{\frac{n+2}{n-2}}=0,\quad 1/2<|z|<2.\end{aligned}$$

We know that $\int_{\frac12\le|z|\le 2}\tilde v_k(z)^{\frac{2n}{n-2}}\le 5(\bar b+1)\epsilon$. Fix some universally small $\epsilon>0$, we apply the Moser iteration technique to obtain $\tilde v_k(z)\le C$ for $\frac34\le|z|\le\frac43$, where C is independent of k. With this, we apply the Harnack inequality to obtain $\max_{|z|=1}\tilde v_k(z)\le C\min_{|z|=1}\tilde v_k(z)$, i.e., $\max_{|y|=r}v_k(y)\le C\min_{|y|=r}v_k(y)$. By Lemma 2.1,

$$\min_{|y|=r}v_k(y)\le (1+\epsilon)U(r).$$

Lemma 2.4 follows from these together with the convergence of v_k to U. $\square$

Proof of Proposition 2.2. The argument below is very similar to the proof of Lemma 3.3 in [35]. Let $\Lambda_k = \max_{|y|\le \delta R_k} |(v_k - U)(y)|$ and let $w_k = \Lambda_k^{-1}(v_k - U)$. We will show that

$$\Lambda_k \le C_2 R_k^{2-n} \tag{97}$$

for some $C_2 > 0$, independent of k. Suppose this is not true, then, along a subsequence,

$$\Lambda_k R_k^{n-2} \to \infty. \tag{98}$$

By Lemma 2.4,

$$w_k(y) \le C\Lambda_k^{-1} U(y), \qquad |y| \le \delta R_k. \tag{99}$$

By (98) and (99),

$$\max_{\partial B(0,\delta R_k)} |w_k| \to 0. \tag{100}$$

Since

$$\begin{aligned} \Delta_{g_k} v_k - \bar{c} v_k + n(n-2) v_k^{\frac{n+2}{n-2}} &= 0, \qquad\quad |y| \le \delta_1 M_k^{\frac{2}{n-2}}, \\ \Delta_{g_k} U - \bar{c} U + n(n-2) U^{\frac{n+2}{n-2}} &= -\bar{c}U, \quad |y| \le \delta_1 M_k^{\frac{2}{n-2}}, \end{aligned}$$

w_k satisfies

$$\Delta_{g_k} w_k - \bar{c} w_k = -n(n+2)\hat{\xi}^{\frac{4}{n-2}} w_k + \Lambda_k^{-1} \bar{c} U, \quad \text{in} \quad |y| \le \delta R_k$$

where $\hat{\xi}$ is between v_k and U. By Lemma 2.4, there is $C > 0$ such that

$$|\hat{\xi}(y)| \le C(1+|y|)^{2-n}, \qquad |y| \le \delta R_k.$$

By (98), $\Lambda_k^{-1} = \circ(1) R_k^{n-2}$.

Fixing $\epsilon > 0$ sufficiently small, we know from (29) and (50) that, for $|y| \le \delta R_k$,

$$|\bar{c}(y)| R_k^{2+\epsilon} + |\bar{b}_i(y)| R_k^{1+\epsilon} + |\bar{d}_{ij}(y)| R_k^{\epsilon} \le C M_k^{-\frac{4}{n-2}} R_k^{2+\epsilon} = \circ(1).$$

By (98), $\Lambda_k^{-1} = \circ(1) R_k^{n-2}$. So we have, using (29) and (50),

$$\Lambda_k^{-1} |\bar{c} U| \le \circ(1) M_k^{-\frac{8}{n-2}} R_k^{\max\{4+\epsilon, n-2\}} (1+|y|)^{-2-\epsilon} = \circ(1)(1+|y|)^{-2-\epsilon}. \tag{101}$$

It follows that w_k satisfies, for $|y| \le \delta R_k$, that

$$\left(\Delta + \frac{\circ(1)\partial_{ij}}{(1+|y|)^{\epsilon}} + \frac{\circ(1)\partial_i}{(1+|y|)^{1+\epsilon}} + \frac{\circ(1)}{(1+|y|)^{2+\epsilon}}\right) w_k(y) = O(1)(1+|y|)^{-2-\epsilon}.$$

Let $\eta(r) = (1+r^2)^{-\frac{\epsilon}{2}}$, we have, for some $C > 0$ depending only on ϵ and n,

$$\left(\Delta + \frac{\circ(1)\partial_{ij}}{(1+|y|)^{\epsilon}} + \frac{\circ(1)\partial_i}{(1+|y|)^{1+\epsilon}} + \frac{\circ(1)}{(1+|y|)^{2+\epsilon}}\right) \eta \le -C^{-1}(1+r)^{-\epsilon-2}.$$

Taking a large constant positive Q, independent of k, we have, for $|y| \le \delta R_k$,

$$\left(\Delta + \frac{\circ(1)\partial_{ij}}{(1+|y|)^{\epsilon}} + \frac{\circ(1)\partial_i}{(1+|y|)^{1+\epsilon}} + \frac{\circ(1)}{(1+|y|)^{2+\epsilon}}\right)(\pm w_k - \max_{|z|=\delta R_k} |w_k(z)| - Q\eta) \ge 0.$$

By the maximum principle,

$$|w_k(y)| \le \eta(|y|) + \max_{|z|=\delta R_k} |w_k(z)|, \qquad |y| \le \delta R_k.$$

Next, we deduce, using standard elliptic estimates, from (101) and the equation of w_k that w_k converges in $C^2_{loc}(\mathbb{R}^n)$ to some w_0 satisfying

$$\begin{cases} \Delta w_0 + n(n+2)U^{\frac{4}{n-2}} w_0 = 0, \quad \text{in } \mathbb{R}^n, \\ w_0(0) = 0, \ \nabla w_0(0) = 0, \ \lim_{|y|\to\infty} w_0(y) = 0. \end{cases}$$

By Lemma 2.4 in [35], $w_0 \equiv 0$.

Let y_k be a maximum point of $w_k(y)$ in $|y| \le \delta R_k$, i.e., $w_k(y_k) = 1, |y_k| \le \delta R_k$. By the above estimates,

$$1 = w_k(y_k) \le C(1 + |y_k|)^{-1} + \circ(1),$$

so, $\{y_k\}$ must be bounded, and therefore, by the convergence of w_k to $w_0 \equiv 0$, $w_k(y_k) \to 0$. This contradicts to $w_k(y_k) = 1$. Thus we have established (97). Using the equation satisfied by w_k, we have, by standard elliptic theories,

$$\|w_k\|_{C^2(B(0,\delta R_k - 1))} \le C\|w_k\|_{L^\infty(B(0,\delta R_k))} \le C.$$

Proposition 2.2 follows from this in view of (97). □

Now we give the

Proof of (18) for $3 \le n \le 7$. For this, we only need to reach a contradiction to (22) in dimension $3 \le n \le 7$. As pointed before, (22) is equivalent to (30). It is easy to see from (30) that $\frac{1}{k} M_k^{\frac{2}{n-2}} \to \infty$. We know from Corollary 2.3 that (45) holds in dimensions $3 \le n \le 6$ for any $0 < \gamma < 2(n-2)$ while in dimension $n = 7$ it holds for $\gamma = 8$. Let, in dimensions $3 \le n \le 7$, $R_k = k^{-\frac{1}{4}} M_k^{\frac{2}{n-2}}$. Then $\{R_k\}$ satisfy (47), (48) and (49) with the above γ and sufficiently small $\bar{\epsilon}$. Thus, by Lemma 2.4,

$$v_k(y) \le CU(y), \qquad \forall\, |y| \le k^{-\frac{1}{2}} M_k^{\frac{2}{n-2}},$$

where $C > 0$ is independent of k. This violates (30). Thus estimate (18) in dimension $3 \le n \le 7$ is established. □

The following is a Pohozaev type identity.

Lemma 2.5. *For $n \geq 3$, let u be a solution of the Yamabe equation, then in a neighborhood of any point $P \in M$, the following identity holds in a normal coordinate of P.*

$$\int_{|x|\leq\sigma} \left\{(-b_i\partial_i u - d_{ij}\partial_{ij}u)\left(\nabla u \cdot x + \frac{n-2}{2}u\right) - \frac{c(n)}{2}u^2(x\cdot\nabla R(x)) - c(n)R(x)u^2\right\} + \frac{\sigma}{2}c(n)\int_{|x|=\sigma} R(x)u^2 - \frac{(n-2)^2}{2}\sigma\int_{|x|=\sigma} u^{\frac{2n}{n-2}} = B(\sigma, u, \nabla u)$$

where

$$B(\sigma, u, \nabla u) = \int_{|x|=\sigma}\left(\left|\frac{\partial u}{\partial \nu}\right|^2\sigma - \frac{1}{2}|\nabla u|^2\sigma + \frac{n-2}{2}u\frac{\partial u}{\partial \nu}\right)$$

Proof. This identity is established for $n = 3$ in [73]. A modification of the proof there yields the above lemma. □

Applying Lemma 2.5 to $u = u_k$ (see (27)) with $\sigma = M_k^{-\frac{2}{n-2}}R_k'$, we have, after a change of variables,

$$I_1[v_k] + I_2[v_k] + I_3[v_k] + I_4[v_k] = I_5[v_k], \tag{102}$$

$$I_1[v_k] = \int_{|y|\leq R_k'}(-\bar{b}_i\partial_i v_k - \bar{d}_{ij}\partial_{ij}v_k)\left(\nabla v_k\cdot y + \frac{n-2}{2}v_k\right),$$

$$I_2[v_k] = -\frac{c(n)}{2}M_k^{-\frac{4}{n-2}}\int_{|y|\leq R_k'}\left\{\left(M_k^{-\frac{2}{n-2}}y\right)\cdot\nabla R\left(M_k^{-\frac{2}{n-2}}y\right) + 2R\left(M_k^{-\frac{2}{n-2}}y\right)\right\}v_k^2(y),$$

$$I_3[v_k] = \frac{c(n)}{2}M_k^{-\frac{4}{n-2}}R_k'\int_{|y|=R_k'}R\left(M_k^{-\frac{2}{n-2}}y\right)v_k^2(y),$$

$$I_4[v_k] = -\frac{(n-2)^2}{2}R_k'\int_{|y|=R_k'}v_k(y)^{\frac{2n}{n-2}},$$

$$I_5[v_k] = \int_{|y|=R_k'}\left\{\left(|\frac{\partial v_k}{\partial\nu}|^2 - \frac{1}{2}|\nabla v_k|^2\right)R_k' + \frac{n-2}{2}v_k\frac{\partial v_k}{\partial \nu}\right\} = O(1)(R_k')^{2-n}.$$

Let $\beta_2, \beta_4, \beta_2'', \beta_2''', \beta_3''' \geq 0$ satisfy, for some constant $C \geq 0$,

$$\bar{R}^{(2)} \leq CM_k^{-\frac{\beta_2}{n-2}}, \quad \bar{R}^{(4)} \leq CM_k^{-\frac{\beta_4}{n-2}}, \tag{103}$$

$$|b_i(x)| \leq CM_k^{-\frac{\beta_2''}{n-2}}|x|^2 + C|x|^3, \ |d_{ij}(x)| \leq C\sum_{l=2}^{3}M_k^{-\frac{\beta_l'''}{n-2}}|x|^l + C|x|^4, \tag{104}$$

or, equivalently,

$$
\begin{aligned}
|\bar{b}_i(y)| &\le CM_k^{-\frac{6+\beta_2''}{n-2}}|y|^2 + CM_k^{-\frac{8}{n-2}}|y|^3,\\
|\bar{d}_{ij}(y)| &\le C\sum_{l=2}^{3} M_k^{-\frac{2l+\beta_l'''}{n-2}}|y|^l + CM_k^{-\frac{8}{n-2}}|y|^4.
\end{aligned}
\tag{105}
$$

We will always take $\beta_3''' = \beta_2''$.

Lemma 2.6. *For $n \ge 7$, $\bar{l} \ge 2$, let v_k satisfy the first line of (29), and we assume (104) holds for some constants $\beta_2'', \beta_2''', \beta_3''' \ge 0$. For $2 \le R_k' \le \frac{1}{4}M_k^{\frac{2}{n-2}}$, we assume, for some constants $\gamma_1, \gamma_2, C \ge 0$,*

$$
v_k(y) \le CU(y), \qquad |y| \le 2R_k', \tag{106}
$$

and

$$
|\nabla^j(v_k - U)| \le CM_k^{-\gamma_1}(1+|y|)^{-\gamma_2-j}, \qquad |y| \le R_k', j = 0,1,2. \tag{107}
$$

Then

$$
\begin{aligned}
&-\frac{c(n)}{2}|\mathbb{S}^{n-1}|\sum_{l=2}^{\bar{l}}(l+2)M_k^{-\frac{4+2l}{n-2}}\bar{R}^{(l)}\int_0^{R_k'} r^{l+n-1}U(r)^2dr\\
&= I_5[v_k] + O(1)\sum_{l=2}^{\bar{l}} M_k^{-\gamma_1-\frac{4+2l}{n-2}}\int_{|y|\le R_k'}(1+|y|)^{2-n-\gamma_2+l}\\
&\quad + O(1)M_k^{-\frac{6+2\bar{l}}{n-2}}\int_{|y|\le R_k'}(1+|y|)^{5-2n+\bar{l}}\\
&\quad + O(1)M_k^{-\gamma_1-\frac{6+\beta_2''}{n-2}}\int_{|y|\le R_k'}(1+|y|)^{3-n-\gamma_2}\\
&\quad + O(1)M_k^{-\gamma_1-\frac{8}{n-2}}\int_{|y|\le R_k'}(1+|y|)^{4-n-\gamma_2}\\
&\quad + O(1)\sum_{l=2}^{3} M_k^{-\gamma_1-\frac{2l+\beta_l'''}{n-2}}\int_{|y|\le R_k'}(1+|y|)^{-n+l-\gamma_2}\\
&\quad + I_3[v_k] + O(1)(R_k')^{-n},
\end{aligned}
\tag{108}
$$

and

$$
I_5[v_k] = O(1)(R_k')^{2-n}, \quad I_3[v_k] = O(1)M_k^{-\frac{8}{n-2}}(R_k')^{6-n}. \tag{109}
$$

Proof. Applying standard elliptic estimates to the equation of v_k and using (106), we obtain

$$|\nabla v_k(y)| \le C(1+|y|)^{1-n}, \quad |\nabla^2 v_k(y)| \le C(1+|y|)^{-n}, \qquad |y| \le R_k', \tag{110}$$

where, and throughout the proof, C denotes various positive constants independent of k.

The desired estimates are deduced from (102). The main term there is

$$\begin{aligned}
I_2[v_k] &= -\frac{c(n)}{2}\sum_{l=2}^{\bar{l}}\sum_{|\alpha|=l}\int_{|y|\le R_k'}\left\{\left(\frac{(l+2)}{\alpha!}\right)M_k^{-\frac{4+2l}{n-2}}(\partial_\alpha R)y^\alpha \right.\\
&\quad \left. +O(1)M_k^{-\frac{6+2\bar{l}}{n-2}}|y|^{\bar{l}+1}\right\}v_k^2\\
&= -\frac{c(n)}{2}\sum_{l=2}^{\bar{l}}\sum_{|\alpha|=l}\left(\frac{(l+2)}{\alpha!}\right)M_k^{-\frac{4+2l}{n-2}}\int_{|y|\le R_k'}(\partial_\alpha R)y^\alpha U(y)^2\\
&\quad +O(1)\sum_{l=2}^{\bar{l}}M_k^{-\gamma_1-\frac{4+2l}{n-2}}\int_{|y|\le R_k'}(1+|y|)^{2-n-\gamma_2+l}\\
&\quad +O(1)M_k^{-\frac{6+2\bar{l}}{n-2}}\int_{|y|\le R_k'}(1+|y|)^{5-2n+\bar{l}}\\
&= -\frac{c(n)}{2}\sum_{l=2}^{\bar{l}}(l+2)M_k^{-\frac{4+2l}{n-2}}|\mathbb{S}^{n-1}|\bar{R}^{(l)}\int_0^{R_k'}r^{l+n-1}U(r)^2dr\\
&\quad +O(1)\sum_{l=2}^{\bar{l}}M_k^{-\gamma_1-\frac{4+2l}{n-2}}\int_{|y|\le R_k'}(1+|y|)^{2-n-\gamma_2+l}\\
&\quad +O(1)M_k^{-\frac{6+2\bar{l}}{n-2}}\int_{|y|\le R_k'}(1+|y|)^{5-2n+\bar{l}}.
\end{aligned} \tag{111}$$

Using (110) and (107), we have

$$\begin{aligned}
|I_1[v_k]| &= |\int_{|y|\le R_k'}[(\Delta-\Delta_{g_k})(v_k-U)]\left(\nabla v_k\cdot y+\frac{n-2}{2}v_k\right)\\
&\le C\int_{|y|\le R_k'}(|\bar{b}_i||\partial_i(v_k-U)|+|\bar{d}_{ij}||\partial_{ij}(v_k-U)|)U(y)\\
&\le CM_k^{-\gamma_1-\frac{6+\beta_2''}{n-2}}\int_{|y|\le R_k'}(1+|y|)^{3-n-\gamma_2}\\
&\quad +CM_k^{-\gamma_1-\frac{8}{n-2}}\int_{|y|\le R_k'}(1+|y|)^{4-n-\gamma_2}\\
&\quad +C\sum_{l=2}^{3}M_k^{-\gamma_1-\frac{2l+\beta_l'''}{n-2}}\int_{|y|\le R_k'}(1+|y|)^{-n+l-\gamma_2}.
\end{aligned}$$

The following estimates are straight forward:

$$|I_3[v_k]| = O(1)M_k^{-\frac{8}{n-2}}(R_k')^{6-n}, \qquad |I_4[v_k]| = O(1)(R_k')^{-n},$$
$$|I_5[v_k]| = O(1)\int_{|y|=R_k'} (|\nabla U(y)|^2 R_k' + U|\nabla U(y)|) = O(1)(R_k')^{2-n}. \qquad \square$$

To prove Theorem 2.1 for $n \geq 6$ we need to establish appropriate decay rates of the Riemannian curvature tensor at the center of the conformal normal coordinate system we use.

Lemma 2.7. *For $3 \leq n \leq 6$, let $\{v_k\}$ satisfy (29), assume that for some $\delta > 0$,*

$$v_k(y) \leq C_1 U(y) \quad \textit{for} \quad |y| \leq \delta M_k^{\frac{2}{n-2}}. \tag{112}$$

Then for any $\epsilon > 0$ there exists some constant C independent of k such that for all $|y| \leq \frac{\delta}{2}$,

$$|\nabla^j (v_k - U)(y)| \leq C M_k^{-2+\frac{2\epsilon}{n-2}} (1+|y|)^{-\epsilon - j}, \quad j = 0, 1, 2. \tag{113}$$

Lemma 2.8. *For $n \geq 7$, $0 \leq \bar{a} < n-6$, let v_k satisfy (29). There exists some small $\delta' > 0$, depending only on n, such that if $\{R_k'\}$ and $\{v_k\}$ satisfy*

$$2 \leq R_k' \leq 2\delta' M_k^{\frac{2}{n-2}}, \tag{114}$$
$$M_k^{\frac{8}{n-2}} = O(1)(R_k')^{4+\bar{a}}, \tag{115}$$

and

$$v_k(y) \leq C U(y), \qquad |y| \leq R_k', \tag{116}$$

then, for $j = 0, 1, 2$, and for some $C' > 0$ independent of k,

$$|\nabla^j (v_k - U)(y)| \leq C' M_k^{-\frac{8}{n-2}} (1+|y|)^{6-n+\bar{a}-j}, \qquad \forall\, |y| \leq \frac{1}{4} R_k'. \tag{117}$$

Remark 2.2. Since we have established (18) for $n \leq 7$, we already know that the hypothesis in Lemma 2.8 is satisfied for $n = 7$ with $R_k' = \delta M_k^{\frac{2}{n-2}}$, $\bar{a} = 0$, where $\delta > 0$ is some number independent of k.

We prove Lemma 2.8 first and Lemma 2.7 next.

Proof of Lemma 2.8. Let $w_k = v_k - U$, consider the equation for w_k:

$$\Delta_{g_k} w_k(y) - \bar{c} w_k(y) + n(n+2)\bar{\xi}^{\frac{4}{n-2}} w_k(y) = \bar{c} U(y), \tag{118}$$

where

$$n(n+2)\bar{\xi}^{\frac{4}{n-2}}(y) = \begin{cases} n(n-2)\dfrac{v_k^{\frac{n+2}{n-2}}(y) - U^{\frac{n+2}{n-2}}(y)}{v_k(y) - U(y)} & \text{if} \quad v_k(y) \neq U(y), \\ n(n+2)U(y)^{\frac{4}{n-2}} & \text{if} \quad v_k(y) = U(y). \end{cases}$$

By (29),

$$|\bar{c}U(y)| \leq CM_k^{-\frac{8}{n-2}}(1+|y|)^{4-n}. \tag{119}$$

For R_1 sufficiently large, we claim that the operator $\Delta_{g_k} - \bar{c} + n(n+2)\bar{\xi}^{\frac{4}{n-2}}$ satisfies maximum principle over $R_1 < |y| < R_k'$. To see this, we estimate the $L^{\frac{n}{2}}$ norm of the coefficients of w_k:

$$\begin{aligned} &\int_{R_1 \leq |y| \leq R_k'} |n(n+2)\bar{\xi}^{\frac{4}{n-2}} - \bar{c}|^{\frac{n}{2}} \\ &\leq C \int_{R_1 \leq |y| \leq \delta' M_k^{-\frac{2}{n-2}}} \left| M_k^{-\frac{8}{n-2}}|y|^2 + |y|^{-4} \right|^{\frac{n}{2}} \leq C[(\delta')^{2n} + (R_1)^{-n}]. \end{aligned}$$

So for R_1 sufficiently large and δ' sufficiently small, the maximum principle holds for $\Delta_{g_k} - \bar{c} + n(n+2)\bar{\xi}^{\frac{4}{n-2}}$ over $R_1 \leq |y| \leq R_k'$.

We compare w_k with

$$f(r) := C_{10}M_k^{-\frac{8}{n-2}}|y|^{6-n+\bar{a}}$$

over $R_1 \leq |y| \leq R_k'$ where C_{10} will be chosen momentarily.

Since $\Delta(r^{6-n+\bar{a}}) = -(n-6-\bar{a})(\bar{a}+4)r^{4-n+\bar{a}}$, we have, using (29), (114) and (116), and taking R_1 larger and δ' smaller if necessary,

$$\begin{aligned} &(\Delta + \bar{b}_i\partial_i + \bar{d}_{ij}\partial_{ij} - \bar{c} + n(n+2)\bar{\xi}^{\frac{4}{n-2}})f(|y|) \\ &\leq -\frac{1}{2}(n-6-\bar{a})(\bar{a}+4)r^{-2}f(r), \quad R_1 \leq |y| \leq R_k'. \end{aligned} \tag{120}$$

Making C_{10} larger if necessary, we have, using (115) and Corollary 2.2,

$$|w_k(y)| \leq f(y), \qquad |y| = R_1 \text{ or } |y| = R_k'.$$

By the maximum principle, applied to the difference of (118) and (120), and using (119) and $\bar{a} \geq 0$, we have

$$|w_k(y)| \leq f(y), \qquad R_1 < |y| \leq R_k'.$$

Estimate (117) for $j = 0$ follows from this.

Applying standard elliptic estimates to the equation of v_k and using (116), we obtain

$$|\nabla v_k(y)| \leq (1+|y|)^{1-n}, \qquad |y| \leq \frac{1}{2}R_k'.$$

With this and (32), we obtain

$$|\nabla \bar{\xi}^{\frac{4}{n-2}}(y)| \le C(1+|y|)^{-5}, \qquad |y| \le \frac{1}{2}R_k'. \tag{121}$$

Let $\hat{w}_k(z) = w_k(y + \frac{|y|}{10}z)$, $|z| \le 1$. Applying standard elliptic estimates to the equation satisfied by $\hat{w}_k$, derived from (118), and using (121), we obtain

$$|\nabla \hat{w}_k(0)| + |\nabla^2 \hat{w}_k(0)| \le C\left(\|\hat{w}_k\|_{L^\infty(B_1)} + \||y|^2(\bar{c}U)\left(y + \frac{|y|}{10}\cdot\right)\|_{C^1(B_1)}\right).$$

Estimate (117) for $j = 1, 2$ follows from the above by using (117) for $j = 0$. Lemma 2.8 is proved. □

Proof of Lemma 2.7. This proof is essentially the same as that of Lemma 2.8. We only need to change the comparison function to

$$f(r) := C_{10}M_k^{-2+\frac{2\epsilon}{n-2}}r^{-2-\epsilon}$$

and to observe that

$$|\bar{c}U(y)| \le CM_k^{-\frac{8}{n-2}}(1+|y|)^{4-n} \le CM_k^{-2+\frac{2\epsilon}{n-2}}(1+|y|)^{-2-\epsilon}.$$ □

Proof of (19) for $n = 6$. By (102) with $R_k' = \frac{\delta}{2}M_k^{\frac{2}{n-2}}$,

$$I_2[v_k] = O(1)\left(|I_1[v_k]| + |I_3[v_k]| + |I_4[v_k]| + |I_5[v_k]|\right).$$

Using (29), (110), (112) and (113), we obtain

$$|I_1[v_k]| + |I_3[v_k]| + |I_4[v_k]| + |I_5[v_k]| \le CM_k^{-2}.$$

In view of the first line in (111), we have, for some $c_6(n) > 0$,

$$\begin{aligned} I_2[v_k] &= -12nc_6(n)\bar{R}^{(2)}M_k^{-2}\log M_k + O\left(M_k^{-2}\right) \\ &= -c_6(n)|W|^2M_k^{-2}\log M_k + O\left(M_k^{-2}\right). \end{aligned}$$

Estimate (19) for $n = 6$ follows from the above.

Proof of (19) for $n = 7$. By Lemma 2.8 and Remark 2.2, (117) holds for $n = 7$, $R_k' = \delta M_k^{\frac{2}{n-2}}$ and $\bar{a} = 0$. Thus, by Lemma 2.6, (108) holds with $\beta_2'' = \beta_2''' = \beta_3''' = 0$, $R_k' = \delta M_k^{\frac{2}{n-2}}$, $\gamma_1 = \frac{8}{n-2}$, $\gamma_2 = n-6$ and $\bar{l} = 3$. It follows that

$$M_k^{-\frac{8}{n-2}}\bar{R}^{(2)}\int_0^{R_k'} r^{1+n}U(r)^2dr = O\left(M_k^{-2}\right)$$

which implies (19) for $n = 7$.

Lemma 2.9. *For $n \ge 8$, let v_k satisfy (29). Then, for any $\epsilon > 0$,*

$$|W| \le M_k^{-\frac{2-\epsilon}{n-2}}.$$

Proof of Lemma 2.9. By Corollary 2.3, (116) is satisfied with $R'_k = M_k^{\frac{12-\epsilon}{(n-2)^2}}$ for any small $\epsilon > 0$. With this R'_k, (114) is satisfied and (115) is satisfied with $\bar{a} = \frac{2(n-8)}{3} + O(1)\epsilon$. Thus by Lemma 2.8, (107) is satisfied with $\gamma_1 = \frac{8}{n-2}$ and $\gamma_2 = n - 6 - \bar{a}$. Applying Lemma 2.6 with the above data and $\bar{l} = 3$, we derive from (108) and (109) that

$$|W|^2 = -12n\bar{R}^{(2)} = O(1)M_k^{-\frac{4-\epsilon}{n-2}}.$$

Lemma 2.9 is established. □

The following properties of conformal normal coordinates are established in [60]: If $W = 0$, then $R_{abcd} = 0$ and, for some constant $c_1(n) > 0$, $\bar{R}^{(4)} = -c_1(n)|R_{abcd,e}|^2$; if $W = 0$ and $\nabla W = 0$, then $R_{abcd,e} = 0$. Examining the proofs there, we arrive at

$$|R_{abcd}| = O(1)|W|, \quad |R_{abcd,e}| = O(|W|) + O(|\nabla_g W|), \tag{122}$$

and

$$\bar{R}^{(4)} = -c_1(n)|R_{abcd,e}|^2 + O(|W|). \tag{123}$$

It follows from Lemma 2.9, (122) and (123) that, for any $\epsilon > 0$,

$$|R_{abcd}| = O(1)M_k^{-\frac{2-\epsilon}{n-2}}, \tag{124}$$

and

$$\bar{R}^{(4)} \le CM_k^{-\frac{2-\epsilon}{n-2}}. \tag{125}$$

We know from Corollary 2.2 that

$$\sigma_k \le CM_k^{-\frac{8}{n-2}}. \tag{126}$$

By (124),

$$|\bar{d}_{ij}(y)| \le \begin{cases} CM_k^{-\frac{6-\epsilon}{n-2}}|y|^{3-\frac{\epsilon}{2}}, & |y| \ge 1, \\ CM_k^{-\frac{6-\epsilon}{n-2}}|y|^2, & |y| \le 1. \end{cases} \tag{127}$$

Following the proof of Proposition 2.1 while using also (126) and (127), we have, instead of (36),

$$E_\lambda(y) \le \bar{c}(y)U^\lambda(y) - \left(\frac{\lambda}{|y|}\right)^{n+2} \bar{c}(y^\lambda)U(y^\lambda) + O(1)M_k^{-\frac{14-\epsilon}{n-2}}|y|^{1-n}.$$

Instead of Corollary 2.1, we now have

$$E_\lambda(y) \le \sum_{l=2}^{4} M_k^{-\frac{4+2l}{n-2}} H_{l,\lambda}(r)\tilde{R}^{(l)}(\theta) + C_0 M_k^{-\frac{14-\epsilon}{n-2}} r^{7-n-\frac{\epsilon}{2}}.$$

Take

$$\tilde{h}_{1,\lambda}(y) = \sum_{l=2}^{4} \sum_{j=1}^{l} \sum_{i=1}^{I_j} \tilde{h}^{(i)}_{l,j,\lambda}(y),$$
$$\tilde{h}_{2,\lambda}(y) = QM_k^{-\frac{14-\epsilon}{n-2}} f_{n,\min\{n-7+\frac{\epsilon}{2},2-\epsilon\}}\left(\frac{r}{\lambda}\right),$$

and

$$h_\lambda(y) = \tilde{h}_{1,\lambda}(y) + \tilde{h}_{2,\lambda}(y).$$

Let

$$R_k = \begin{cases} k^{-\frac{1}{4}} M_k^{\frac{2}{n-2}}, \, n = 8, \\ M_k^{\frac{14-\sqrt{\epsilon}}{(n-2)^2}}, \, n \geq 9. \end{cases} \tag{128}$$

Then we can follow the proof of Lemma 2.1 to show that

$$\min_{|y|=r} v_k(y) \leq (1+\epsilon) U(r), \qquad \forall \, 0 < r \leq R_k. \tag{129}$$

Indeed we only need to verify a few things. First we still have (73). As before, we can show that

$$(\Delta_{g_k} - \bar{c} + n(n+2)\xi^{\frac{4}{n-2}})\tilde{h}_{2,\lambda}(y) \leq -\frac{Q}{4}\bar{D}_{k,\epsilon}(r),$$

where

$$\bar{D}_{k,\epsilon}(r) := M_k^{-\frac{14-\epsilon}{n-2}} r^{-\min\{n-7+\frac{\epsilon}{2},2-\epsilon\}}.$$

We can verify, using the strengthened estimate (127), that

$$|\bar{c}||\tilde{h}_{1,\lambda}(y)| + |\bar{b}_i \partial_i \tilde{h}_{1,\lambda}(y)| + |\bar{d}_{ij}\partial_{ij}\tilde{h}_{1,\lambda}(y)| \leq C\bar{D}_{k,\epsilon}(r). \tag{130}$$

Recall that

$$|\nabla^j (v_k - U)(y)| \leq C' M_k^{-\frac{8}{n-2}} |y|^{6-n+\bar{a}-j}, \qquad \forall \, |y| \leq \frac{1}{4} M_k^{\frac{12-\epsilon}{(n-2)^2}}, \tag{131}$$

where $\bar{a} = \frac{2(n-8)}{3} + \sqrt{\epsilon}$. With (131) we have, instead of (90),

$$|n(n+2)\xi^{\frac{4}{n-2}}(y) - V_\lambda(|y|)||\tilde{h}_{1,\lambda}(y)| \leq CM_k^{-\frac{8}{n-2}}|y|^{\bar{a}}, \quad \lambda \leq |y| \leq \frac{1}{4} M_k^{\frac{12-\epsilon}{(n-2)^2}}, \tag{132}$$

which can be shown by following the arguments in the proof of Lemma 2.3 together with the improved bounds

$$|a(y)| + |b(y)| \leq M_k^{-\frac{8}{n-2}} |y|^{6-n+\bar{a}}$$

given by (131).

With (91) and the improved estimate (132), we can show that

$$
\begin{aligned}
&(\Delta+n(n+2)\xi^{\frac{4}{n-2}})\tilde{h}_{1,\lambda}\\
&\le(\Delta+V_\lambda)\tilde{h}_{1,\lambda}+|n(n+2)\xi^{\frac{4}{n-2}}-V_\lambda||\tilde{h}_{1,\lambda}|\\
&\le-\sum_{l=2}^{4}M_k^{-\frac{4+2l}{n-2}}H_{l,\lambda}(r)\tilde{R}^{(l)}(\theta)+O(1)\bar{D}_{k,\epsilon}(r),\qquad \text{in } \hat{O}_\lambda.
\end{aligned}
$$

Thus, in view of (130),

$$
\begin{aligned}
&(\Delta_{g_k}-\bar{c}+n(n+2)\xi^{\frac{4}{n-2}})\tilde{h}_{1,\lambda}\\
&\le-\sum_{l=2}^{4}M_k^{-\frac{4+2l}{n-2}}H_{l,\lambda}(r)\tilde{R}^{(l)}(\theta)+C\bar{D}_{k,\epsilon}(r),\qquad \text{in } \hat{O}_\lambda.
\end{aligned}
$$

Fixing a large Q, we obtain

$$
(\Delta_{g_k}-\bar{c}+n(n+2)\xi^{\frac{4}{n-2}})h_\lambda+E_\lambda\le 0\quad\text{in}\quad \hat{O}_\lambda. \tag{133}
$$

With (133), estimate (129) with R_k given by (128) is established, as in the proof of Lemma 2.1.

Once (129) is established, the proof of Lemma 2.4 yields

$$
v_k(y)\le CU(y),\qquad |y|\le R_k \tag{134}
$$

for the R_k given by (128).

Proof of (18) for $n=8$. Since estimate (134) holds for $R_k=k^{-\frac{1}{2}}M_k^{\frac{2}{n-2}}$, which violates (30), estimate (18) in dimension $n=8$ is established.

Now we turn to $n\ge 9$. Since (134) holds with $R_k=M_k^{\frac{14-\epsilon}{(n-2)^2}}$ for any $\epsilon>0$, hypotheses (114), (115) and (116) hold with $R_k'=M_k^{\frac{14-\epsilon}{(n-2)^2}}$ and $\bar{a}=\frac{4(n-9)}{7}+\sqrt{\epsilon}$. Thus, by Lemma 2.8,

$$
|\nabla^j(v_k-U)(y)|\le C'M_k^{-\frac{8}{n-2}}|y|^{6-n+\bar{a}-j},\qquad \forall\,|y|\le M_k^{\frac{14-\epsilon}{(n-2)^2}},\ j=0,1,2. \tag{135}
$$

With this, we can apply Lemma 2.6 with $\bar{l}=5$, $\beta_2''=\beta_3'''=0$, $\beta_2'''=2-\epsilon$, $R_k'=M_k^{\frac{14-\epsilon}{(n-2)^2}}$, $\gamma_1=\frac{8}{n-2}$ and $\gamma_2=n-6-\bar{a}$ to deduce, in view of (123), from (108) and (109) that

$$
M_k^{-\frac{8}{n-2}}|W|^2+M_k^{-\frac{12}{n-2}}\left[|R_{abcd,e}|^2+O(|W|)\right]=O\left(M_k^{-\frac{14-\epsilon}{n-2}}\right).
$$

Consequently, we have, using (122), that for any $\epsilon>0$,

$$
|W|=O(1)M_k^{-\frac{3-\epsilon}{n-2}},\qquad |R_{abcd,e}|=O(1)M_k^{-\frac{1-\epsilon}{n-2}}. \tag{136}
$$

Now we consider $n \geq 9$. By (136),

$$|b_i(x)| \leq CM_k^{-\frac{\beta_2''}{n-2}}|x|^2 + C|x|^3,\ |d_{ij}(x)| \leq C\sum_{l=2}^{3} M_k^{-\frac{\beta_l'''}{n-2}}|x|^l + C|x|^4,$$

or, equivalently,

$$\begin{cases} |\bar{b}_i(y)| \leq CM_k^{-\frac{6+\beta_2''}{n-2}}|y|^2 + CM_k^{-\frac{8}{n-2}}|y|^3, \\ |\bar{d}_{ij}(y)| \leq C\sum_{l=2}^{3} M_k^{-\frac{2l+\beta_l'''}{n-2}}|y|^l + CM_k^{-\frac{8}{n-2}}|y|^4, \end{cases} \tag{137}$$

where

$$\beta_2'' = \beta_3''' = 1-\epsilon,\ \beta_2''' = 3-\epsilon,\ \epsilon > 0.$$

With (137), we follow the proof of Proposition 2.1 to obtain, using also (126), that

$$E_\lambda(y) = \bar{c}(y)U^\lambda(y) - \left(\frac{\lambda}{|y|}\right)^{n+2}\bar{c}(y^\lambda)U(y^\lambda) + O(1)M_k^{-\frac{15-\epsilon}{n-2}}|y|^{2-n}.$$

We also know that for $n \geq 9$,

$$\bar{R}^{(4)} \leq CM_k^{-\frac{3-\epsilon}{n-2}}.$$

Thus we have, using also $\bar{R}^{(2)} \leq 0$ and (41) with $\bar{l} = 5$,

$$E_\lambda(y) \leq \sum_{l=2}^{5} M_k^{-\frac{4+2l}{n-2}} H_{l,\lambda}(r)\tilde{R}^{(l)}(\theta) + CM_k^{-\frac{15-\epsilon}{n-2}}|y|^{-\frac{3}{2}-\frac{\epsilon}{2}}. \tag{138}$$

Now we take

$$\tilde{h}_{1,\lambda}(y) = \sum_{l=2}^{5}\sum_{j=1}^{l}\sum_{i=1}^{I_j}\tilde{h}_{l,j,\lambda}^{(i)}(y),$$
$$\tilde{h}_{2,\lambda}(y) = QM_k^{-\frac{15-\epsilon}{n-2}} f_{n,\frac{3}{2}+\epsilon}\left(\frac{r}{\lambda}\right),$$

and

$$h_\lambda(y) = \tilde{h}_{1,\lambda}(y) + \tilde{h}_{2,\lambda}(y).$$

Let

$$R_k = k^{-\frac{1}{4}} M_k^{\frac{2}{n-2}}. \tag{139}$$

Then we can follow the proof of Lemma 2.1 to show, with the above $\{R_k\}$, that

$$\min_{|y|=r} v_k(y) \leq (1+\epsilon)U(r), \qquad \forall\, 0 < r \leq R_k. \tag{140}$$

Indeed we only need to verify a few things. First we still have (73). As before, we can show that

$$(\Delta_{g_k} - \bar{c} + n(n+2)\xi^{\frac{4}{n-2}})\tilde{h}_{2,\lambda}(y) \le -\frac{Q}{4}\bar{D}_{k,\epsilon}(r),$$

where

$$\bar{D}_{k,\epsilon}(r) := M_k^{-\frac{15-\epsilon}{n-2}} r^{-\frac{3}{2}-\frac{\epsilon}{2}}.$$

We now have the strengthened estimates:

$$|\bar{b}_i(y)| \le C M_k^{-\frac{7-\epsilon}{n-2}} |y|^{\frac{5-\epsilon}{2}}, \quad |\bar{d}_{ij}(y)| \le C M_k^{-\frac{7-\epsilon}{n-2}} |y|^3.$$

With this we can show that

$$|\bar{c}||\tilde{h}_{1,\lambda}(y)| + |\bar{b}_i \partial_i \tilde{h}_{1,\lambda}(y)| + |\bar{d}_{ij} \partial_{ij} \tilde{h}_{1,\lambda}(y)| \le C \bar{D}_{k,\epsilon}(r). \tag{141}$$

Because of (135), we have, instead of (90),

$$|n(n+2)\xi^{\frac{4}{n-2}}(y) - V_\lambda(|y|)| \le C M_k^{-\frac{8}{n-2}} |y|^{\bar{a}}, \qquad \lambda \le |y| \le \frac{1}{4} M_k^{\frac{14-\epsilon}{(n-2)^2}}, \tag{142}$$

which can be shown by following the arguments in the proof of Lemma 2.3 together with the improved bounds

$$|a(y)| + |b(y)| \le M_k^{-\frac{8}{n-2}} |y|^{6-n+\bar{a}}$$

given by (135). Recall that $\bar{a} = \frac{4(n-9)+\sqrt{\epsilon}}{7}$.

With (91) and the improved estimate (142), we can show that

$$\begin{aligned}
&(\Delta + n(n+2)\xi^{\frac{4}{n-2}})\tilde{h}_{1,\lambda} \\
&\le (\Delta + V_\lambda)\tilde{h}_{1,\lambda} + |n(n+2)\xi^{\frac{4}{n-2}} - V_\lambda||\tilde{h}_{1,\lambda}| \\
&\le -\sum_{l=2}^{5} M_k^{-\frac{4+2l}{n-2}} H_{l,\lambda}(r)\tilde{R}^{(l)}(\theta) + O(1)\bar{D}_{k,\epsilon}(r), \qquad \text{in } \hat{O}_\lambda.
\end{aligned}$$

Thus, in view of (141),

$$\begin{aligned}
&(\Delta_{g_k} - \bar{c} + n(n+2)\xi^{\frac{4}{n-2}})\tilde{h}_{1,\lambda} \\
&\le -\sum_{l=2}^{5} M_k^{-\frac{4+2l}{n-2}} H_{l,\lambda}(r)\tilde{R}^{(l)}(\theta) + C\bar{D}_{k,\epsilon}(r), \qquad \text{in } \hat{O}_\lambda.
\end{aligned}$$

Fixing a large Q, we obtain

$$(\Delta_{g_k} - \bar{c} + n(n+2)\xi^{\frac{4}{n-2}})h_\lambda + E_\lambda \le 0 \quad \text{in} \quad \hat{O}_\lambda. \tag{143}$$

With (143), estimate (140) with R_k given by (139) is established, as in the proof of Lemma 2.1.

Once (140) is established, the proof of Lemma 2.4 yields

$$v_k(y) \le CU(y), \qquad |y| \le R_k \tag{144}$$

for the R_k given by (139).

Proof of (18) for $n = 9$. Since estimate (144) holds for $R_k = k^{-\frac{1}{2}} M_k^{\frac{2}{n-2}}$, which violates (30), estimate (18) in dimension $n = 9$ is established. □

Now we make use of the Pohozaev identity (102) (with appropriate R'_k) to prove (19), (20) and (21). Since we have established (18), the assumption in Lemma 2.7 is satisfied for some $\delta > 0$ and the assumption of Lemma 2.8 is satisfied for $7 \le n \le 9$ for $\bar{a} = 0$ and for some $\delta' > 0$ Thus, for some $\delta > 0$, we have , for some $\epsilon > 0$, any $|y| \le \delta M_k^{\frac{2}{n-2}}$,

$$|\nabla^j (v_k - U)(y)| \le C(\delta, \epsilon) M_k^{-2+\frac{2\epsilon}{n-2}} (1+|y|)^{-\epsilon-j}, j=0,1,2, \quad 3\le n \le 6, \tag{145}$$

$$|\nabla^j (v_k - U)(y)| \le C(\delta, \epsilon) M_k^{-\frac{8}{n-2}} (1+|y|)^{6-n-j}, j=0,1,2, \quad 7\le n\le 9. \tag{146}$$

Taking $R'_k = \sigma M_k^{\frac{2}{n-2}}$ in (102), $0 < \sigma < \delta$, using (145) and (29), we have, for some $C_1(n) > 0$,

$$|M_k^2 I_2[v_k] - C_1(n)|W|^2 M_k^{\frac{2(n-6)}{n-2}} \int_{|y|\le \sigma M_k^{\frac{2}{n-2}}} |y|^2 U^2(y)| \le \begin{cases} C\sigma^2, \ n=6, \\ C\sigma, \quad n=7. \end{cases} \tag{147}$$

$$|M_k^2 I_1[v_k]| \le \begin{cases} C\sigma^{2-\epsilon}, & n=6, \\ C\sigma, & n=7. \end{cases} \tag{148}$$

$$|M_k^2 I_3[v_k]| \le C, \quad 6 \le n \le 9. \tag{149}$$

$$\limsup_{k\to\infty} M_k^2 |I_4[v_k]| = 0, \quad 6 \le n \le 9. \tag{150}$$

$$\limsup_{k\to\infty} M_k^2 |I_5[v_k]| = 0, \quad 6 \le n \le 9. \tag{151}$$

Estimate (19) for $n = 6, 7$ follows from (102), (147), (148),(149),(150) and (151). By Lemma 2.9, (122), (136), assumption (104) holds for

$$\beta''_2 = \beta'''_3 = \begin{cases} 0, & n=8, \\ 1-\epsilon, & n=9, \end{cases}$$

$$\beta'''_2 = \begin{cases} 2-\epsilon, & n=8, \\ 3-\epsilon, & n=9, \end{cases}$$

where $\epsilon > 0$ is any number. Let $R'_k = \sigma M_k^{\frac{2}{n-2}}, 0 < \sigma < \delta, \gamma_1 = \frac{8}{n-2}, \gamma_2 = n-6$, $\bar{l} = 5$, we deduce from (108), in view of (123), that

$$|W|^2 M_k^{\frac{2(n-6)}{n-2}} + |R_{abcd,e}|^2 M_k^{\frac{2(n-8)}{n-2}} \int_0^{\sigma M_k^{\frac{2}{n-2}}} r^{3+n} U^2(r) dr = O(1)$$

for $n = 8, 9$. Estimate (19) and (20) for $n = 8, 9$ follows from the above.

Proof of (21). For $n \geq 10$, we claim that we have

$$v_k(y) \leq CU(y), \quad |y| \leq R'_k = M_k^{\frac{16-\epsilon}{(n-2)^2}}. \tag{152}$$

$$\beta''_2 = \beta'''_3 = 2 - \epsilon, \quad \beta'''_2 = 4 - \epsilon. \tag{153}$$

Given this and (117) with $\bar{a} = \frac{n-10+\sqrt{\epsilon}}{2}$, an application of Lemma 2.6 yields

$$c_1(n)|W(x_k)|^2 M_k^{\frac{2(n-6)}{n-2}} + c_2(n)|\nabla W(x_k)|^2 M_k^{\frac{2(n-8)}{n-2}} = O(M_k^2 (R'_k)^{2-n}).$$

Then estimate (21) follows from the above.

To show (152), we need to improve the decay rate of $|W(x_k)|$ and $|\nabla W(x_k)|$ by iteration. Right now for the rate of β''_2, β'''_2 and β'''_3 we have (137), for E_λ we have (138). By exactly the same way of constructing auxiliary functions we have, for $n \geq 10$, that

$$v_k(y) \leq CU(y), \quad |y| \leq R'_k := M_k^{\frac{15-5\epsilon}{(n-2)^2}}. \tag{154}$$

Since ϵ indicates an arbitrarily small positive number, we just replace 5ϵ by ϵ. Then we apply Lemma 2.6 with this R'_k to get

$$c_1(n)|W|^2 M_k^{-\frac{8}{n-2}} + c_2(n)|\nabla W|^2 M_k^{-\frac{12}{n-2}} = O((R'_k)^{2-n}) = O(1) M_k^{-\frac{15-\epsilon}{n-2}}.$$

So after this step we have

$$|W| = O\left(M_k^{-\frac{3.5-\epsilon}{n-2}}\right), \quad |\nabla W| = O\left(M_k^{-\frac{1.5-\epsilon}{n-2}}\right), \quad \text{for any } \epsilon > 0.$$

By this stronger decay rate we can show (154) for $R'_k = M_k^{\frac{15.5-\epsilon}{(n-2)^2}}$. Then Pohozaev identity gives $|W| = O(M_k^{-\frac{3.75-\epsilon}{n-2}})$ and $|\nabla W| = O(M_k^{-\frac{1.75-\epsilon}{n-2}})$. The corresponding rates for $\beta''_2, \beta'''_2, \beta'''_3$ improve too. These new rates lead to (152) except that $R'_k = M_k^{\frac{15.75-\epsilon}{(n-2)^2}}$. Now we see that for any fixed $\epsilon' > 0$, after doing this iteration finite times we can derive (152) for $R'_k = M_k^{\frac{16-\epsilon'}{(n-2)^2}}$. (21) is established. So is Theorem 2.1. □

3. Proof of Theorem 1.3 for $p = \frac{n+2}{n-2}$

In this section we prove Theorem 1.3 for $p = \frac{n+2}{n-2}$ by contradiction argument. By Remark 1.3, we only need to consider $3 \leq n \leq 9$. Suppose the contrary, for each $k = 1, 2, \cdots, C = k$ does not work for some solution u_k of (1) satisfying the hypotheses of the theorem. Clearly $u_k(\bar{P}) \to \infty$. A point Q in Ω is called a blow up point of $\{u_k\}$ if $\{u_k\}$ is not locally bounded in some fixed neighborhood of Q. Let $\mathcal{S}^*$ denote the set of all blow up points of $\{u_k\}$ in Ω.

Lemma 3.1. *There exists some positive constant δ', depending only on (M, g), Ω, $dist_g(\bar{P}, \partial\Omega)$, $\bar{b}$ and ϵ such that*

$$dist_h(Q, Q') \geq \delta', \qquad \forall\, Q, Q' \in \mathcal{S}^* \cap \Omega_\epsilon, Q \neq Q'.$$

Proof of Lemma 3.1. $\forall\, Q \in (\mathcal{S}^* \cap \Omega_\epsilon) \setminus \{\bar{P}\}$, there exists a subsequence of $\{u_k\}$, still denoted as $\{u_k\}$, and a sequence of points $P_k' \to Q$ such that $u_k(P_k') \to \infty$. Let $\delta_k = u_k(P_k')^{-\frac{1}{n-2}}$, then $\delta_k^{\frac{n-2}{2}} u_k(P_k') \to \infty$.

Consider

$$\tilde{u}_k(P) = (\delta_k - dist_g(P, P_k'))^{\frac{n-2}{2}} u_k(P), \qquad P \in B(P_k', \delta_k),$$

and let P_k'' be a maximum point of $\tilde{u}_k$ in the closure of $B(P_k', \delta_k)$. Let $r_k = \frac{1}{2}(\delta_k - dist_g(P_k'', P_k')) \in (0, \frac{\delta_k}{2})$. Then

$$\begin{aligned}\gamma_k^{\frac{2}{n-2}} &:= (r_k)^{\frac{n-2}{2}} u_k(P_k'') = 2^{\frac{2-n}{2}} \tilde{u}_k(P_k'') \geq 2^{\frac{2-n}{2}} \tilde{u}_k(P_k') \\ &= 2^{\frac{2-n}{2}} (\delta_k)^{\frac{n-2}{2}} u_k(P_k') \to \infty,\end{aligned} \tag{155}$$

and

$$(2r_k)^{\frac{n-2}{2}} u_k(P_k'') = \tilde{u}_k(P_k'') \geq \tilde{u}_k(P) \geq (r_k)^{\frac{n-2}{2}} u_k(P), \quad \forall\, P \in B(P_k'', r_k),$$

i.e.

$$\sup_{B(P_k'', r_k)} u_k \leq 2^{\frac{n-2}{2}} u_k(P_k''). \tag{156}$$

Let $\{x^1, \cdots, x^n\}$ be some geodesic normal coordinates centered at P_k'', so $x = 0$ corresponds to P_k''. Consider

$$w_k(y) = \frac{1}{u_k(0)} u_k\left(\frac{y}{u_k(0)^{\frac{2}{n-2}}}\right), \qquad |y| < \Gamma_k := \frac{1}{2} dist_g(Q, \partial\Omega) u_k(0)^{\frac{2}{n-2}}.$$

Then, with g_k denoting the rescaled metric,

$$\begin{aligned} -L_{g_k} w_k &= n(n-2) w_k^{\frac{n+2}{n-2}}, \qquad |y| < \Gamma_k, \\ 1 &= w_k(0) \geq 2^{\frac{2-n}{2}} w_k(y), \qquad |y| < \gamma_k, \\ \Gamma_k &\geq \gamma_k \to \infty. \end{aligned}$$

As usual, $w_k \to w$ in $C^2_{loc}(\mathbb{R}^n)$ with

$$w(y) = \left(\frac{\bar{\lambda}}{1 + \bar{\lambda}^2 |y - \bar{y}|^2}\right)^{\frac{n-2}{2}},$$

and $0 < \bar{\epsilon} \leq \bar{\lambda} \leq 1/\bar{\epsilon}$, $|\bar{y}| \leq 1/\bar{\epsilon}$ for some constant $\bar{\epsilon}$ depending only on n. It follows that $\nabla w_k(y) = 0$ for some $y_k = \bar{y} + \circ(1)$. Let $P_k''' = u_k(P_k'')^{-\frac{2}{n-2}} y_k$, then

$$u_k(P_k'') \leq u_k(P_k''') \leq (2^{\frac{n-2}{2}} + \circ(1)) u_k(P_k''), \qquad \nabla u_k(P_k''') = 0. \tag{157}$$

By the convergence of w_k to w, we have, for some $\hat{\epsilon} > 0$ depending only on n,

$$u_k(P) \geq \hat{\epsilon} u_k(P_k'''), \qquad \forall\, dist_g(P, P_k''') = u_k(P_k''')^{-\frac{2}{n-2}}. \tag{158}$$

Let G denote the Green$'$s function of $-L_g$ on Ω with zero Dirichlet boundary condition. Then G is positive in Ω and

$$\lim_{k\to\infty} \max_{dist_g(P_k''',P)=u_k(P_k''')^{-\frac{2}{n-2}}} \left| G(P_k''', P) dist_g(P_k''', P)^{n-2} - \frac{1}{(n-2)\omega_n} \right| = 0, \tag{159}$$

where ω_n denotes the volume of the standard $(n-1)-$sphere.

By (158) and (159),

$$u_k(P) \geq [(n-2)\omega_n \hat{\epsilon} + \circ(1)] u_k(P_k''')^{-1} G(P_k''', P),$$
$$\forall\, dist_g(P_k''', P) = u_k(P_k''')^{-\frac{2}{n-2}}.$$

Since $L_g u_k \leq 0$, we have, using the maximum principle,

$$u_k(P) \geq [(n-2)\omega_n \hat{\epsilon} + \circ(1)] u_k(P_k''')^{-1} G(P_k''', P) \tag{160}$$

for all $P \in \Omega$ satisfying $\mathrm{dist}_g(P_k''', P) \geq u_k(P_k''')^{-\frac{2}{n-2}}$. By Theorem 2.1, there exist some universal constants $C_0, \delta > 0$, i.e., they depend only on (M, g), $\Omega, dist_g(\bar{P}, \partial\Omega)$, $\bar{b}$ and ϵ, such that

$$u_k(\bar{P}) u_k(P) dist_g(\bar{P}, P)^{n-2} \leq C_0, \qquad \forall\, 0 < dist_g(\bar{P}, P) \leq \delta. \tag{161}$$

Taking a P in $B(\bar{P}, \delta) \setminus B(\bar{P}, \frac{\delta}{2})$ satisfying $dist_g(P_k''', P) \geq \frac{\delta}{9}$, we obtain, using (161) and (160),

$$u_k(\bar{P}) \leq b' u_k(P_k'''),$$

where $b' \geq 1$ is some universal constant. Thus, by (4),

$$\mathrm{sup}_\Omega \mathrm{u_k} \leq \bar{\mathrm{b}}\mathrm{b}' \mathrm{u_k}(\mathrm{P_k'''}).$$

Now, applying again Theorem 2.1, we have, for some universal constants $C_0', \delta' > 0$,

$$u_k(P_k''') u_k(P) dist_g(P_k''', P)^{n-2} \leq C_0', \qquad \forall\, 0 < dist_g(P_k'', P) \leq 2\delta'.$$

It follows, using the fact $P_k''' \to Q$, that

$$\mathcal{S}^* \cap B(Q, \delta') = \{Q\}.$$

Lemma 3.1 is established. □

Let $\mathcal{S}^* \cap \Omega_\epsilon = \{P_1, \cdots, P_m\}$. Because of Lemma 3.1, we may choose local maximum points $\{P_1^{(k)}, \cdots, P_m^{(k)}\}$ of u_k such that $u_k(P_i^{(k)}) \to \infty$ and $P_i^{(k)} \to P_i$. By the arguments in the proof of Lemma 3.1,

$$0 < \liminf_{k\to\infty} \frac{u_k(P_i^{(k)})}{u_k(\bar{P})} \leq \limsup_{k\to\infty} \frac{u_k(P_i^{(k)})}{u_k(\bar{P})} < \infty, \quad 1 \leq i \leq m. \tag{162}$$

By Theorem 2.1, there exist some positive constants $C_0 > 0$ and $0 < \delta < \frac{\epsilon}{2}$ such that

$$u_k(P_i^{(k)})u_k(P)dist_g(P_i^{(k)}, P) \le C_0, \quad \forall\, dist_g(P_i^{(k)}, P) \le \delta,\ 1 \le i \le m. \tag{163}$$

Since $\{u_k\}$ is bounded in $\Omega_{2\epsilon} \setminus \cup_{i=1}^m B(P_i^{(k)}, \frac{\delta}{2})$, we have, by the Harnack inequality and (163),

$$u_k(P) \le C u_k(\bar{P})^{-1}, \qquad \forall\, P \in \Omega_{2\epsilon} \setminus \cup_{i=1}^m B(P_i^{(k)}, \delta). \tag{164}$$

Now applying Theorem 2.1 together with some standard elliptic estimates, we obtain (6), (7), (8), (9) and (10), with u and $\{P_i\}$ replaced respectively by u_k and $\{P_i^{(k)}\}$, for some constant C independent of k. Estimate (11), still with u and $\{P_i\}$ replaced respectively by u_k and $\{P_i^{(k)}\}$, for some constant C independent of k, is simply a rewritten of the estimates on the rescaled v_k obtained in Sect. 2. Theorem 1.3 is established. □

4. Proof of Theorem 1.1 for $p = \frac{n+2}{n-2}$

In this section we establish Theorem 1.1 for $p = \frac{n+2}{n-2}$. The case $1 < 1 + \epsilon \le p \le \frac{n+2}{n-2}$ will be discussed in Sect. 5. Suppose the contrary of (2), let $\{u_k\}$ be a sequence of solutions of (1) with $p = \frac{n+2}{n-2}$ satisfying

$$u_k(P^{(k)}) = \max_M u_k \to \infty, \tag{165}$$

with $P^{(k)} \to \bar{P} \in M$.

By Theorem 1.3, there exist local maximum points $\{P_1^{(k)}, \cdots, P_m^{(k)}\}$ of u_k, such that (6), (7), (8), (9), (10), and (11) hold with P_i replaced by $P_i^{(k)}$. We may assume without loss of generality that $P_1^{(k)} = P^{(k)}$ and $P_i^{(k)} \to \bar{P}_i$ as $k \to \infty$. As explained before we may work in conformal normal coordinates centered at $P_1^{(k)} = P^{(k)}$, and we rescale u_k to v_k as in (27). Multiplying the equation of u_k by $u_k(0)$, we obtain

$$u_k(0)u_k \to G := \sum_{i=1}^m a_i G(\cdot, \bar{P}_i), \qquad \text{in } C^2_{\text{loc}}(M \setminus \{\bar{P}_1, \cdots, \bar{P}_m\}), \tag{166}$$

where $a_i > 0$.

By (8) and (9)

$$W_g(\bar{P}_1) = 0, \qquad \text{if } n \ge 6, \tag{167}$$

and

$$\nabla_g W_g(\bar{P}_1) = 0, \qquad \text{if } n \ge 8.$$

We take a small ball $B_\sigma := B(\bar{P}_1, \sigma)$. The Pohozaev identity is computed over $\bar{B}_\sigma$. The following lemma will be combined with the positive mass theorem to get a contradiction.

Lemma 4.1.

$$\lim_{\sigma\to 0+} B(\sigma, G, \nabla G) \geq 0, \quad 3 \leq n \leq 9.$$

Proof of Lemma 4.1. Since $u_k(0)u_k \to G$ in $C^2_{\text{loc}}(B_\sigma \setminus \{0\})$, $M_k^2 I_5[v_k] \to B(\sigma, G, \nabla G)$ over ∂B_σ. Thus by (102) we only need to show

$$\lim_{\sigma\to 0+} \liminf_{k\to\infty} M_k^2(I_1[v_k] + I_2[v_k] + I_3[v_k] + I_4[v_k]) \geq 0. \tag{168}$$

By (166),

$$\lim_{k\to\infty} M_k^2 I_4[v_k] = -\frac{(n-2)^2}{2}\sigma \lim_{k\to\infty} u_k(0)^2 \int_{|x|=\sigma} u_k^{\frac{2n}{n-2}} = 0, \quad 3 \leq n \leq 9. \tag{169}$$

By (29), (37), (145) and (146)

$$M_k^2|I_1[v_k]| \leq C M_k^2 \int_{|y|\leq \sigma M_k^{\frac{2}{n-2}}} |(\bar{b}_i\partial_i + \bar{d}_{ij}\partial_{ij})(v_k - U)|U \leq C\sigma, \quad 3 \leq n \leq 7. \tag{170}$$

By (44)

$$\begin{aligned} &M_k^2 I_2[v_k] \\ &= -\frac{c(n)}{2} M_k^{2-\frac{4}{n-2}} \int_{|y|\leq \sigma M_k^{\frac{2}{n-2}}} \left\{2R_{,ij} M_k^{-\frac{4}{n-2}} y^i y^j + O\left(|M_k^{-\frac{6}{n-2}}|y|^3\right)\right\} v_k^2 \\ &\geq -C M_k^{2-\frac{8}{n-2}} \int_{|y|\leq \sigma M_k^{\frac{2}{n-2}}} |y|^2 |v_k^2 - U^2| - C M_k^{2-\frac{10}{n-2}} \int_{|y|\leq \sigma M_k^{\frac{2}{n-2}}} |y|^3 U^2 \\ &\geq -C\sigma, \quad 3 \leq n \leq 6. \end{aligned} \tag{171}$$

By (44) and $|W| \to 0$ for $n = 6$,

$$\begin{aligned} &\limsup_{k\to\infty} M_k^2 |I_3[v_k]| \\ &= \limsup_{k\to\infty} \frac{c(n)}{2} M_k^{2-\frac{4}{n-2}} \left(\sigma M_k^{\frac{2}{n-2}}\right) \\ &\times \int_{|y|=\sigma M_k^{\frac{2}{n-2}}} \left\{\frac{1}{2} R_{,ij} M_k^{-\frac{4}{n-2}} y^i y^j + O\left(M_k^{-\frac{6}{n-2}} |y|^3\right)\right\} v_k^2 \\ &= \limsup_{k\to\infty} \left\{\frac{c(n)}{4} \sigma M_k^{\frac{2(n-5)}{n-2}} \int_{|y|=\sigma M_k^{\frac{2}{n-2}}} R_{,ij} M_k^{-\frac{4}{n-2}} y^i y^j U^2 \right. \\ &\quad \left. + C\sigma M_k^{\frac{2(n-5)}{n-2}} \int_{|y|=\sigma M_k^{\frac{2}{n-2}}} |y|^2 |v_k - U| U + C M_k^{\frac{2(n-6)}{n-2}} \int_{|y|=\sigma M_k^{\frac{2}{n-2}}} |y|^3 U^2 \right\} \\ &\leq C\sigma, \quad 3 \leq n \leq 6. \end{aligned} \tag{172}$$

Estimate (168) for $3 \leq n \leq 6$ follows from (169), (170), (171) and (172). To prove (168) for $7 \leq n \leq 9$, we make use of (108) with $\bar{l} = 5$.

For $7 \le n \le 9$, let

$$\beta_2'' = \beta_3''' = \begin{cases} 0, & 7 \le n \le 8, \\ 1, & n = 9, \end{cases}$$
$$\beta_2''' = n - 6, \quad 7 \le n \le 9.$$

By (19), (20) and (122), (105) holds for the above defined β_2'', β_3''' and β_2'''. By (19), (44) and (123),

$$\begin{aligned} &-M_k^2 \frac{c(n)}{2} |S^{n-1}| \sum_{l=2}^{5} (l+2) M_k^{-\frac{4+2l}{n-2}} \bar{R}^{(l)} \int_0^{\sigma M_k^{\frac{2}{n-2}}} r^{l+n-1} U(r)^2 dr \\ &\ge M_k^{\frac{2(n-8)}{n-2}} O(|W|) \int_0^{\sigma M_k^{\frac{2}{n-2}}} r^{3+n} U^2(r) dr \\ &\to 0, \quad \text{as} \quad k \to \infty, \quad \text{for} \quad 7 \le n \le 9. \end{aligned} \tag{173}$$

Multiplying (108) by M_k^2 with $\gamma_1 = \frac{8}{n-2}$, $\gamma_2 = n - 6$, $\bar{l} = 5$ and β_2'',β_3''' given above, we obtain, using also (173) that

$$\liminf_{k \to \infty} M_k^2 (I_5[v_k] + I_2[v_k]) \ge 0. \tag{174}$$

We know

$$\begin{aligned} \lim_{k \to \infty} M_k^2 I_3[v_k] &= \lim_{k \to \infty} \frac{\sigma}{2} c(n) \int_{|x|=\sigma} R(x) (M_k u_k)^2 \\ &= \frac{\sigma}{2} c(n) \int_{|x|=\sigma} R(x) G^2. \end{aligned} \tag{175}$$

For the last term we are in conformal normal coordinates centered at $\bar{P}_1$. It is elementary to see that for some constant $C_4(n) > 0$,

$$G = C_4(n) r^{2-n} + O(r^{6-n}) \tag{176}$$

For $n = 7$, $W(\bar{P}_1) = 0$, so

$$\begin{aligned} &\sigma \int_{|x|=\sigma} R(x) G^2 \\ &= \sigma \int_{|x|=\sigma} \left[\sum_{l=2}^{3} \sum_{|\alpha|=l} \frac{\partial_\alpha R(\bar{P}_1)}{\alpha!} x^\alpha + O(|x|^4) \right] \cdot [C_4(n)^2 r^{4-2n} + O(r^{8-2n})] \\ &= \sigma \int_{|x|=\sigma} O(r^{8-2n}) = O(\sigma), \quad n = 7. \end{aligned} \tag{177}$$

For $n = 8, 9$,

$$|R_{abcd}(\bar{P}_1)| = |R_{abcd,e}(\bar{P}_1)| = 0.$$

Thus by (123)

$$\int_{|x|=\sigma} \sum_{|\alpha|=4} \frac{\partial_\alpha R(\bar{P}_1)}{\alpha!} x^\alpha = |S^{n-1}| \bar{R}^{(4)}(\bar{P}_1) = 0.$$

It follows that

$$\begin{aligned}
&\sigma \int_{|x|=\sigma} R(x) G^2 \\
&= \sigma \int_{|x|=\sigma} \left[\sum_{l=2}^{5} \sum_{|\alpha|=l} \frac{\partial_\alpha R(\bar{P}_1)}{\alpha!} x^\alpha + O(|x|^6) \right] \cdot [C_4(n)^2 r^{4-2n} + O(r^{8-2n})] \\
&= \sigma \int_{|x|=\sigma} O(r^{10-2n}) = O(\sigma), \quad n = 8, 9. \qquad (178)
\end{aligned}$$

By (175), (177), (178) and (174)

$$\lim_{\sigma \to 0} \liminf_{k \to \infty} M_k^2 I_5[v_k] \geq 0, \quad 7 \leq n \leq 9.$$

Lemma 4.1 is established. □

For $n = 3, 4, 5$,

$$G(\cdot, \bar{P}_1) = a_1 (r^{2-n} + A + \text{higher order})$$

where $a_1 > 0$ and A are constants. Thus, for some $\bar{A} \geq A$,

$$G = a_1 (r^{2-n} + \bar{A} + \text{higher order}).$$

For $n = 6, 7$, since $|W(\bar{P}_1)| = 0$, we have

$$G(\cdot, \bar{P}_1) = \begin{cases} a_2 (r^{-4} + \psi(\theta) + O(r \log r), & n = 6, \\ a_3 r^{-5} (1 - a_4 R_{,ij}(\bar{P}_1) x^i x^j r^2] + A + O(r), & n = 7 \end{cases}$$

where $x = r\theta$, $a_2, a_3 > 0$ and A are constants. ψ is a smooth function on θ. Thus

$$G = \begin{cases} a_2 (r^{-4} + \bar{\psi}(\theta) + O(r \log r), & n = 6, \\ a_3 r^{-5} (1 - a_6 R_{,ij}(\bar{P}_1) x^i x^j r^2) + \bar{A} + O(r), & n = 7, \end{cases}$$

where $\bar{A} \geq A$ and $\bar{\psi} \geq \psi$.

A computation yields

$$\lim_{\sigma \to 0} B(\sigma, G, \nabla G) = \begin{cases} -a_5 \bar{A}, & n = 3, 4, 5, 7 \\ -a_6 \int_{S^5} \bar{\psi}(\theta), & n = 6, \end{cases} \qquad (179)$$

where $a_5, a_6 > 0$ are constants.

Consider on M,

$$\hat{g} = G(\cdot, \bar{P}_1)^{\frac{4}{n-2}} g.$$

We make a change of variables $z = |x|^{-2}x$ and write $\hat{g} = \hat{g}_{ij}(z)dz^i dz^j$. Since $W(\bar{P}_1) = 0$ for $n = 6, 7$, it is not difficult to verify that the mass of $(M, \hat{g})$ is given, modulo a positive constant multiple, by, see [13],

$$\text{mass} = \lim_{\rho\to\infty} \int_{|z|=\rho} (\partial_i \hat{g}_{ij} - \partial_j \hat{g}_{ii}) \frac{z^i}{|z|} dz.$$

A computation yields, modulo some positive constant multiple,

$$\text{mass} = \begin{cases} A, & n = 3, 4, 5, 7 \\ \int_{S^5} \psi, & n = 6. \end{cases}$$

By the Positive Mass theorem of Schoen and Yau, $A > 0$ for $n = 3, 4, 5, 7$ and $\int_{S^5} \psi > 0$ for $n = 6$. Since $\bar{A} \geq A$ and $\bar{\psi} \geq \psi$, the above violates, in view of (179) Lemma 4.1. Theorem 1.1 is established. □

Comments on Remark 1.1. For $n = 8, 9$, we not only have $|W(\bar{x})| = 0$, but also $|\nabla W(\bar{x})| = 0$. Since we work in conformal local coordinates, $|\nabla R(\bar{x})| = 0$. Then G can be written as follows:

$$G = \begin{cases} a_7 \left(r^{-6} + \sum_{l=2}^{4} \psi_l(\theta) r^{l-4} + A + O(r \log r) \right), & n = 8, \\ a_8 \left(r^{-7} + \sum_{l=2}^{4} \psi_l(\theta) r^{l-5} + A + O(r) \right), & n = 9. \end{cases}$$

where $a_7, a_8 > 0$ and $\{\psi_l\}$ are spherical harmonics that have integral 0 on S^{n-1}. A contradiction can be obtained similarly from the positive mass theorem for high dimensions. Remark 1.1 is proved. □

5. A discussion of the case $1 < 1 + \epsilon \leq p \leq \frac{n+2}{n-2}$

The proofs for the case $1 < 1 + \epsilon \leq p \leq \frac{n+2}{n-2}$ are modifications of our proofs for $p = \frac{n+2}{n-2}$. Modifications of similar nature can be found in [84, 66] and [73]. We point out some of these modifications for the proof of Theorem 2.1. We still prove it by contradiction argument, so associated with $\{u_k\}$, we now have $\{p_k\} \subset [1+\epsilon, \frac{n+2}{n-2}$. Using standard blow up arguments together with the well known result in [52], we only need to consider the case $\tau_k := \frac{n+2}{n-2} - p_k \to 0$. To avoid introducing new definitions we still let v_k be defined as in (27). Then the equation for v_k is

$$(\Delta + \bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij} - \bar{c}) v_k + n(n-2) M_k^{-\tau_k} v_k^{p_k} = 0, \quad |y| \leq \delta M_k^{\frac{2}{n-2}}. \quad (180)$$

where $\bar{b}_i$, $\bar{d}_{ij}$, $\bar{c}$ are defined as before. Let v_k^λ and w_λ be defined as before, then direct computation shows that

$$(\Delta + \bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij} - \bar{c}) w_\lambda + n(n-2) p_k M_k^{-\tau_k} \xi_k^{p_k - 1} w_\lambda \leq E_\lambda \quad \text{in} \quad \Sigma_\lambda, \quad (181)$$

where E_λ and Σ_λ are defined as before. In this more general context, the estimate of σ_k is related to that of τ_k. First by Proposition 2.1 and assuming $\sigma_k = \circ(1)$, we obtain

$$E_\lambda = \bar{c}(y)U^\lambda(y) - \left(\frac{\lambda}{|y|}\right)^{n+2} \bar{c}(y^\lambda)U(y^\lambda) + \circ(1)M_k^{-\frac{4}{n-2}}|y|^{-n}.$$

Then we have

$$\min_{|y|=L_k} v_k(y) \le (1+\epsilon)U(L_k), \quad L_k = \begin{cases} \delta M_k^{\frac{2}{n-2}}, & n=3,4, \\ M_k^{\frac{4}{(n-2)^2}}, & n \ge 5. \end{cases}$$

where $\epsilon > 0$ is an arbitrarily small positive constant, $\delta > 0$ depends on ϵ. The above inequality leads to

$$v_k(y) \le CU(y), \quad |y| \le \delta_1 L_k \tag{182}$$

for some $\delta_1 > 0$ by the Moser iteration technique used previously. The following Pohozaev identity will be used to obtain the vanishing rate of τ_k:

$$\begin{aligned} &\int_{|x|\le\sigma} \{(-b_i\partial_i u - d_{ij}\partial_{ij}u)(\nabla u \cdot x \\ &\quad + \frac{n-2}{2}u) - \frac{c(n)}{2}u^2(x\cdot\nabla R) - c(n)Ru^2\Big\} \\ &\quad + \frac{\sigma}{2}c(n)\int_{|x|=\sigma} Ru^2 - \frac{n(n-2)}{p+1}\sigma\int_{|x|=\sigma} u^{p+1} \\ &\quad + \left(\frac{n^2(n-2)}{p+1} - \frac{n(n-2)^2}{2}\right)\int_{B_\sigma} u^{p+1} \\ &= B(\sigma, u, \nabla u) \end{aligned}$$

where $B(\sigma, u, \nabla u)$ is defined as before. Then by using (29) to evaluate the Pohozaev identity over $|y| \le L_k$ we obtain $\tau_k = O(M_k^{-2})$ for $n = 3, 4$ and $\tau_k = O(M_k^{-\frac{4}{n-2}})$ for $n \ge 5$. ϵ is arbitrarily small. Note that in the evaluation of Pohozaev identity, say $n \ge 5$, we first have $\tau_k = O(M_k^{-\frac{4}{n-2}+\tau_k})$, this implies $M_k^{\tau_k} = O(1)$, consequently, $\tau_k = O(M_k^{-\frac{4}{n-2}})$.

For $n \ge 5$, by comparing the equations for v_k and U we know

$$\begin{aligned} &(\Delta + \bar{b}_i\partial_i + \bar{d}_{ij}\partial_{ij} - \bar{c})w_k + n(n-2)M_k^{-\tau_k}p_k\xi_k^{p_k-1}w_k \\ &= \bar{c}U + n(n-2)\left(U^{\frac{n+2}{n-2}} - M_k^{-\tau_k}U^{p_k}\right), \end{aligned} \tag{183}$$

where $w_k = v_k - U$. It is clear that for all $\epsilon > 0$,

$$\begin{aligned} &\bar{c}U + n(n-2)\left(U^{\frac{n+2}{n-2}} - M_k^{-\tau_k}U^{p_k}\right) \\ &= O\left(M_k^{-\frac{8}{n-2}}\right)(1+r)^{4-n} + O\left(M_k^{-\frac{4}{n-2}+\epsilon}\right)(1+r)^{-2-n}. \end{aligned}$$

Then by the Proof of Proposition 2.2 we obtain

$$\sigma_k \le C M_k^{-\frac{4}{n-2}+\epsilon}, \quad \forall \epsilon > 0.$$

Once we have this new estimate of σ_k we can improve the estimate of E_λ to

$$E_\lambda = O(1) M_k^{-\frac{8}{n-2}} |y|^{4-n} + O(1) M_k^{-\frac{8-\epsilon}{n-2}} |y|^{-n}.$$

So we can prove (182) for a new L_k, which is

$$L_k := \begin{cases} \delta M_k^{\frac{2}{n-2}} & n = 5, \\ M_k^{\frac{8-\epsilon}{(n-2)^2}} & n \ge 6. \end{cases}$$

where $\epsilon > 0$ is arbitrarily small.

Now we want to use the new estimate of σ_k to get a new estimate of τ_k. To do this, first for $n = 5$ we compare w_k with

$$f(y) = Q M_k^{-2+\frac{2\epsilon}{3}} |y|^{-\epsilon} + Q M_k^{-\frac{4-\epsilon}{3}} |y|^{\epsilon-3}, \quad R < |y| < \delta M_k^{\frac{2}{3}}$$

where $\epsilon > 0$ is small and $Q(\epsilon) > 1$ is large, $R >> 1$ is to make the maximum principle possible. Then from the maximum principle we get

$$|v_k(y) - U(y)| \le f(y), \quad \text{for} \quad R \le |y| \le \delta M_k^{\frac{2}{3}}.$$

Estimates for $|\nabla^j (v_k - U)|, j = 1, 2$ can be obtained similarly. Consequently we get

$$v_k = U + O\left(M_k^{-2+\frac{2\epsilon}{3}}\right)(1+|y|)^{-\epsilon} + O\left(M_k^{-\frac{4-\epsilon}{3}}\right)(1+|y|)^{\epsilon-3}, \quad |y| < \delta M_k^{\frac{2}{n-2}}.$$

Then we apply the Pohozaev identity over $|y| \le \delta M_k^{\frac{2}{3}}$ to get $\tau_k = O(M_k^{-2})$.

For $n \ge 6$, we follow the same procedure except that we use

$$f(y) = Q M_k^{-\frac{8-2\epsilon}{n-2}} |y|^{-\frac{\epsilon}{10}} + Q M_k^{-\frac{4-\epsilon}{n-2}} |y|^{2-n+\epsilon}, \quad R \le |y| \le M_k^{\frac{8-\sqrt{\epsilon}}{(n-2)^2}}.$$

Then the expansion for v_k becomes

$$\begin{aligned} v_k = U &+ O\left(M_k^{-\frac{8-2\epsilon}{n-2}}\right)(1+|y|)^{-\frac{\epsilon}{10}} \\ &+ O\left(M_k^{-\frac{4-\epsilon}{n-2}}\right)(1+|y|)^{2-n+\epsilon}, \quad |y| < M_k^{\frac{8-\sqrt{\epsilon}}{(n-2)^2}}. \end{aligned}$$

From the Pohozaev identity over $|y| \le M_k^{\frac{8-\sqrt{\epsilon}}{(n-2)^2}}$ we have $\tau_k = O(M_k^{-\frac{8-2\sqrt{\epsilon}}{n-2}})$, consequently we obtain, by (183), that $\sigma_k = O(M_k^{-\frac{8-3\sqrt{\epsilon}}{n-2}})$. From now on we just replace $3\sqrt{\epsilon}$ by ϵ for convenience. Now the right hand side of (183) becomes

$$\bar{c} U + n(n-2)\left(U^{\frac{n+2}{n-2}} - M_k^{-\tau_k} U^{p_k}\right)$$

$$= O\left(M_k^{-\frac{8}{n-2}}\right)(1+r)^{4-n} + O\left(M_k^{-\frac{8-\epsilon}{n-2}}\right)(1+r)^{-2-n}$$

Also the new estimate of σ_k for $n \geq 6$ leads to a new estimate of E_λ :

$$E_\lambda = \bar{c}(y)U^\lambda(y) - (\frac{\lambda}{|y|})^{n+2}\bar{c}(y^\lambda)U(y^\lambda) + O(1)M_k^{-\frac{12-\epsilon}{n-2}}|y|^{-n}.$$

Then we obtain (182) again except that

$$L_k = \begin{cases} \delta M_k^{\frac{2}{n-2}}, & n = 6, 7 \\ M_k^{\frac{12-\epsilon}{(n-2)^2}}, & n \geq 8. \end{cases}$$

For $n = 7$, we improve the estimate of $v_k - U$ by using

$$f(y) = QM_k^{-\frac{8}{n-2}}|y|^{6-n} + QM_k^{-\frac{8-\epsilon}{n-2}}|y|^{2-n+\epsilon}, \quad R \leq |y| \leq \delta M_k^{\frac{2}{n-2}}$$

Then the Pohozaev identity gives $\tau_k = O(M_k^{-2})$. Note that in the computation, the term that contributes Weyl tensor and the term that contributes τ_k are both positive, therefore the existence of W does not affect the rate of τ_k. After this we improve the rate of σ_k to $O(M_k^{-\frac{8}{n-2}})$.

For $n \geq 8$, to control $v_k - U$ over $L_k = M_k^{\frac{12-\epsilon}{(n-2)^2}}$ we use function

$$f(y) = QM_k^{-\frac{8}{n-2}}|y|^{6-n+\bar{a}} + QM_k^{-\frac{8-\epsilon}{n-2}}|y|^{2-n+\epsilon}$$

where $\bar{a} = \frac{2}{3}(n-8) + O(1)\epsilon$. Then from Pohozaev identity we get $\tau_k = O(M_k^{-\frac{12-\epsilon}{n-2}})$. Then $\sigma_k = O(M_k^{-\frac{8}{n-2}})$ for $n \geq 8$. Once we have this rate of τ_k, it does not affect the estimate of σ_k any more (for $n = 8, 9$). Then in the evaluation of the Pohozaev identity, the term that contributes τ_k has the same sign as the term that contributes all the curvature tensors. So τ_k does not affect the vanishing rate of the Weyl or ∇R_{abcd} for $n = 8, 9$. We get the estimate of τ_k when we get the vanishing rate of curvature tensors. Eventually we have $\tau_k = O(M_k^{-2})$ for $n = 8, 9$ and $\tau_k = O(M_k^{-\frac{16-\epsilon}{n-2}})$ for $n \geq 10$. Other estimates on v_k (for $n = 8, 9$) are just the same as those in the special case $p_k \equiv \frac{n+2}{n-2}$. The proofs of Theorem 1.1 and Theorem 1.3 can proceed as the special case $p_k \equiv \frac{n+2}{n-2}$.

6. Appendix A: A useful function and its estimate

For $n = 3, 4, 5, \cdots$, let $A > 2$, $a, b, \mu > 0$, and $0 \leq \gamma \leq n-2$ be constants, we consider functions V and H satisfying

$$-\mu^{-1}(1+r)^{-2-\mu} \leq V(r) \leq n(n+2)U(r)^{\frac{4}{n-2}} + \frac{a}{2}r^{-4}, \quad 1 \leq r \leq A, \tag{184}$$

$$|V'(r)| \leq \mu^{-1}r^{-3}, \quad 1 \leq r \leq A, \tag{185}$$

$$0 \leq H(r) \leq br^{\gamma-n}, \quad 1 \leq r \leq A, \tag{186}$$

and

$$|H'(r)| \le br^{\gamma-n-1}, \quad 1 \le r \le A, \tag{187}$$

where $U(r) = (1+r^2)^{\frac{2-n}{2}}$.

Proposition 6.1. *For $n = 3, 4, 5, \cdots$, let $A > 2$, $a, b > 0$ and $0 \le \gamma \le n-2$ be constants and let V satisfy (184)-(185) and H satisfy (186)-(187). Then there exists a unique solution of*

$$\begin{cases} \eta''(r) + \dfrac{n-1}{r}\eta'(r) + \Big(V(r) - \dfrac{a}{r^2}\Big)\eta(r) = -H(r), \quad 1 < r < A, \\ \eta(1) = \eta(A) = 0. \end{cases} \tag{188}$$

Moreover

$$0 \le \eta(r) \le Cr^{\gamma+2-n}, \quad 1 < r < A, \tag{189}$$

and

$$|\eta'(r)| \le Cr^{\gamma+1-n}, \quad |\eta''(r)| \le Cr^{\gamma-n}, \quad 1 \le r \le A, \tag{190}$$

where $C > 0$ depends only on n, a, b, μ and γ.

Thinking of $r = |x|$, $x \in \mathbb{R}^n$, equation (188) takes the form

$$\begin{cases} \Delta\eta + \Big(V(|x|) - \dfrac{a}{|x|^2}\Big)\eta = -H(|x|), & \text{in } B_A \setminus B_1, \\ \eta = 0, & \text{on } \partial(B_A \setminus B_1). \end{cases} \tag{191}$$

Lemma 6.1. *For $A > 2$, let $\lambda_1 = \lambda_1(n, A)$ be the first eigenvalue of $-\Delta - n(n+2)U^{\frac{4}{n-2}}$ on $B_A \setminus B_1$ with respect to zero Dirichlet boundary value. Then $\lambda_1 > 0$.*

Proof of Lemma 6.1. Since $U_t(x) := t^{\frac{n-2}{2}}U(tx)$ satisfies, for all $t > 0$,

$$-\Delta U_t - n(n-2)U_t^{\frac{n+2}{n-2}} = 0, \qquad \text{in } \mathbb{R}^n,$$

$\frac{d}{dt}U_t|_{t=1}$ satisfies the linearized equation, i.e.,

$$-\Delta\varphi - n(n+2)U^{\frac{4}{n-2}}\varphi = 0, \qquad \text{in } \mathbb{R}^n,$$

where $\varphi(r) := \frac{r^2-1}{r^2+1}U(r)$ is positive in $1 < r < \infty$.

Let $\bar\eta$ be a positive eigenfunction with respect to λ_1, so

$$-\Delta\bar\eta - n(n+2)U^{\frac{4}{n-2}}\bar\eta = \lambda_1\bar\eta, \qquad 1 < |x| < A,$$

and $\bar\eta = 0$ on $\partial(B_A \setminus B_1)$. Multiplying the above equation of $\bar\eta$ by φ and integrating by parts lead to

$$\lambda_1 \int_{B_A\setminus B_1} \bar\eta\varphi > \int_{B_A\setminus B_1} \Big[-\bar\eta\Delta\varphi - n(n+2)U^{\frac{4}{n-2}}\bar\eta\varphi\Big] = 0.$$

Lemma 6.1 is established. □

Corollary 6.1. *Under the hypotheses of Proposition 6.1, equation (188) has a unique solution η, which is non-negative.*

Proof of Corollary 6.1. By (184),

$$V(|x|) - \frac{a}{|x|^2} \leq n(n+2)U(x)^{\frac{4}{n-2}} - \frac{a}{2|x|^2}, \qquad 1 < |x| < A. \tag{192}$$

Corollary 6.1 follows from Lemma 6.1 and standard elliptic theories.

Proof of Proposition 6.1. Fix an $R > 2$, depending only on n and a, such that

$$n(n+2)U(r)^{\frac{4}{n-2}} + \frac{a}{2}r^{-4} - ar^{-2} \leq -\frac{a}{2}r^{-2}, \qquad \forall\, r \geq R. \tag{193}$$

If $A \leq 3R$, we know from Lemma 6.1 that the first eigenvalue λ_1 of $-\Delta-(V-\frac{a}{r^2})$ on $B_A \setminus B_1$, with respect to zero Dirichlet boundary value, has a positive lower bound which depends only on n and R. Thus the L^2 norm of η on $B_A \setminus B_1$ is under control. By standard elliptic estimates, the L^∞ norm of η on $B_A \setminus B_1$ is also under control. In the following we assume that $A > 3R$. By (184) and (193),

$$V(r) - ar^{-2} \leq -\frac{a}{2}r^{-2}, \qquad \forall R \leq r \leq A. \tag{194}$$

Since $0 < \gamma \leq n-2$, we can pick some constant $C > 1$, depending only on γ, n, b, a such that $w(r) := Cr^{\gamma+2-n}$ satisfies

$$[\Delta + (V - ar^{-2})]w \leq -br^{\gamma-n} \leq -H(r), \qquad \forall\, R \leq r \leq A. \tag{195}$$

Fix some smooth function $f(r)$, depending only on the usual parameters (i.e. n, a, b, μ and γ), satisfying

$$f(r) \equiv 0, \qquad 2R < r < \infty,$$
$$f(r) \leq -[\Delta + (V - ar^{-2})]w - H(r), \qquad 1 \leq r \leq R,$$

and

$$f(r) \leq 0, \qquad \forall\, 1 \leq r < \infty.$$

To prove (189), we only need to find some non-negative function $w_1(r)$ in $1 \leq r < \infty$ satisfying

$$w_1(r) \leq C_1 r^{2+\gamma-n}, \qquad 1 \leq r < \infty, \tag{196}$$

and

$$[\Delta + (V - ar^{-2})]w_1(r) \leq f(r), \qquad \forall\, 1 \leq r \leq A, \tag{197}$$

where $C_1 > 0$ is some constant depending only on the usual parameters.

Indeed, let w_1 be as above, then

$$[\Delta + (V - ar^{-2})](w + w_1) \leq -H_1(r), \qquad 1 \leq r \leq A,$$

and therefore, in view of (188),

$$[\Delta + (V - ar^{-2})](w + w_1 - \eta) \leq 0, \qquad \text{on } B_A \setminus B_1.$$

We also have, using the non-negativity of w and w_1,

$$w + w_1 - \eta \geq 0, \qquad \text{on } \partial(B_A \setminus B_1).$$

Because of Lemma 6.1 and (192), we may apply the maximum principle to obtain

$$w + w_1 - \eta \geq 0, \qquad 1 \leq r \leq A.$$

This gives the desired estimate (189).

Now we construct such a w_1. Consider

$$\begin{cases} \Delta \tilde{w}_1(y) + \left[n(n+2)U(y)^{\frac{4}{n-2}} - \frac{a}{2}\right] \tilde{w}_1(y) = |y|^{-n-2} f\left(\frac{y}{|y|^2}\right), & |y| < 1, \\ \tilde{w}_1(y) = 0, & |y| = 1. \end{cases} \tag{198}$$

We know from the proof of Lemma 6.1 that $\varphi(r) := \frac{r^2-1}{r^2+1} U(r)$ satisfies

$$\begin{aligned} &[\Delta + n(n+2)U(y)^{\frac{4}{n-2}}]\varphi(y) = 0, \qquad |y| < 1, \\ &\varphi(r) < 0, \quad \forall\, 0 \leq r < 1, \qquad \text{and} \qquad \varphi(1) = 0. \end{aligned}$$

It follows that the first eigenvalue, with respect to zero Dirichlet boundary data, of $-\Delta - n(n+2)U^{\frac{4}{n-2}}$ on B_1 is zero. So the first eigenvalue of $-\Delta - n(n+2)U^{\frac{4}{n-2}} + \frac{a}{2}$ on B_1 is equal to $\frac{a}{2} > 0$. We also know that $|y|^{-n-2} f(\frac{y}{|y|^2})$ is non-positive for all $|y| \leq 1$ and is equal to zero for $|y| \leq \frac{1}{2R}$. By standard elliptic theories, (198) has a unique radial solution $\tilde{w}_1$ satisfying

$$0 \leq \tilde{w}_1(y) \leq C, \qquad \forall\, |y| \leq 1,$$

where C is some positive constant depending only on the usual parameters.

Let

$$w_1(x) = \frac{1}{|x|^{n-2}} \tilde{w}_1\left(\frac{x}{|x|^2}\right), \qquad |x| \geq 1.$$

Then, because of (198), w_1 satisfies

$$\left\{\Delta + \left[n(n+2)U(r)^{\frac{4}{n-2}} - \frac{a}{2r^4}\right]\right\} w_1(r) = f, \quad r > 1,$$

and

$$0 \leq w_1(r) \leq C r^{2-n}, \qquad \forall r \geq 1.$$

Finally, using (192), we have, for $1 \leq r \leq A$,

$$\begin{aligned} [\Delta + (V(r) - ar^{-2})]w_1 &\leq \left[\Delta + \left(n(n+2)U(r)^{4/(n-2)} - \frac{a}{2r^2}\right)\right] w_1 \\ &\leq \left[\Delta + \left(n(n+2)U(r)^{4/(n-2)} - \frac{a}{2r^4}\right)\right] w_1 = f. \end{aligned}$$

Thus, the w_1 has the desired properties, and (189) is established. Using (185), (187), (189) and (188), estimate (190) follows from standard elliptic theories with the help of a standard scaling argument. Proposition 6.1 is established. $\square$

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