The Distance Function to the Boundary, Finsler Geometry, and the Singular Set of Viscosity Solutions of Some Hamilton-Jacobi Equations

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Dedicated in memory to Jacques Louis Lions

1 Introduction

This paper is concerned with viscosity solutions of Hamilton-Jacobi (H-J) equations of the form

$$H(x, u, \nabla u) = 1 \quad \text{in } \Omega\,, \tag{1.1}$$

a $C^{2,1}$ bounded domain (connected open set) in $\mathbb{R}^n$, and

$$H(x, t, p) \in C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)\,. \tag{1.2}$$

We consider positive solutions u satisfying

$$u\big|_{\partial\Omega} = 0\,. \tag{1.3}$$

For definitions and properties of viscosity solutions we refer to [5, 12]. Our main results are for special $H = H(x, p)$, i.e.,

$$H(x, \nabla u) = 1 \quad \text{in } \Omega\,. \tag{1.4}$$

Under suitable conditions we show that the $(n-1)$–dimensional Hausdorff measure of the singular set of solutions (the complement of the open set where $u \in C^{1,1}$) is finite.

In addition, we prove the corresponding result for $H(x, t, p)$ but under very special conditions. See Theorem 10.5 and its simple consequences, Propositions 1.8, 1.10, and 1.11.

We were brought to the problem by first studying the singular set of the distance function to the boundary of Ω. This set is sometimes called the ridge of Ω, or medial axes. Our interest in the set arises in connection with nonlinear elliptic boundary value problems [11]. We first describe this set Σ.

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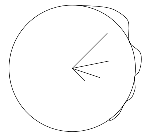

FIGURE 1.1

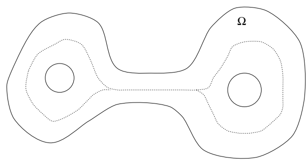

FIGURE 1.2

Let G be the largest open subset of Ω such that every point x in G has a unique closest point on $\partial\Omega$. The set Σ is defined to be

$$\Sigma = \Omega \setminus G\,.$$

In G, the distance u to the boundary is smooth (i.e., of class $C^{1,1}$, or C^∞ in case $\partial\Omega$ is in C^∞).

In case Ω is a ball, Σ is just one point, its center. If we perturb the boundary of the ball by many small (but C^∞) perturbations as in Figure 1.1, we see that the set Σ consists of segments coming from the origin.

Another typical situation, with Ω not simply connected, is shown in Figure 1.2. In this case Σ is the dotted curve.

It is well known that Σ is always a connected set. See, for instance, W. D. Evans and D. J. Harris [9], which treats general domains and a slightly different notion of ridge, which is not always closed. In Appendix C we will include a fairly short proof that it is arcwise connected.

Concerning the set Σ we proved that the $(n-1)$–dimensional Hausdorff measure

$$H^{n-1}(\Sigma) \quad \text{is finite}\,.$$

This is an immediate consequence of the following result:

THEOREM 1.1 *From every point y on $\partial\Omega$, move along the inner normal until first hitting a point $m(y)$ on Σ. The length $\bar{s}(y)$ of the resulting segment is Lipschitz-continuous in y.*

Remark 1.2. The condition $C^{2,1}$ is sharp. In Appendix A we present a convex domain Ω in the plane with $C^{2,\alpha}$ boundary, $0 < \alpha < 1$, for which the conclusion of Theorem 1.1 does not hold.

If the domain Ω is unbounded, the set Σ may be empty, for example, if Ω is the half-space $x_n > 0$. However, the following form of Theorem 1.1 holds for general Ω with G and Σ defined as before.

THEOREM 1.1′ *For $y \in \partial\Omega$ let $\bar{s}(y)$ be defined as in Theorem* 1.1 (*it may be infinite*). *For any $N > 0$,* $\min(N, \bar{s}(y))$ *is Lipschitz-continuous in y in any compact subset of $\partial\Omega$.*

After proving these theorems, we extend them to complete Riemannian manifold (M^n, g).

THEOREM 1.1″ *For any domain Ω in M, with $\partial\Omega$ locally in $C^{2,1}$, the conclusion of Theorem* 1.1′ *holds. Here $\bar{s}(y)$ represents the length of the geodesic going from a point y on $\partial\Omega$, normal to $\partial\Omega$, until it hits Σ.*

COROLLARY 1.3 *For Ω as above in (M^n, g), $H^{n-1}(\Sigma \cap B) < \infty$ for any bounded set B in M.*

We then discovered that Theorem 1.1″ had already been proven by J.-I. Itoh and M. Tanaka [10] in 2001. In fact, their domain Ω may be the complement of a smooth submanifold X of M of any dimension. However, the result for such X follows from the case $\dim X = n - 1$ by taking for Ω the exterior of a tubular neighborhood of X.

CUT POINT In Theorem 1.1″ we considered a geodesic from a point y going into Ω in the normal direction until it first hits a point $x = m(y)$ in Σ. The point $m(y)$ is called the *cut point* of y on $\partial\Omega$, meaning that if we go beyond x on the geodesic to any point x', then x' has a closer point on $\partial\Omega$ than y. The collection of these points $m(y)$ on Σ for all y (namely Σ itself) is called the *cut locus* of $\partial\Omega$. That Σ is the set of cut points is established in Section 4; see Corollary 4.11.

Recall the analogous notion of *conjugate* point of y: This is the first point $\bar{x}$ on the normal geodesic such that any point x'' on the geodesic beyond $\bar{x}$ has, in any neighborhood of the normal geodesic, a point on $\partial\Omega$ in the neighborhood that can be connected to it by a path in the neighborhood with length shorter than the arc length of the normal geodesic from y to it.

Remark 1.4. In case Ω is a domain in $\mathbb{R}^n$, the distance from a point y to the conjugate point is the smallest of the principal radii of curvature of $\partial\Omega$ at y.

In Corollary 4.15 we give an analogous characterization for Finsler spaces. It says that $m(y)$ is a conjugate point if and only if the (Finsler) sphere about $m(y)$, of radius $s(y)$, has second-order contact with the boundary of Ω at y in some direction. This result is not used in the paper.

Remark 1.4 will be used in the construction given in Appendix A.

Our proof of Theorem 1.1″ is different from that of [10]. Some time ago Walter Craig suggested that we might prove an analogue of Corollary 1.3 for viscosity solutions of Hamilton-Jacobi equations, and we express our thanks to him. The extension is what we do in the paper. As we learned to our surprise, for the problem (1.4) and (1.3) it involves an extension of Theorem 1.1″ to Finsler geometry, and we now proceed to describe this.

1.1 Hamilton-Jacobi Equation

Consider the problem

$$H(x, \nabla u) = 1 \quad \text{in } \Omega\,, \tag{1.5}$$

$$u\big|_{\partial\Omega} = 0\,. \tag{1.6}$$

Here $H(x, p) \in C^\infty(\overline{\Omega} \times \mathbb{R}^n)$. We assume that for every $x \in \overline{\Omega}$ the set

$$V_x = \{p \in \mathbb{R}^n : H(x, p) < 1\} \tag{1.7}$$

is a bounded convex surface containing 0, with smooth, strictly convex boundary S_x (i.e., having positive principal curvatures). For some $r > 0$ we assume that

$$B_r(0) \subset V_x \quad \forall x \in \overline{\Omega}\,. \tag{1.8}$$

What is important are the sets V_x rather than the particular function $H(x, p)$.

Theorem 5.3 of [12] gives an explicit formula for the viscosity solution u of (1.5)–(1.6). It involves, for each $x \in \overline{\Omega}$, the support function $\varphi(x; \cdot\,)$ of S_x, i.e.,

$$\varphi(x; v) = \max\{v \cdot p : p \in S_x\}\,, \qquad v \in \mathbb{R}^n\,.$$

The function φ is in $C^\infty(\overline{\Omega} \times (\mathbb{R}^n \setminus \{0\}))$ and is positive homogeneous of degree 1 in v. It is also a convex function of v; in fact, for each $x \in \overline{\Omega}$, the set

$$\{v \in \mathbb{R}^n : \varphi(x; v) = 1\}$$

is a smooth convex hypersurface (with positive principal curvatures) containing the origin in its interior. Furthermore, φ satisfies the triangle inequality in v. Thus for any curve $\xi(t)$, $0 < t < T$, in $\overline{\Omega}$,

$$\varphi(\xi(t); \dot{\xi}(t))dt$$

is a Finsler metric. The length of the curve, if $\dot{\xi} \in L^1$, is

$$\int_0^T \varphi(\xi(t); \dot{\xi}(t))dt\,.$$

Because of the homogeneity it is independent of its t-parametrization.

Note that the length of the curve depends on the direction in which it is transversed, so we talk of its length from $\xi(0)$ *to* $\xi(T)$.

For any $x, y \in \overline{\Omega}$ we denote by $L(x, y)$ the infimum of length of curves in $\overline{\Omega}$ going from y *to* x,

$$L(x, y) = \inf\left\{ \int_0^1 \varphi(\xi(t); \dot{\xi}(t))dt : \xi(t) \in \overline{\Omega} \text{ for } 0 \le t \le 1, \right.$$
$$\left. \dot{\xi} \in L^\infty(0, 1) \text{ and } \xi(0) = y, \xi(1) = x \right\}.$$

Then for $x \in \overline{\Omega}$,

$$u(x) := \inf_{y \in \partial\Omega} L(x, y)$$

is the viscosity solution of (1.5)–(1.6), $u > 0$ in Ω and $u \in W^{1,\infty}$; see [12, theorem 5.3].

Thus the solution $u(x)$ is the distance from $\partial\Omega$ to x measured in the Finsler metric. What we do is to extend Theorem 1.1$''$ to a general Finsler manifold.

1.2

Consider an n-dimensional smooth manifold M with a complete, smooth Finsler metric. Let Ω be a domain in M with

$$\partial\Omega \in C^{2,1}_{\mathrm{loc}} .$$

Let G be the largest open subset of Ω such that for every x in G there is a unique closest point y on $\partial\Omega$ to x, where we measure lengths of curves in $\overline{\Omega}$ going from $\partial\Omega$ to x in the Finsler metric. It is easy to see that the distance function from $\partial\Omega$ to x is in $C^{1,1}(G \cup \partial\Omega)$. Moreover, u belongs to $C^{k-1,\alpha}(G \cup \partial\Omega)$ if $\partial\Omega$ is $C^{k,\alpha}$ for $k \geq 3$ and $0 < \alpha \leq 1$. But of course it never belongs to C^1.

Set

$$\Sigma = \Omega \setminus G .$$

As for Riemannian manifolds, Σ is called the cut locus of $\partial\Omega$. The cut point of y on $\partial\Omega$ is defined as in the Riemannian case, and the collection of $m(y)$ for all $y \in \partial\Omega$ is Σ itself. The cut point of y on $\partial\Omega$ is usually defined differently as follows: We consider the geodesic from y going into Ω in the "normal" direction with unit speed, denoted as $\xi(y, s)$. The set of $s > 0$ satisfying

$$\mathrm{dist}(\partial\Omega \text{ to } \xi(y, s)) = s$$

is either $(0, \infty)$ or $(0, \tilde{s}(y)]$ for some $0 < \tilde{s}(y) < \infty$. In the latter case, $\tilde{m}(y) := \xi(s, \tilde{s}(y))$ is the cut point of y on $\partial\Omega$, and the collection of $\tilde{m}(y)$ for all $y \in \partial\Omega$, denoted as $\tilde{\Sigma}$, is called the cut locus of $\partial\Omega$. The two definitions are the same, i.e., $\tilde{m}(y) = m(y)$ for all $y \in \partial\Omega$, and $\tilde{\Sigma} = \Sigma$. This will be proven in Section 4.

The geodesic equations for the Finsler metric $\varphi(\xi; v)$ are

$$\varphi_{\xi^i}(\xi(t); \dot{\xi}(t)) = \frac{d}{dt}\varphi_{v^i}(\xi(t); \dot{\xi}(t)) , \quad i = 1, \ldots, n. \tag{1.9}$$

A C^1 solution with nonvanishing $\dot{\xi}$ is called a geodesic. A geodesic locally minimizes

$$\int_a^b \varphi(\xi(t); \dot{\xi}(t))dt .$$

From any point y on $\partial\Omega$ there is a unique geodesic, in the metric, going into Ω, "normally" at $\partial\Omega$. This means that for a point on the geodesic close to y, y is the unique closest point on $\partial\Omega$ to it. This will be explained further below (see Lemma 2.2).

THEOREM 1.5 *Let $\ell(y)$ denote the length of the "normal" geodesic from y until it first hits a point $m(y) \in \Sigma$. So $\Sigma = m(\partial\Omega)$. Then, for any $N > 0$,*

$$\min(N, \ell(y))$$

is Lipschitz-continuous in y on any compact subset of $\partial\Omega$.

COROLLARY 1.6 *$H^{n-1}(\Sigma \cap B) < \infty$ for any bounded set B.*

Returning to our viscosity solution of (1.5)–(1.6), it means that for its singular set Σ,

$$H^{n-1}(\Sigma) < \infty .$$

1.3 Remarks on the General H-J Equations (1.1) in a Bounded Ω

Many authors have studied boundary value problems

$$u(x) = u_0(x) \quad \text{on } \partial\Omega ;$$

see, for example, papers below and references therein. Usually it is considered that H is convex in p. Sometimes it is also assumed that H is convex in (t, p). And it is sometimes assumed that H is nondecreasing in t; this is usually used in proving uniqueness of the viscosity solution. Adimurthi and Gowda (see [1, 2] and references therein) do not require H nondecreasing in t. In [12, theorem 5.5], positive viscosity solutions of (1.1) and (1.3) are obtained assuming H is convex in (t, p) and nondecreasing in t (and some additional conditions).

There are also a number of papers that study the singular set of solutions, which go back at least to [15] by Ting. A. C. Mennucci [14] studied the singular set for viscosity (and what he calls "minimal") solutions u for equation (1.4) on a smooth n-dimensional manifold, with the value of u prescribed to be u_0 on a closed subset K of M. K and u_0 are usually assumed to be in C^2. Among other things, he gives a very fine characterization of the set A where the solution u is not differentiable; namely, A is the union of a countable number of smooth $(n-1)$–dimensional manifolds with a set having zero $(n-1)$–dimensional Hausdorff measure. Such sets are called "rectifiable." This result does not contain ours, since it does not show that the total $(n-1)$–dimensional measure is finite. In an earlier paper [13], he and C. Mantegazza studied the distance function to the boundary and showed that the singular set is "rectifiable" if K is in C^2. In addition, they presented an example of a closed convex curve K in $\mathbb{R}^2$, K of class $C^{1,1}$, such that the singular set has positive Lebesgue measure. These papers contain many more excellent results, including some for the initial value problem, as well as many references to earlier work. P. T. Chruściel, J. H. G. Fu, G. J. Galloway and R. Howard in [8], working with "horizons" in relativity theory and applications to Riemannian geometry, obtained results closely related to those of Mennucci [14]. They all make use of deep results of G. Alberti [3] on singular sets of convex functions.

In connection with Theorem 1.1, A. Cellina in [6] treated a variational problem of the form $\min \int_\Omega h(|\nabla u| + u)$. In the paper he considered normals from points y

on $\partial\Omega$ inside Ω, for a distance t and showed regularity of the map, in its dependence on y for $t < \bar{s}(y)$. In [7] he and Perrotta treated more general variational problems. In particular, they considered functions u with $|\nabla u| = 1$ and $u = u_0$, given on $\partial\Omega$.

1.4

We wish to stress that what is important are the sets

$$V_x = \{(t, p) \in \mathbb{R}^{n+1} : H(x, t, p) < 1\} \quad \forall x \in \overline{\Omega}, \tag{1.10}$$

and

$$S_x = \partial V_x = \{(t, p) : H(x, t, p) = 1\} \quad \forall x \in \overline{\Omega}.$$

For example, consider the following:

SITUATION A Suppose H is smooth in a neighborhood of $\bigcup_x S_x$ and that $\forall x \in \overline{\Omega}$, V_x is convex, and S_x is a smooth, strictly convex hypersurface with positive principal curvatures, and that

$$\operatorname{dist}(0, S_x) \geq r_0 > 0 \quad \forall x \in \overline{\Omega}. \tag{1.11}$$

Suppose, furthermore, that each V_x lies in a fixed downward cone: for some $k, C_1 > 0$,

$$|p| \leq k(C_1 - t), \quad t < C_1. \tag{1.12}$$

Thus t may be unbounded below in V_x.

Without loss of generality we may replace the given H by one that is homogeneous in (t, p) of degree 1.

Remark 1.7. If $\tilde{H}(x, t, p)$ is another function satisfying the condition above, with the same sets V_x as H, then a continuous viscosity solution of the problem (1.1), (1.3) for H is also one for $\tilde{H}$, as is easily verified.

For H and V_x as above, we take H to be homogeneous of degree 1 in (t, p); there is a viscosity solution. See Claim 10.3. However we do not know if $H^{n-1}(\Sigma) < \infty$ for the singular set Σ.

In Section 10 we present a result, Theorem 10.5, with this setup, for which a viscosity solution exists and its singular set Σ satisfies

$$H^{n-1}(\Sigma) < \infty.$$

Here are three special cases of that theorem. In the first two, $h(x, p)$ is a function such that $\forall x \in \overline{\Omega}$,

$$V(x) = \{p : h(x, p) < 1\}$$

is a bounded convex set with smooth boundary S_x, strictly convex with positive principal curvatures. The function h is assumed to be smooth in a neighborhood of $\bigcup_x S_x$.

PROPOSITION 1.8 *There exists $\lambda_0 > 0$, depending on h and on Ω, such that for any $0 < \lambda < \lambda_0$, for the function*

$$H(x, t, p) = \lambda t + h(x, p), \tag{1.13}$$

problem (1.1), (1.3) *has a positive viscosity solution and its singular set Σ satisfies*

$$H^{n-1}(\Sigma) < \infty\,. \tag{1.14}$$

The existence of a positive viscosity solution for any $\lambda > 0$ is, of course, part of theorem 5.4 in [12]. For large λ we have not succeeded in proving (1.14).

Remark 1.9. One may ask what happens for H given in (1.13) if $\lambda < 0$. Then there exists a negative viscosity solution, namely $u = -v$, where v is the viscosity solution for

$$\hat{H} = |\lambda| v + h(x, -\nabla v) = 1\,,$$

as is easily verified.

PROPOSITION 1.10 *There exists $\epsilon_0 > 0$ depending on h and on Ω such that $\forall\epsilon$, $0 < \epsilon < \epsilon_0$, for*

$$H = \epsilon t^2 + h(x, p)\,,$$

problem (1.1), (1.3) *has a viscosity solution for which*

$$H^{n-1}(\Sigma) < \infty\,.$$

PROPOSITION 1.11 *Let $H(x, t, p)$, with corresponding V_x and S_x, satisfy the conditions of Situation* A *in a domain Ω. Then there exists a number $d_0 > 0$ depending on H such that if Ω' is any bounded subdomain of Ω, with $\partial\Omega' \in C^{2,1}$, and such that the distance of any point x in Ω' to $\partial\Omega'$ is less than d_0 (i.e., Ω' is narrow), then in Ω' problem* (1.1), (1.3) *has a positive viscosity solution. Furthermore, for its singular set Σ,*

$$H^{n-1}(\Sigma) < \infty\,.$$

The proofs of Propositions 1.8, 1.10, and 1.11 follow easily from Theorem 10.5 and will be presented in Section 10.

We present one more proposition; it is proven in Section 10. Here we consider H independent of x,

$$H = H(t, p)\,,$$

satisfying the conditions of Situation A, in a bounded domain Ω. Let $\bar{t}$ be the positive number satisfying $H(\bar{t}, 0) = 1$ and let

$$\hat{t} = \max_{H(t,p)=1} t\,,$$

clearly $\bar{t} \le \hat{t}$.

PROPOSITION 1.12 *Suppose $\bar{t} < \hat{t}$. Then there is a positive viscosity solution of* (1.1), (1.3) *for this H, whose singular set Σ satisfies*

$$H^{n-1}(\Sigma) < \infty\,.$$

In case $\bar{t} = \hat{t}$, we believe the same conclusion holds but, as we explain in Section 10, our method of proof cannot work.

1.5

Theorem 10.5, which concerns general $H(x, t, p)$, is derived from Theorem 1.5, where H does not involve t, by introducing an extra independent variable τ and by considering the function

$$z(\tau, x) = e^{\tau} u(x)\,. \tag{1.15}$$

We conclude the introduction by giving a brief description of our proof of Theorem 1.5. For simplicity we assume $\overline{\Omega}$ is compact.

Consider a geodesic for the Finsler metric $\varphi(\xi; v)$, starting at a point y on $\partial\Omega$ and going in the direction "normal" to $\partial\Omega$. The geodesic is given by $\xi(t)$, with $\xi(0) = y$, and satisfies the geodesic equation

$$\varphi_{\xi^i}(\xi(t); \dot{\xi}(t)) = \frac{d}{dt}\varphi_{v^i}(\xi(t); \dot{\xi}(t))\,.$$

We may parametrize the geodesic using arc length s, i.e.,

$$\varphi(\xi(s); \dot{\xi}(s)) \equiv 1\,.$$

Denote the geodesic by

$$\xi(y, s)\,.$$

We have to explain the "normal" direction

$$V(y) = \dot{\xi}(y, 0)\,.$$

Let $\nu(y)$ be the unit inner normal to $\partial\Omega$ at y. Then $V(y)$ is the unique vector-valued function on $\partial\Omega$ satisfying

$$\begin{cases} V(y) \cdot \nu(y) > 0 \\ \varphi(y; V(y)) = 1 \\ \nabla_v \varphi(y, V(y)) \text{ is parallel to } \nu(y). \end{cases} \tag{1.16}$$

From y on $\partial\Omega$ we go along the geodesic until we hit a point $m(y)$ and set

$$\bar{s}(y) = \operatorname{dist}(y, m(y))\,.$$

Without loss of generality we may assume that $\bar{s}(\bar{y}) = 1$, i.e., $m(\bar{y}) = \xi(\bar{y}, 1)$. We will show that there exist some large constant $K \geq 1$ and some small constant $\delta > 0$ such that for all $y \in \partial\Omega$ satisfying $0 < |y - \bar{y}| \leq \delta$, we can find $z = z(\bar{y}, y) \in \partial\Omega$ that satisfies

$$\operatorname{dist}(z \text{ to } \xi(y, 1 + K|y - \bar{y}|)) < 1 + K|y - \bar{y}| = \bar{s}(\bar{y}) + K|y - \bar{y}|\,. \tag{1.17}$$

This implies that

$$\bar{s}(y) \leq \bar{s}(\bar{y}) + K|y - \bar{y}| \quad \forall |y - \bar{y}| \leq \delta\,.$$

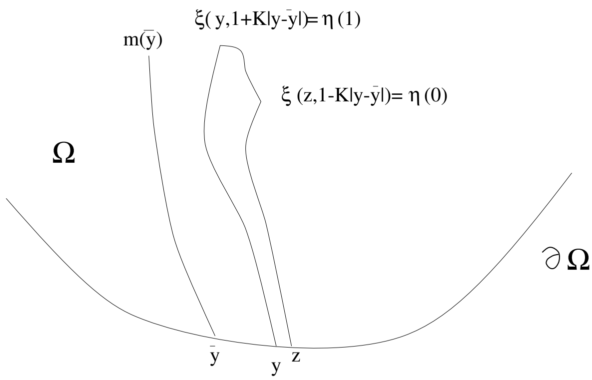

FIGURE 1.3

Since K and δ are independent of $\bar{y}$ and y, we also have, by switching the roles of $\bar{y}$ and y, that

$$\bar{s}(\bar{y}) \le \bar{s}(y) + K|y-\bar{y}| \quad \forall |y-\bar{y}| \le \delta\,.$$

Thus

$$|\bar{s}(y) - \bar{s}(z)| \le K|y-z| \quad \forall y, z \in \partial\Omega,\ |y-z| \le \delta\,.$$

It follows, possibly for a larger K, that

$$|m(y) - m(z)| \le K|y-z| \quad \forall y, z \in \partial\Omega,\ |y-z| \le \delta\,.$$

To establish (1.17), we first use the triangle inequality (see Figure 1.3),

$$\begin{aligned}
&\mathrm{dist}(z \text{ to } \xi(y, 1+K|y-\bar{y}|)) \\
&\quad \le \mathrm{dist}(z \text{ to } \xi(z, 1-K|y-\bar{y}|)) \\
&\qquad + \mathrm{dist}(\xi(z, 1-K|y-\bar{y}|) \text{ to } \xi(y, 1+K|y-\bar{y}|)) \\
&\quad \le (1-K|y-\bar{y}|) + \mathrm{dist}(\xi(z, 1-K|y-\bar{y}|) \text{ to } \xi(y, 1+K|y-\bar{y}|))\,.
\end{aligned}$$

We then construct a curve $\eta(t)$, $0 \le t \le 1$, satisfying

$$\eta(0) = \xi(z, 1-K|y-\bar{y}|)\,, \quad \eta(1) = \xi(y, 1+K|y-\bar{y}|)\,,$$

and

$$\int_0^1 \varphi(\eta(t); \dot{\eta}(t))dt < 2K|y-\bar{y}|\,,$$

from which we deduce

$$\begin{aligned}
\mathrm{dist}(z \text{ to } \xi(y, 1+K|y-\bar{y}|)) &\le (1-K|y-\bar{y}|) \\
&\quad + \int_0^1 \varphi(\eta(t); \dot{\eta}(t))dt < 1 + K|y-\bar{y}|\,.
\end{aligned}$$

To construct the η, we make, for some small $\epsilon_0 > 0$, a diffeomorphism to map a neighborhood of $\{\xi(\bar{y}, \tau)\}_{-\epsilon_0 \le \tau \le 1+\epsilon_0}$ to a neighborhood of $\{\tau e_n\}_{-\epsilon_0 \le \tau \le 1+\epsilon_0}$ so that in the new coordinates, $\{\tau e_n\}_{-\epsilon_0 \le \tau \le 1+\epsilon_0}$ is a geodesic for the new φ, and the new φ has better properties. Such new coordinates will be called special coordinates, and they are produced in Section 3. In the special coordinates, our η is a straight segment connecting $\xi(z, 1 - K|y - \bar{y}|)$ to $\xi(y, 1 + K|y - \bar{y}|)$.

2 Preliminaries

2.1

It is convenient to extend φ so that it satisfies

$$\begin{cases} \varphi \in C^{2,1}(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \text{ with derivatives smooth in } v \text{ for } v \neq 0 \\ \varphi(\xi; sv) \equiv s\varphi(\xi; v) \ \ \forall s > 0, \xi \in \mathbb{R}^n, v \in \mathbb{R}^n \setminus \{0\} \\ 0 < \inf_{\xi \in \mathbb{R}^n, \|v\|=1} \varphi(\xi; v) \leq \sup_{\xi \in \mathbb{R}^n, \|v\|=1} \varphi(\xi, v) < \infty \end{cases} \tag{2.1}$$

and

$$\begin{aligned} 0 &< \inf_{\xi \in \mathbb{R}^n, \|v\|=1, \|w\|=1} \frac{\partial(\varphi^2)}{\partial v^i \partial v^j}(\xi; v) w^i w^j \\ &\leq \sup_{\xi \in \mathbb{R}^n, \|v\|=1, \|w\|=1} \frac{\partial(\varphi^2)}{\partial v^i \partial v^j}(\xi; v) w^i w^j < \infty . \end{aligned} \tag{2.2}$$

Define, for $x, y \in \mathbb{R}^n$,

$$\operatorname{dist}(y \text{ to } x) = \inf \left\{ \int_0^1 \varphi(\xi(t), \dot{\xi}(t)) dt : \xi(0) = y, \ \xi((1) = x, \ \dot{\xi} \in L^1(0, 1) \right\} .$$

Then $\mathbb{R}^n$, equipped with dist(y to x), is a complete (both forward and backward) Finsler manifold (see, e.g., [4]).

Again, the geodesic equation for the Finsler metric is

$$\varphi_{\xi^i}(\xi(t), \dot{\xi}(t)) = \frac{d}{dt} \varphi_{v^i}(\xi(t), \dot{\xi}(t)) .$$

We may always introduce a new t-variable so that

$$\varphi(\xi; \dot{\xi}) \equiv 1 ;$$

i.e., t is the arc length.

It is not difficult to see that

$$u(x) = \inf_{y \in \partial\Omega} L(x, y) = \inf_{y \in \partial\Omega} \operatorname{dist}(y \text{ to } x) , \quad x \in \overline{\Omega} .$$

Let

$$\psi = \varphi^2 .$$

For $y \in \partial\Omega$, the vector $V(y)$ given in (1.16) is simply

$$V(y) = \mu[\nabla_v \psi(y, \cdot)]^{-1}(\nu(y)) ,$$

where $\mu > 0$ is uniquely determined by

$$\mu^2 \psi \left(y, [\nabla_v \psi(y, \cdot)]^{-1}(\nu(y)) \right) = 1 .$$

See Figure 2.1.

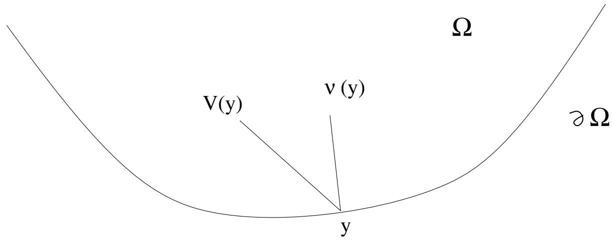

FIGURE 2.1

For $y \in \partial\Omega$, we consider the following ODE:

$$\psi_{\xi^i}(\xi(y,s);\dot\xi(y,s)) = \frac{\partial}{\partial s}\psi_{v^i}(\xi(y,s);\dot\xi(y,s))\,, \quad s \geq 0\,,$$
$$\xi(y,0) = y\,, \tag{2.3}$$

and

$$\dot\xi(y,0) = V(y)\,. \tag{2.4}$$

Solutions $\xi(y,s)$ are geodesics starting from y with unit speed, i.e.,

$$\dot\xi(y,s) \neq 0\,, \quad \varphi(\xi(y,s);\dot\xi(y,s)) \equiv 1\,, \quad s \geq 0\,,$$

and

$$\varphi_{\xi^i}(\xi(y,s);\dot\xi(y,s)) = \frac{\partial}{\partial s}\varphi_{v^i}(\xi(y,s);\dot\xi(y,s))\,, \quad s \geq 0\,,$$

with initial conditions (2.3) and (2.4).

For any $x, y \in \mathbb{R}^n$, let

$$X_1 = \{\xi \in C([0,1],\mathbb{R}^n) : \xi(0) = y, \xi(1) = x, \dot\xi \in L^1(0,1)\}\,,$$
$$X_2 = \{\xi \text{ in } X_1 \text{ with } \dot\xi \in L^2(0,1)\}\,,$$
$$I_1 = \int_0^1 \varphi(\xi(t);\dot\xi(t))dt\,, \qquad \xi \in X_1\,,$$

and

$$I_2 = \int_0^1 \varphi^2(\xi(t);\dot\xi(t))dt\,, \quad \xi \in X_2\,.$$

For any $\xi \in X_1$ and any $t = t(\tau) \in C^1[0,1]$ satisfying $t(0) = 0$, $t(1) = 1$, and $t'(\tau) > 0$, $0 \leq \tau \leq 1$, let $\eta(\tau) = \xi(t(\tau))$. It is easy to see that $\eta \in X_1$ and

$$I_1(\eta) = I_1(\xi)\,.$$

We list some elementary facts that can be found in, e.g., [4].

Fact 1. If $\bar\xi \in X_2$ is a critical point of I_2 in the sense that

$$\frac{d}{d\epsilon} I_2(\bar\xi + \epsilon h)|_{\epsilon=0} = 0 \quad \forall h \in C_c^\infty((0,1),\mathbb{R}^n)\,,$$

then $\bar{\xi}$ belongs to $C^\infty([0,1],\mathbb{R}^n)$,

$$\dot{\bar{\xi}}(t) \neq 0 \quad \forall 0 \le t \le 1\,,$$

and $\bar{\xi}$ satisfies

$$\psi_{\xi^i}(\bar{\xi}(t);\dot{\bar{\xi}}(t)) = \frac{d}{dt}\psi_{v^i}(\bar{\xi}(t);\dot{\bar{\xi}}(t)) \quad \text{on } [0,1]\,,$$

where $\psi = \varphi^2$. Moreover, if φ is independent of ξ, then

$$\bar{\xi}(t) \equiv y + t(x-y)\,.$$

Fact 2. $I_1(\xi) \le \sqrt{I_2(\xi)}\ \ \forall \xi \in X_2$.

Fact 3. $\inf_{X_1} I_1$ and $\inf_{X_2} I_2$ are achieved, and

$$\inf_{X_2} I_2 = \left(\inf_{X_1} I_1\right)^2\,.$$

Fact 4. Let $\bar{\xi} \in X_2$ be a minimum point of I_2, i.e.,

$$I_2(\bar{\xi}) = \min_{X_2} I_2\,.$$

Then $\bar{\xi}$ is also a minimum point of I_1, i.e.,

$$I_1(\bar{\xi}) = \min_{X_1} I_1\,.$$

Fact 5. For $-\infty < a < b < \infty$, assume that $\xi \in C^2(a,b)$ satisfies

$$\psi_{\xi^i}(\xi;\dot{\xi}) = \frac{d}{dt}\psi_{v^i}(\xi;\dot{\xi}) \quad \text{on } (a,b)$$

where, as usual, $\psi = \varphi^2$. Then

$$\frac{d}{dt}\psi(\xi;\dot{\xi}) \equiv 0 \quad \text{on } (a,b)\,,$$

and consequently ξ satisfies the geodesic equation

$$\varphi_{\xi^i}(\xi;\dot{\xi}) = \frac{d}{dt}\varphi_{v^i}(\xi;\dot{\xi}) \quad \text{on } (a,b)\,.$$

Moreover, either $\dot{\xi} \equiv 0$ on (a,b) or $\dot{\xi}(t) \neq 0$ for all $t \in (a,b)$.

The following is a simple but useful lemma.

LEMMA 2.1 *Let $\xi(s,\sigma)$ be a C^1 family of geodesics with s as arc length depending on some parameters $\sigma = (\sigma_1,\dots,\sigma_k)$, and assume that ξ_{σ_α} are twice continuously differentiable in s. Then*

$$\frac{\partial}{\partial s}\left(\xi^i_{\sigma_\alpha}\varphi_{v^i}(\xi;\dot{\xi})\right) \equiv 0\,. \tag{2.5}$$

Here $\cdot = \partial_s$.

PROOF: Differentiating $\varphi(\xi;\dot\xi)=1$ with respect to σ_α we find

$$\varphi_{\xi^i}\xi^i_{\sigma_\alpha}+\varphi_{v^i}\dot\xi^i_{\sigma_\alpha}=0\,.$$

Identity (2.5) then follows with the aid of the geodesic equations. □

2.2

We now turn to a point on $\partial\Omega$. We may assume it is the origin and that Ω is given by

$$x_n>f(x')\,,\quad x'\in\mathbb{R}^{n-1}\,,$$

with f a $C^{2,1}$ function defined on $|x'|\le\epsilon_1$, with

$$f(0')=0\,,\quad \nabla f(0')=0\,.$$

Throughout, when we say that some constant depends on f, we mean it depends on the $C^{2,1}$ norm of f:

$$\|f\|_{C^{2,1}}=\|f\|_{C^2}+\sup_{x'\neq y'}\frac{|D^2f(x')-D^2f(y')|}{|x'-y'|}\,.$$

We consider geodesics $\xi=\xi(x',s)$ that are $C^{1,1}$ functions of x' and s, with $\nabla_{x'}\xi$ smooth in s, with unit speed starting at $z=(x',f(x'))$; i.e., ξ satisfies

$$\varphi_{\xi^i}(\xi;\dot\xi)=\frac{\partial}{\partial s}\varphi_{v^i}(\xi;\dot\xi)\,,\quad |x'|\le\epsilon_1,\ 0\le s<a\,, \tag{2.6}$$

$$\varphi(\xi;\dot\xi)\equiv 1\,,\quad |x'|<\epsilon_1,\ 0\le s<a, \tag{2.7}$$

and

$$\xi(x',0)=z=(x',f(x'))\,,\quad |x'|<\epsilon_1\,,$$

and entering Ω,

$$\dot\xi(x',0)\cdot(-\nabla f(0'),1)>0\,.$$

We have changed notation: before, the geodesic $\xi(x',s)$ was denoted by $\xi((x',f(x')),s)$.

LEMMA 2.2 *Suppose that for some fixed $w=(x',f(x'))$ and $\bar s$ small, w is the closest point on $\partial\Omega$ to $\xi(x',\bar s)$. Then*

$$\dot\xi(x',0)=V(x')\,, \tag{2.8}$$

where $V(x')$ is the vector satisfying (1.16)*, i.e.,*

$$V(x')\cdot(-\nabla f(x'),1)>0\,,\quad \psi(w;V(x'))=1\,,$$

$$\nabla_v\psi(w;V(x'))\quad \text{is parallel to } (-\nabla f(x'),1)\,.$$

The vector $V(x')$ is simply

$$V(x')=\mu[\nabla_v\psi(w;\cdot)]^{-1}(-\nabla f(x'),1)$$

with μ determined by $\psi(w;V(x'))=1$. Here we have abused the notation a little since by our earlier convention, $V(x')$ should be denoted as $V(w)$.

PROOF: For any $0 < s < \bar{s}$, w is the closest point on $\partial\Omega$ to $\xi(x', s)$, so we may take $\bar{s}$ so small that for every y' close to x' there is a minimal geodesic $\eta(y', t)$, $0 \le t \le \bar{s}$, with

$$\eta(y', 0) = (y', f(y')), \quad \eta(y', \bar{s}) = \xi(x', \bar{s}). \tag{2.9}$$

Note that except for $\eta(x', t)$, t may not be the arc length on the geodesics η. By assumption,

$$\int_0^{\bar{s}} \varphi(\eta(y', t); \dot{\eta}(y', t))dt$$

has a minimum at $y' = x'$; so at x', for $\alpha < n$, its y_α-derivative is zero:

$$\begin{aligned} 0 &= \int_0^{\bar{s}} \varphi_{\xi^i}(\eta; \dot{\eta})\eta^i_{y_\alpha} + \varphi_{v^i}(\eta; \dot{\eta})\dot{\eta}^i_{y_\alpha}\, dt \\ &= \int_0^{\bar{s}} \frac{\partial}{\partial t}\big[\varphi_{v^i}\eta^i_{y_\alpha}\big]dt = \big(\varphi_{v^i}\eta^i_{y_\alpha}\big)(\bar{s}) - \big(\varphi_{v^i}\eta^i_{y_\alpha}\big)(0). \end{aligned} \tag{2.10}$$

Here we have used the geodesic equations satisfied by η. By (2.9),

$$\eta^i_{y_\alpha}(y', \bar{s}) \equiv 0.$$

Also, for $1 \le \alpha, \beta \le n-1$,

$$\begin{cases} \xi^\beta_{x_\alpha}(x', 0) = \eta^\beta_{y_\alpha}(x', 0) = \delta^\beta_\alpha \\ \xi^n_{x_\alpha}(x', 0) = \eta^n_{y_\alpha}(x', 0) = f_{x_\alpha}(x'). \end{cases} \tag{2.11}$$

Inserting these into (2.10) we find, for $\alpha \le n-1$,

$$\varphi_{v^\alpha}(\xi(x', 0); \dot{\xi}(x', 0)) + f_{x_\alpha}\varphi_{v^n}(\xi(x', 0); \dot{\xi}(x', 0)) = 0,$$

i.e.,

$$\nabla_v\varphi(z; \dot{\xi}(x', 0)) \quad \text{is parallel to } (-\nabla f(x'), 1) \tag{2.12}$$

so (2.8) is proven. □

Note that, from (2.11),

$$\xi^i_{x_\alpha}(x', 0)\varphi_{v^i}(\xi(x', 0); \dot{\xi}(x', 0)) = 0. \tag{2.13}$$

In the following we continue to use $\xi(x', s)$ to denote the solution of

$$\begin{gathered} \psi_{\xi^i}(\xi(x', s); \dot{\xi}(x', s)) = \frac{\partial}{\partial s}\psi_{v^i}(\xi(x', s); \dot{\xi}(x', s)), \\ \xi(x', 0) = (x', f(x')), \quad \dot{\xi}(x', 0) = V(x'). \end{gathered}$$

By the choice of $V(x')$, $\psi(\xi(x', 0); \dot{\xi}(x', 0)) = 1$, so $\psi(\xi(x', \cdot); \dot{\xi}(x', \cdot)) \equiv 1$ by Fact 5. By the smooth dependence of solutions of ODEs on initial data, we have, for some smooth χ, that $\xi(x', s) = \chi((x', f(x')), V(x'), s)$. Since f is in $C^{2,1}$,

$V(x')$ is in $C^{1,1}$, and therefore, for some constant E, depending only on φ, f, and a, we have, for all $1 \le \alpha, \beta \le n-1$, $|x'| \le \epsilon_1$, and $-\epsilon_1 \le s \le a$, that

$$\sum_{k=0}^{3} \left(\left| \frac{\partial^k}{\partial s^k} \xi(x', s) \right| + \left| \frac{\partial^k}{\partial s^k} \xi_{x_\alpha}(x', s) \right| + \left| \frac{\partial^k}{\partial s^k} \xi_{x_\alpha x_\beta}(x', s) \right| \right) \le E$$

and

$$\sum_{k=0}^{3} \left| \frac{\partial^k}{\partial s^k} \xi_{x_\alpha}(x', s) - \frac{\partial^k}{\partial s^k} \xi_{x_\alpha}(0', s) \right| \le E|x'| .$$

The conditions of Lemma 2.1 therefore hold, and it follows from the lemma and (2.13) that

$$\xi^i_{x_\alpha}(x', s)\varphi_{v^i}(\xi(x', s); \dot\xi(x', s)) \equiv 0 . \tag{2.14}$$

We now show, in some sense, the converse of Lemma 2.2.

LEMMA 2.3 *Consider* $|x'| \le \epsilon_1$. *For some positive constant* ϵ_2, *depending only on* φ *and* f, *we have*

$$\text{dist}(0 \text{ to } \xi(0', s)) < \text{dist}((x', f(x')) \text{ to } \xi(0', s)) \ \ \forall s,\ 0 < s < \epsilon_2,\ 0 < |x'| \le \epsilon_1 ,$$
$$\text{dist}(\xi(0', s) \text{ to } 0) < \text{dist}(\xi(0', s) \text{ to } (x', f(x'))) \ \ \forall s, -\epsilon_2 < s < 0,\ 0 < |x'| \le \epsilon_1 .$$

PROOF: For simplicity we assume $s > 0$. There exists $\epsilon_2 > 0$, depending only on f and φ, such that

$$\psi_{\xi^i}(\xi(x', s); \dot\xi(x', s)) = \frac{\partial}{\partial s}\psi_{v^i}(\xi(x', s); \dot\xi(x', s)) , \quad |x'| \le \frac{\epsilon_1}{2},\ |s| \le 2\epsilon_2 ,$$
$$\xi(x', 0) = (x', f(x')) , \qquad |x'| \le \frac{\epsilon_1}{2} ,$$

and

$$\dot\xi(x', 0) = V(x') , \qquad |x'| \le \frac{\epsilon_1}{2} ,$$

has unique smooth solutions. Moreover, for any $|x'| \le \epsilon_1/2$, $\xi(x'; s)$ is the shortest geodesic for $|s| \le \epsilon_2$. From Lemma 2.2 and (2.13) we see that for $|x'| < \epsilon_1$, the Jacobian of the map $(x', s) \to \xi(x', s)$ is positive at $s = 0$. Hence for ϵ_2 small, the map $(x', s) \to \xi(x'; s)$ is a diffeomorphism for $|x'| \le \epsilon_1/2$ and $|s| \le \epsilon_2$, and

$$s = \text{dist}(0 \text{ to } \xi(0', s)) < \text{dist}((x', f(x')) \text{ to } \xi(0', s)) \quad \forall |x'| \ge \frac{\epsilon_1}{4} ,$$
$$\text{dist}(0 \text{ to } \xi(0', s)) = \min_{|x'| \le \epsilon_1/4} \text{dist}((x', f(x')) \text{ to } \xi(0', s)) .$$

Let $\bar x'$ be a minimum point, i.e., $|\bar x'| \le \epsilon_1/4$ and

$$s = \text{dist}(0 \text{ to } \xi(0', s)) = \text{dist}((\bar x', f(\bar x')) \text{ to } \xi(0', s)) .$$

By Lemma 2.2, $\xi(0', s) = \xi(\bar x', s)$. Since the map $(x', s) \to \xi(x', s)$ is a diffeomorphism, we must have $\bar x' = 0'$. Lemma 2.3 is established. □

3 Special Coordinates

Let $\varphi(\xi; v)$ be as in Section 2, and let $\xi = \xi(t)$ be a geodesic with $\dot{\xi}(t) \neq 0$ and

$$\varphi_{\xi^i}(\xi(t); \dot{\xi}(t)) = \frac{d}{dt}\varphi_{v^i}(\xi(t); \dot{\xi}(t)) \quad \forall i,\ 1 \le i \le n\,.$$

For a nonsingular change of variables $\xi = \xi(\eta)$ in $\mathbb{R}^n$, let

$$\tilde{\varphi}(\eta; w) = \varphi(\xi(\eta); \xi_\eta w)) \quad \text{where } \xi_\eta := \left\{\frac{\partial \xi^i}{\partial \eta^j}\right\}.$$

Such a change of variables maps geodesics to geodesics.

With $\partial\Omega$ locally as in Section 2.2, so that $\nu(0) = e_n = (0, \dots, 0, 1)$, we consider the geodesics $\xi(x', s)$ of that section. In view of the above one may make a smooth change of variables so that in the new variables the geodesic $\xi(0', s)$, with s as arc length, runs on the x_n-axis and such that we still have $\nu(0) = e_n$. We start with this situation.

Throughout, Greek letters α and β run from 1 to $n-1$, while indices i, j, k, etc., run from 1 to n.

LEMMA 3.1 *Let* $\{te_n : 0 \le t \le 1\}$ *be a geodesic for* $\varphi(\xi; v)$ *with unit speed, i.e.,*

$$\varphi_{\xi^i}(te_n; e_n) \equiv \partial_t \varphi_{v^i}(te_n; e_n) \quad \forall t,\ 0 \le t \le 1,\ 1 \le i \le n\,,$$

and

$$\varphi(te_n; e_n) \equiv 1\,, \quad 0 \le t \le 1\,. \tag{3.1}$$

Then, in an open neighborhood of the geodesic segment, there exists some nonsingular change of variables $\xi = \xi(\eta)$ *such that*

$$\xi_\eta(0) = \text{Id}\,, \quad \xi(te_n) = te_n\,, \quad \xi_\eta(te_n)e_n = e_n\,, \quad 0 \le t \le 1\,, \tag{3.2}$$

and $\tilde{\varphi}(\eta; w) = \varphi(\xi(\eta); \xi_\eta w))$ *satisfies* (3.1) *and*

$$\tilde{\varphi}_{\eta^j}(te_n; e_n) = 0\,, \quad 1 \le j \le n,\ 0 \le t \le 1\,, \tag{3.3}$$

$$\tilde{\varphi}_{w^\alpha}(te_n; e_n) = 0\,, \quad 1 \le \alpha \le n-1,\ 0 \le t \le 1\,, \tag{3.4}$$

$$\tilde{\varphi}_{\eta^j w^k}(te_n; e_n) = 0\,, \quad 1 \le j,k \le n,\ 0 \le t \le 1\,. \tag{3.5}$$

By the homogeneity, it then follows that

$$\tilde{\varphi}_{w^n}(te_n; e_n) \equiv 1\,. \tag{3.6}$$

The reader may choose to postpone reading the long proof of the lemma and go on to the next section.

PROOF: By the chain rule, $\tilde{\varphi}_{\eta^j} = \varphi_{\xi^i}\xi^i_j + \varphi_{v^i}\xi^i_{lj}w^l$, where we have used the notation

$$\xi^i_j := \frac{\partial \xi^i}{\partial \eta^j} \quad \text{and} \quad \xi^i_{lj} := \frac{\partial^2 \xi^i}{\partial \eta^l \partial \eta^j}\,.$$

Step 1. Let

$$b_\beta(t) := -\int_0^t \varphi_{\xi^\beta}(\tau e_n; e_n)d\tau .$$

We take

$$\xi = \xi(\eta) := \left(\eta^1, \dots, \eta^{n-1}, \eta^n + \sum_{\beta=1}^{n-1} b_\beta(\eta^n)\eta^\beta\right).$$

It is easy to check that

$$\begin{aligned} &\xi^\alpha_\beta(te_n) \equiv \delta^\alpha_\beta , \quad \xi^\alpha_n(te_n) \equiv 0 ,\\ &\xi^n_\beta(te_n) \equiv b_\beta(t) , \quad \xi^n_n(te_n) \equiv 1 ,\\ &\xi^\alpha_{\beta\gamma}(te_n) \equiv \xi^\alpha_{\beta n}(te_n) \equiv \xi^\alpha_{n\beta}(te_n) \equiv \xi^\alpha_{nn}(te_n) \equiv 0 ,\\ &\xi^n_{\beta\gamma}(te_n) \equiv \xi^n_{nn}(te_n) \equiv 0 , \quad \xi^n_{\beta n}(te_n) \equiv \xi^n_{n\beta}(te_n) \equiv b'_\beta(t) . \end{aligned}$$

Identity (3.2) follows from the above. Also, from the above,

$$\det\left(\xi^i_j(te_n)\right) \equiv 1 .$$

Thus the change of variables is nonsingular near $\{te_n : 0 \le t \le 1\}$.

For $1 \le \beta \le n-1$,

$$\begin{aligned} \tilde\varphi_{\eta^\beta}(te_n; e_n) &= \varphi_{\xi^i}(te_n; e_n)\xi^i_\beta(te_n) + \varphi_{v^i}(te_n; e_n)\xi^i_{n\beta}(te_n)\\ &= \varphi_{\xi^\alpha}\xi^\alpha_\beta + \varphi_{\xi^n}\xi^n_\beta + \varphi_{v^n}\xi^n_{n\beta} = \varphi_{\xi^\beta} + \varphi_{\xi^n}\xi^n_\beta + \varphi_{v^n}\xi^n_{n\beta} . \end{aligned}$$

Differentiating (3.1) in t, we find

$$\varphi_{\xi^n}(te_n; e_n) \equiv 0 . \tag{3.7}$$

By (3.1) and the homogeneity of φ in v,

$$\varphi_{v^n}(te_n; e_n) \equiv \varphi(te_n; e_n) \equiv 1 . \tag{3.8}$$

Using (3.7) and (3.8), we have

$$\tilde\varphi_{\eta^\beta}(te_n; e_n) = \varphi_{\xi^\beta}(te_n; e_n) + \xi^n_{n\beta}(te_n) = \varphi_{\xi^\beta}(te_n; e_n) + b'_\beta(t) = 0 .$$

Next, by (3.7),

$$\tilde\varphi_{\eta^n}(te_n; e_n) = \varphi_{\xi^i}(te_n; e_n)\xi^i_n(te_n) + \varphi_{v^i}\xi^i_{nn}(te_n) = \varphi_{\xi^n}(te_n; e_n) = 0 .$$

We have verified (3.3).

Step 2. Since we have verified (3.3) for $\tilde\varphi$ and the change of variables also preserve the hypotheses on φ, we may assume without loss of generality that, to start, the φ satisfies the additional hypothesis

$$\varphi_{\xi^j}(te_n; e_n) = 0 , \quad 1 \le j \le n,\ 0 \le t \le 1 . \tag{3.9}$$

Now we try to make a change of variables such that $\tilde\varphi$ also satisfies (3.2), (3.3), and, in addition, (3.4). Later we do another transformation to ensure also (3.5).

Since $\{te_n\}$ is a geodesic, we deduce from the geodesic equations together with (3.9) that

$$\varphi_{v^i}(te_n; e_n) \equiv \varphi_{v^i}(0; e_n) \quad \forall i,\ 1 \le i \le n . \tag{3.10}$$

Let

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \mathbf{0} & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \mathbf{0} & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 \\ -\varphi_{v^1}(0; e_n) & -\varphi_{v^2}(0; e_n) & \cdots & \cdots & -\varphi_{v^{n-1}}(0; e_n) & 1 \end{pmatrix},$$

and consider a linear change of variables $\xi = \xi(\eta) := A\eta$. Let

$$\tilde{\varphi}(\eta; w) = \varphi(\xi(\eta); \xi_\eta w)) = \varphi(A\eta; Aw)\,.$$

Clearly the change of variables satisfies (3.2). By (3.8) and (3.10), we have

$$\begin{aligned} \tilde{\varphi}_{w^\alpha}(te_n; e_n) &= \varphi_{v^i}(te_n; e_n) A^i_\alpha \\ &= \varphi_{v^\alpha}(te_n; e_n) + \varphi_{v^n}(te_n; e_n) A^n_\alpha \\ &= \varphi_{v^\alpha}(te_n; e_n) + A^n_\alpha \\ &= \varphi_{v^\alpha}(te_n; e_n) - \varphi_{v^\alpha}(0; e_n) = 0\,. \end{aligned}$$

We have verified that $\tilde{\varphi}$ satisfies (3.4). Clearly $\tilde{\varphi}$ satisfies (3.3), since φ satisfies (3.9).

So from now on, we may assume without loss of generality that φ further satisfies equation (3.9) and

$$\varphi_{v^\alpha}(te_n; e_n) = 0\,, \quad 1 \le \alpha \le n-1,\ 0 \le t \le 1\,. \tag{3.11}$$

Step 3. Let $\psi := \varphi^2$. For $1 \le \alpha, \beta \le n-1$, we have, by (3.1) and (3.11),

$$\psi_{v^\alpha v^\beta}(te_n; e_n) = 2\varphi(te_n, e_n)\varphi_{v^\alpha v^\beta}(te_n; e_n) = 2\varphi_{v^\alpha v^\beta}(te_n; e_n)\,.$$

So, by the positivity of $(\psi_{v^\alpha v^\beta})$, $A := (\varphi_{v^\alpha v^\beta}(te_n; e_n))$ is real, symmetric, and positive definite.

Let

$$E(t) := (\varphi_{\xi^\alpha v^\beta}(te_n; e_n))\,.$$

By Lemma B.2 in Appendix B, the dimension of the space of solutions of

$$X^\top A - AX = E^\top - E$$

is $(n-1)n/2$. For fixed t, let $X(t)$ be the solution of the above equation with the least Euclidean norm. Clearly $X(t)$ depends smoothly on t.

Let $B(t)$ be the solution of

$$\begin{cases} \dot{B}(t) := \frac{d}{dt}B(t) = XB, & 0 \le t \le 1, \\ B(0) = I. \end{cases}$$

Clearly $\det(B(t)) \ne 0$, $0 \le t \le 1$. Let

$$M(t) := B^\top E^\top B + B^\top A \dot{B}\,.$$

It is easy to see that M is symmetric, i.e.,

$$M^{\mathsf{T}} \equiv M\,.$$

We introduce a final change of variables $\xi = \xi(\eta)$ by

$$\begin{cases} \xi^\alpha = \sum_{1\le\beta\le n-1} B^\alpha_\beta(\eta^n)\eta^\beta, \\ \xi^n = \eta^n - \frac{1}{2}\sum_{1\le\gamma,\mu\le n-1} M_{\gamma\mu}(\eta^n)\eta^\gamma\eta^\mu. \end{cases}$$

Then (3.2) holds,

$$\xi_\eta(0) = \mathrm{Id}\,, \quad \xi(te_n) = te_n\,, \quad \xi_\eta(te_n)e_n = e_n\,,$$

and

$$\det(\xi_\eta(te_n)) = \det(B(t)) \neq 0\,,$$

$$\xi^\alpha_n(te_n) \equiv \xi^\alpha_{nn}(te_n) \equiv \xi^n_{jn}(te_n) \equiv \xi^n_{nj}(te_n) \equiv \xi^n_\alpha(te_n) \equiv 0\,.$$

Let $\tilde\varphi(\eta;\, w) = \varphi(\xi(\eta);\, \xi_\eta w))$. Using (3.9), (3.11), and the above listed properties of the change of variables, we find

$$\tilde\varphi_{\eta^j}(te_n;\, e_n) = \varphi_{v^l}\xi^l_{nj} = \varphi_{v^n}\xi^n_{nj} = 0\,, \quad 1 \le j \le n\,,$$

$$\tilde\varphi_{w^\alpha}(te_n;\, e_n) = \varphi_{v^i}\xi^i_\alpha = \varphi_{v^n}\xi^n_\alpha = 0\,, \quad 1 \le \alpha \le n-1\,.$$

We have verified that (3.2), (3.3), and (3.4) continue to hold in the new variables.

Step 4. Finally, to verify (3.5), consider, at (te_n, e_n),

$$\tilde\varphi_{w^i\eta^j} = \varphi_{\xi^l v^m}\xi^m_i\xi^l_j + \xi^n_{ij} + \varphi_{v^l v^m}\xi^m_i\xi^l_{nj}\,.$$

By (3.11),

$$\varphi_{\xi^n v^\alpha}(te_n;\, e_n) \equiv 0\,, \quad 1 \le \alpha \le n-1\,, \tag{3.12}$$

and, using the homogeneity of φ in v,

$$\varphi_{v^\alpha v^n}(te_n;\, e_n) \equiv 0\,, \quad 1 \le \alpha \le n-1\,. \tag{3.13}$$

By (3.8) and the homogeneity of φ in v,

$$\varphi_{v^n v^n}(te_n;\, e_n) \equiv 0\,. \tag{3.14}$$

By (3.9) and the homogeneity of φ in v,

$$\varphi_{\xi^j v^n}(te_n;\, e_n) \equiv 0\,, \quad 1 \le j \le n\,. \tag{3.15}$$

Simplifying the expression of $\tilde\varphi_{w^i\eta^j}$ by using (3.12), (3.13), (3.14), and (3.15), we have

$$\begin{aligned}\tilde\varphi_{w^i\eta^j}(te_n;\, e_n) &= \varphi_{\xi^l v^\alpha}\xi^\alpha_i\xi^l_j + \xi^n_{ij} + \varphi_{v^\alpha v^\beta}\xi^\beta_i\xi^\alpha_{nj} \\ &= \varphi_{\xi^\beta v^\alpha}\xi^\alpha_i\xi^\beta_j + \xi^n_{ij} + \varphi_{v^\alpha v^\beta}\xi^\beta_i\xi^\alpha_{nj}\,.\end{aligned}$$

Since $\xi^\beta_n(te_n) \equiv \xi^\alpha_{nn}(te_n) \equiv \xi^n_{in}(te_n) \equiv 0$ for all $1 \le \alpha, \beta \le n-1$ and $1 \le i \le n$, we have

$$\tilde\varphi_{w^i\eta^n} \equiv 0\,, \quad 1 \le i \le n\,.$$

Similarly,

$$\tilde{\varphi}_{w^n \eta^j} \equiv 0\,, \quad 1 \le j \le n\,.$$

Finally, for $1 \le \gamma, \mu \le n-1$, as one may check,

$$\tilde{\varphi}_{w^\gamma \eta^\mu}(te_n; e_n) = \varphi_{\xi^\beta v^\alpha} \xi^\alpha_\gamma \xi^\beta_\mu + \xi^n_{\gamma\mu} + \varphi_{v^\alpha v^\beta} \xi^\beta_\gamma \xi^\alpha_{n\mu} = M_{\gamma\mu} + \xi^n_{\gamma\mu} = 0\,.$$

In the above, we have used

$$\dot{B}^\alpha_\mu(t) = \frac{d}{dt} B^\alpha_\mu(t) = \frac{d}{dt} \xi^\alpha_\mu(te_n) = \xi^\alpha_{n\mu}(te_n)\,.$$

We have thus verified (3.5). Lemma 3.1 is established. □

4 Cut Points and Conjugate Points

In this section we establish some properties of the cut points and conjugate points of y on $\partial\Omega$. In particular, we first prove the continuity of the map $m(y)$, defined on $\partial\Omega$, and then prove that $m(y) = \tilde{m}(y)$ for all $y \in \partial\Omega$ and, consequently, $\Sigma = \tilde{\Sigma}$.

4.1

For $y \in \partial\Omega$, without loss of generality, we may assume $\bar{s}(y) = \bar{s}(0) = 1$. Then we use our special coordinates of Section 3; near the origin Ω is given by $x_n > f(x')$ with

$$f(0') = 0\,, \quad \nabla f(0') = 0\,.$$

Then $m(y) = m(0) = e_n$. The "normal" geodesic from 0 lies along the x_n-axis.

For $\epsilon_0 > 0$, let $\Gamma := \{te_n : -\epsilon_0 \le t \le 1 + \epsilon_0\}$ be the geodesic for $\varphi(\xi; v)$ satisfying, for $-\epsilon_0 \le t \le 1 + \epsilon_0$, the conclusions of Lemma 3.1 and (3.6):

$$\varphi(te_n; e_n) \equiv 1\,, \tag{4.1}$$

$$\varphi_{\xi^j}(te_n; e_n) = 0\,, \quad 1 \le j \le n\,, \tag{4.2}$$

$$\varphi_{v^\alpha}(te_n; e_n) = 0\,, \quad 1 \le \alpha \le n-1, \tag{4.3}$$

$$\varphi_{\xi^j v^k}(te_n; e_n) = 0\,, \quad 1 \le j,\ k \le n\,. \tag{4.4}$$

By (4.3) and the homogeneity of φ in v, we have

$$\varphi_{v^\alpha v^n}(te_n; e_n) \equiv 0\,, \quad 1 \le \alpha \le n-1,\ -\epsilon_0 \le t \le 1 + \epsilon_0\,. \tag{4.5}$$

Differentiating (4.2), we have

$$\varphi_{\xi^j \xi^n}(te_n; e_n) \equiv 0\,, \quad 1 \le j \le n,\ -\epsilon_0 \le t \le 1 + \epsilon_0\,. \tag{4.6}$$

For $y \in \partial\Omega$, let $\xi = \xi(y, \tau)$ denote the geodesic satisfying

$$\varphi(\xi; \dot{\xi}) \equiv 1\,, \quad \xi(y, 0) = y\,, \quad \dot{\xi}(y, 0) = V(y)\,,$$

where $V(y)$ is as in (1.16).

Recall that for $|x'| < \epsilon_1$, we write $\xi((x', f(x')), \tau)$ as $\xi(x', \tau)$, i.e., $\xi = \xi(x', \tau)$ is the geodesic satisfying

$$\varphi(\xi; \dot{\xi}) \equiv 1\,, \quad \xi(x', 0) = (x', f(x'))\,, \quad \dot{\xi}(x', 0) = V(x')\,,$$

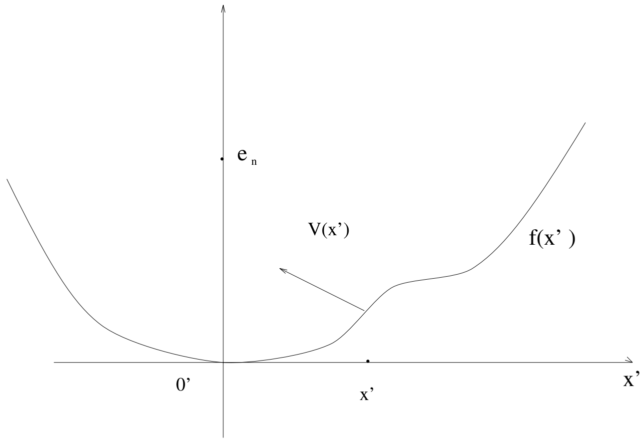

FIGURE 4.1

where $V(x')$ is the vector-valued function defined in Section 2; see also Figure 4.1.

The following lemma establishes the continuity of the map $m(y)$:

LEMMA 4.1 *Suppose, as above,* $m(0) = e_n$. *Then* $\lim_{|x'|\to 0} m((x', f(x')) = e_n$; *i.e., m is continuous at* 0.

PROOF: We prove it by contradiction. Suppose the contrary, that there exist $x'_i \to 0$ such that $m((x'_i, f(x'_i)) = \xi(x'_i, t_i)$ with $t_i \to \bar{t} \neq 1$. We know that $\xi(x'_i, t_i) \to \xi(0', \bar{t}) \in \Sigma$, so we must have $\bar{t} \geq 1$. On the other hand, if $\bar{t} > 1$, then, by compactness, there exists some $\delta > 0$, independent of i, such that the δ-neighborhood of $\{\xi(x'_i, t) : 0 \leq t \leq (1+\bar{t})/2\}$ belongs to G, the complement of Σ, for large i. Since $(1+\bar{t})/2 > 1$, this set would contain e_n for large i, a contradiction. Lemma 4.1 is established. □

4.2

We will prove that $m(0) = \tilde{m}(0)$. We first show the following:

LEMMA 4.2 *Suppose* $m(0) = e_n$. *Then* $\tilde{m}(0) = \tilde{t}e_n$ *for some* $\tilde{t} \geq 1$.

PROOF: We argue by contradiction. Suppose $0 < \tilde{t} < 1$; then, since $\tilde{t}e_n \in G$ and G is open, G contains a neighborhood of $\tilde{t}e_n$. For X close to $\tilde{t}e_n$, X in G, there exists a unique $z = z(X) \in \partial\Omega$ such that

$$\text{dist}(z \text{ to } X) = \text{dist}(\partial\Omega \text{ to } X)\,.$$

Clearly, the map X to $z(X)$ is continuous near $\tilde{t}e_n$. Since dist(0 to $\tilde{t}e_n$) = dist($\partial\Omega$ to $\tilde{t}e_n$) and since $\tilde{t}e_n \in G$, we find $z(\tilde{t}e_n) = 0$ and, by the continuity of the map, $z(X)$ is close to 0 for X close to $\tilde{t}e_n$. So we can write

$$z(X) = (x'(X), f(x'(X)))\,,$$

where $x'(X)$ is continuous near $\tilde{t}e_n$ with $x'(\tilde{t}e_n) = 0'$.

For X close to $\tilde{t}e_n$, consider a geodesic, with unit speed, joining $z(X) = (x'(X), f(x'(X)))$ to X which realizes

$$\ell(X) := \text{dist}(z(X) \text{ to } X) = \text{dist}(\partial\Omega \text{ to } X)\,.$$

By Lemma 2.2 and the fact that the geodesic must enter Ω (otherwise it would not realize the distance of $\partial\Omega$ to X since it has to enter Ω), the geodesic is $\xi(x'(X), s)$, and

$$X = \xi(x'(X), \ell(X))\,. \tag{4.7}$$

It is easy to see that $\ell(X)$ is a continuous function near $\tilde{t}e_n$. Consider the following map defined in a neighborhood of $\tilde{t}e_n$:

$$F(X) := (x'(X), \ell(X))\,.$$

One verifies that F is one-to-one near $\tilde{t}e_n$. A continuous one-to-one map is open; i.e., it maps open sets to open sets. So F maps a neighborhood of $\tilde{t}e_n$ to a neighborhood of $(0', \tilde{t})$. For t close to $\tilde{t}$, let $X_t = F^{-1}(0', t)$. Then $(0', t) = F(X_t)$ and therefore $x'(X_t) = 0'$, $\ell(X_t) = t$. By (4.7), $X_t = \xi(0', t) = te_n$, i.e.,

$$t = \ell(te_n) = \text{dist}(\partial\Omega \text{ to } te_n) \quad \text{for } t \text{ close to } \tilde{t},$$

violating $\tilde{t}e_n = \tilde{m}(0)$. Lemma 4.2 is established. □

Sometimes, for convenience, we normalize so that $\tilde{m}(0) = e_n$ instead of $m(0) = e_n$. We still have the same properties of our special coordinates stated at the beginning of this section.

4.3

LEMMA 4.3 *Assume $\tilde{m}(0) = e_n$. Then there exists some $\mu > 0$, and for all $H \in C_0^1([0,1], \mathbb{R}^{n-1})$,*

$$\int_0^1 \left(\varphi_{\xi^\beta\xi^\gamma}(te_n; e_n) H^\beta H^\gamma + \varphi_{v^\beta v^\gamma}(te_n; e_n) \dot{H}^\beta \dot{H}^\gamma\right) dt \geq \mu \int_0^1 H^2\, dt\,.$$

An easy consequence is the following:

COROLLARY 4.4 *Under the same hypotheses of Lemma* 4.3*, there exists $\mu_1 > 0$ such that for all $H \in C_0^1([0,1], \mathbb{R}^{n-1})$,*

$$\int_0^1 \left(\varphi_{\xi^\beta\xi^\gamma}(te_n; e_n) H^\beta H^\gamma + \varphi_{v^\beta v^\gamma}(te_n; e_n) \dot{H}^\beta \dot{H}^\gamma\right) dt \geq \mu_1 \int_0^1 \dot{H}^2\, dt\,.$$

Remark 4.5. One sees from the proof that the conclusion of Lemma 4.3 and Corollary 4.4 holds when replacing $e_n = \tilde{m}(0)$ by $\hat{t}e_n$ for any $0 < \hat{t} < 1$.

PROOF OF LEMMA 4.3: Let μ be the first eigenvalue of the quadratic form; i.e., μ is the largest number such that for all $H \in C_0^1([0,1], \mathbb{R}^{n-1})$. We have

$$\int_0^1 \left(\varphi_{\xi^\beta\xi^\gamma}(te_n; e_n) H^\beta H^\gamma + \varphi_{v^\beta v^\gamma}(te_n; e_n) \dot{H}^\beta \dot{H}^\gamma\right) dt \geq \mu \int_0^1 H^2\, dt\,.$$

We only need to show that $\mu > 0$. If not, then for $\epsilon > 0$, there exists $\bar{H} \in C_0^1([-\epsilon, 1], \mathbb{R}^{n-1})$ such that

$$\int_{-\epsilon}^{1} \big(\varphi_{\xi^\beta \xi^\gamma}(te_n; e_n)\bar{H}^\beta \bar{H}^\gamma + \varphi_{v^\beta v^\gamma}(te_n; e_n)\dot{\bar{H}}^\beta \dot{\bar{H}}^\gamma\big)dt < 0\,.$$

We identify $\bar{H}(t)$ with $(\bar{H}(t), 0)$ in $\mathbb{R}^{n+1}$ and perturb the geodesic te_n by considering $\zeta(\tau, t) = te_n + \tau\bar{H}(t)$, $-\epsilon < t \leq 1$. Then at $\tau = 0$, we have

$$\frac{d}{d\tau}\int_{-\epsilon}^{1} \varphi(\zeta; \dot{\zeta})dt = 0 \quad \text{and} \quad \frac{d^2}{d\tau^2}\int_{-\epsilon}^{1} \varphi(\zeta; \dot{\zeta})dt < 0\,.$$

It follows that for $\tau > 0$ small, we have

$$\int_{-\epsilon}^{1} \varphi(\zeta; \dot{\zeta})dt < 1 + \epsilon. \tag{4.8}$$

On the other hand, let $\bar{t} = \bar{t}(\tau) > 0$ be such that $\zeta(\tau, \bar{t}) = (x', f(x'))$ for some x'. Since $\tilde{m}(0) = e_n$, we find

$$\int_{\bar{t}}^{1} \varphi(\zeta; \dot{\zeta})dt \geq 1\,,$$

and, by Lemma 2.3,

$$\int_{-\epsilon}^{1} \varphi(\zeta; \dot{\zeta})dt \geq \epsilon$$

for ϵ sufficiently small. The above two estimates violate (4.8), a contradiction. □

4.4

We still assume that $e_n = \tilde{m}(0)$, and we now consider geodesics ending at e_n. For $\sigma' = (\sigma_1, \dots, \sigma_{n-1}) \in \mathbb{R}^{n-1}$ satisfying $|\sigma'| \leq \frac{1}{2}$, let $\tau = \tau(\sigma')$ be defined by

$$\varphi(e_n; (\sigma', \tau)) = 1 \quad \text{and} \quad \tau(0') = 1\,.$$

Since $\varphi_{v^n}(e_n; e_n) = 1$, by the the implicit function theorem, τ exists as a smooth function of σ'.

Let $\eta = \eta(\sigma', t)$ be the unique smooth solution of

$$\psi_{\xi^i}(\eta; \dot{\eta}) = \frac{d}{dt}\psi_{v^i}(\eta; \dot{\eta})\,, \quad t \leq 1\,,$$

satisfying

$$\eta(\sigma', 1) = e_n\,, \quad \dot{\eta}(\sigma', 1) = (\sigma', \tau(\sigma'))\,.$$

The solution exists for all time until it hits the boundary $(x', f(x'))$ (in fact, it goes further since φ has been extended to a fixed open neighborhood of the domain).

Clearly $\eta(\sigma', t)$ is a geodesic and (see Fact 5)

$$\psi(\eta(\sigma', t); \dot{\eta}(\sigma', t)) \equiv \psi(\eta(\sigma', 1); \dot{\eta}(\sigma', 1)) = \psi(e_n; (\sigma', \tau(\sigma')) = 1\,.$$

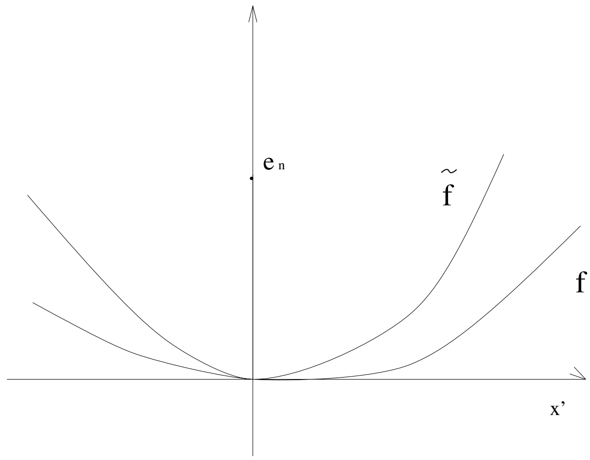

FIGURE 4.2

Applying $\partial/\partial\sigma_\alpha$ to the geodesic equations and setting $\sigma' = 0$, we have, by our special coordinates,

$$\varphi_{\xi^\beta\xi^\gamma}(te_n; e_n)\eta^\gamma_{\sigma_\alpha}(0', t) \equiv \frac{d}{dt}\left(\varphi_{\upsilon^\beta\upsilon^\gamma}(te_n; e_n)\dot{\eta}^\gamma_{\sigma_\alpha}(0', t)\right), \quad 0 \le t \le 1 .$$

We remark that $(\varphi_{\upsilon^\beta\upsilon^\gamma}(te_n; e_n)) = \frac{1}{2}(\psi_{\upsilon^\beta\upsilon^\gamma}(te_n; e_n))$ is positive definite and

$$\eta^\gamma(0', 1) = 0 , \quad \dot{\eta}^\gamma_{\sigma_\alpha}(0', 1) = \delta^\gamma_\alpha .$$

With the aid of Lemma 4.3, one sees that $\{\eta_{\sigma_1}(0', 0), \ldots, \eta_{\sigma_{n-1}}(0', 0)\}$ are linearly independent.

By compactness, for some positive number $\delta > 0$, depending only on f and φ, we have

$$\det(\eta_{\sigma_1}(0', 0), \ldots, \eta_{\sigma_{n-1}}(0', 0)) \ge c > 0 . \tag{4.9}$$

Let

$$x_\alpha = \eta^\alpha(\sigma_1, \ldots, \sigma_{n-1}, 0) , \quad 1 \le \alpha \le n-1 .$$

We know from the above, using the implicit function theorem, that the map σ' to x' is a diffeomorphism in a fixed neighborhood of $0'$ (the size of the neighborhood depends only on f and φ).

Define

$$\tilde{f}(x_1, \ldots, x_{n-1}) = \eta^n(\sigma_1, \ldots, \sigma_{n-1}, 0) ;$$

see Figure 4.2. Then for some positive constants $\tilde{\epsilon}_1 0$ and C, depending only on f and φ, we have

$$\|\tilde{f}\|_{C^{2,1}(B_{\tilde{\epsilon}_1})} \le C . \tag{4.10}$$

In fact, the parameter sphere we have constructed is a distance sphere near the origin; i.e., for possibly a smaller positive constant $\tilde{\epsilon}_1$, still depending only on f and φ, we have

$$\text{dist}((x', \tilde{f}(x')) \text{ to } e_n) = 1\,, \quad |x'| < \tilde{\epsilon}_1\,. \tag{4.11}$$

Indeed, if the above does not hold for any $\tilde{\epsilon}_1$, then there exist $x_i' \to 0$ such that

$$b_i := \text{dist}((x_i', \tilde{f}(x_i')) \text{ to } e_n) < 1\,.$$

It may appear that the above statement is negating (4.11) for $\tilde{\epsilon}_1$, which depends on the initial base point we pick (the origin), but this can be taken care by an easy compactness argument.

Let ζ_i be a shortest geodesic, with unit speed, joining $(x_i', \tilde{f}(x_i'))$ to e_n. We know that $e_n = \zeta_i(b_i)$. After passing to a subsequence, $b_i \to b \le 1$, $\zeta_i \to \zeta$ in the C^1-norm. Clearly ζ is a geodesic with unit speed, $\zeta(0) = 0$, $\zeta(b) = e_n$. Since

$$\text{dist}(0 \text{ to } e_n) = 1\,,$$

we have $b \ge \text{dist}(0 \text{ to } e_n) = 1$. Since we also know $b \le 1$, we find $b = 1$. Now we know that $\text{dist}(0 \text{ to } \zeta(1)) = \text{dist}(\partial\Omega \text{ to } \zeta(1))$, and we find, by Lemma 2.2, that $\zeta(t)$ is normal to $\partial\Omega$ at the origin. Since ζ must enter Ω (otherwise it would not realize the distance of $\partial\Omega$ to $\zeta(1)$), $\zeta(t) \equiv \xi(0', t) \equiv te_n$. Thus $\dot{\zeta}_i(b_i)$ is, for large i, close to e_n, and the geodesics ζ_i comes from the spreading geodesics from e_n we have constructed; i.e., for some $\sigma_i' \to 0'$,

$$\zeta_i(t) \equiv \eta(\sigma_i', t + 1 - b_i)\,.$$

On the other hand, we know that $\zeta_i(0)$ is on the graph of $\tilde{f}$, so $\eta(\sigma_i', 1 - b_i)$ is on the graph of $\tilde{f}$. It follows that $b_i = 1$, a contradiction. (4.11) is established.

Summarizing the above, we have established the following:

LEMMA 4.6 *Under the hypotheses stated at the beginning of Section* 4*, though assuming* $\tilde{m}(0) = e_n$ *instead of* $m(0) = e_n$*, there exists a smooth function* $\tilde{f}$ *satisfying* (4.10) *and* (4.11) *for some positive constants* $\tilde{\epsilon}_1$ *and* C *depending only on* f *and* φ.

Remark 4.7. The distance sphere centered at $\tilde{m}(0) = e_n$ can be constructed the same way with the center being any point before $\tilde{m}(0)$, i.e., with center $\hat{t}e_n$ for any $0 < \hat{t} < 1$, though in this case, the $\tilde{\epsilon}_1$ also depends on the positive lower bound of $\hat{t}$.

Remark 4.8. Clearly, under the assumption of Lemma 4.6,

$$\tilde{f}(x') - f(x') \ge 0\,, \quad |x'| < \tilde{\epsilon}_1\,,$$
$$\tilde{f}(0') = 0\,, \quad \tilde{f}_{x_\alpha}(0') = 0\,, \quad 1 \le \alpha \le n-1\,.$$

Let λ denote the smallest eigenvalue of $(\tilde{f}_{x_\alpha x_\beta}(0') - f_{x_\alpha x_\beta}(0'))$; we know that $\lambda \geq 0$.

We may carry out the above for points X near e_n instead of for e_n only. Indeed, for X close to e_n and for small σ', let $\tau = \tau(\sigma', X)$ be defined by

$$\varphi(X; (\sigma', \tau)) = 1 \quad \text{and} \quad \tau(0', 0) = 1 .$$

τ is a smooth function of σ' and X.

Let $\eta = \eta(\sigma', X, t)$ be the unique smooth solution of

$$\psi_{\xi^i}(\eta; \dot{\eta}) = \frac{d}{dt}\psi_{v^i}(\eta; \dot{\eta}) , \quad t \leq 1 ,$$

satisfying

$$\eta(\sigma', X, 1) = X , \quad \dot{\eta}(\sigma', X, 1) = (\sigma', \tau(\sigma', X)) .$$

Because of (4.9), there exists some positive constant ϵ such that for every $|t| \leq \epsilon$ and $|X - e_n| \leq \epsilon$, $\{\eta(\,\cdot\,, X, t)\}$ is locally represented as a graph, and the gradient and Hessian of the function representing the graph converges to those of $\tilde{f}$ as ϵ tends to 0.

Let us still assume that $\tilde{m}(0) = e_n$. Then $\tilde{f}$ is defined by Lemma 4.6, with the nonnegative least eigenvalue λ of $(\tilde{f}_{x_\alpha x_\beta}(0') - f_{x_\alpha x_\beta}(0'))$. For $0 < \epsilon < \frac{1}{2}$, an application of Lemma 4.6 together with Remark 4.7 yields a smooth function $\tilde{f}^{(\epsilon)}$ satisfying, for some constants $\delta, C > 0$ depending only on φ and f,

$$\operatorname{dist}((x', \tilde{f}^{(\epsilon)}(x')) \text{ to } (1-\epsilon)e_n) = 1 - \epsilon , \quad |x'| < \delta ,$$

$$\tilde{f}^{(\epsilon)}(0') = 0 , \quad \|\tilde{f}^{(\epsilon)}\|_{C^{2,1}(B_\delta)} \leq C ,$$

and, by the triangle inequality for the Finsler metric,

$$\tilde{f}^{(\epsilon_2)}(x') \geq \tilde{f}^{(\epsilon_1)}(x') \geq \tilde{f}(x') \quad \forall |x'| < \delta, \ 0 < \epsilon_1 < \epsilon_2 < \tfrac{1}{2} .$$

Consequently,

$$\tilde{f}^{(\epsilon)}_{x_\alpha}(0') = 0 , \quad 1 \leq \alpha \leq n-1 .$$

Let $\lambda^{(\epsilon)}$ denote the least eigenvalue of $(\tilde{f}^{(\epsilon)}_{x_\alpha x_\beta}(0') - f_{x_\alpha x_\beta}(0'))$, and let $\gamma^{(\epsilon)} \geq 0$ be the least eigenvalue of $(\tilde{f}^{(\epsilon)}_{x_\alpha x_\beta}(0') - \tilde{f}_{x_\alpha x_\beta}(0'))$. Clearly,

$$\lambda^{(\epsilon)} \geq \lambda + \gamma^{(\epsilon)} .$$

LEMMA 4.9 *Assuming $\tilde{m}(0) = e_n$. For $0 < \epsilon < \frac{1}{2}$, let $\gamma^{(\epsilon)}$, $\lambda^{(\epsilon)}$, and λ be as above. Then for some constant $c > 0$, depending only on f and φ, we have*

$$\lambda^{(\epsilon)} - \lambda \geq \gamma^{(\epsilon)} \geq c\epsilon .$$

PROOF: Let, as usual, $\tilde{\xi}(x', t)$ denote the geodesics, with unit speed, starting from $(x', \tilde{f}(x'))$ and "normal" to the graph of $\tilde{f}$. By the property of $\tilde{f}$, $\tilde{\xi}(x', 1) = e_n$. Similarly, let $\tilde{\xi}^{(\epsilon)}(x', t)$ denote the geodesics for $\tilde{f}^{(\epsilon)}$ instead of for $\tilde{f}$. Let $\zeta^{(\epsilon)}$

be a unit eigenvector of $(\tilde{f}^{(\epsilon)}_{x_\alpha x_\beta}(0') - \tilde{f}_{x_\alpha x_\beta}(0'))$ associated with the least eigenvalue $\gamma^{(\epsilon)}$, and let x' be a multiple of $\zeta^{(\epsilon)}$, we find

$$|\tilde{\xi}(x',t) - \tilde{\xi}^{(\epsilon)}(x',t)| \le C(\gamma^{(\epsilon)}|x'| + |x'|^2) \quad \forall 0 \le t \le 1\,. \tag{4.12}$$

For $t = 1-\epsilon$, $\tilde{\xi}^{(\epsilon)}(x', 1-\epsilon) = (1-\epsilon)e_n$, and therefore

$$\begin{aligned} \tilde{\xi}(x', 1-\epsilon) - \tilde{\xi}^{(\epsilon)}(x', 1-\epsilon) &= \tilde{\xi}(x', 1-\epsilon) - e_n + \epsilon e_n \\ &= \tilde{\xi}(x', 1) - \dot{\tilde{\xi}}(x', 1)\epsilon + O(|x'|\epsilon^2) - e_n + \epsilon e_n \\ &= \epsilon(e_n - \dot{\tilde{\xi}}(x', 1)) + O(|x'|\epsilon^2)\,. \end{aligned} \tag{4.13}$$

In the above, we have used, as usual, Taylor expansions and the fact that $\ddot{\tilde{\xi}}(0', t) \equiv 0$.

Since $\tilde{\xi}(x', \cdot\,)$ satisfies the geodesic equations, and since $\tilde{\xi}(0', 1) = \tilde{\xi}(x', 1) = \dot{\xi}(0', 1) = e_n$, we have, for some positive constants a and b, depending only on f and φ, that

$$|e_n - \dot{\tilde{\xi}}(x', 1)| = |\dot{\tilde{\xi}}(0', 1) - \dot{\tilde{\xi}}(x', 1)| \ge b|\tilde{\xi}(0', 0) - \tilde{\xi}(x', 0)| \ge a|x'|.$$

This, together with (4.12) and (4.13), yields

$$a\epsilon|x'| \le C(|x'|\epsilon^2 + \gamma^{(\epsilon)}|x'| + |x'|^2)\,.$$

Dividing the above by $|x'|$ and sending $|x'|$ to 0, we find

$$a\epsilon \le C\epsilon^2 + C\gamma^{(\epsilon)}\,.$$

The desired estimate follows if $C\epsilon \le \frac{a}{2}$. If $C\epsilon > \frac{a}{2}$, the desired estimate follows from the estimate for $\epsilon = \frac{a}{2C}$ and the monotonicity of $\gamma^{(\epsilon)}$ in ϵ. □

4.5

To establish $m = \tilde{m}$, we need, in addition to Lemma 4.2, the following:

LEMMA 4.10 *If* $\tilde{m}(0) = e_n$, *then* $te_n \in G$ *for all* $0 < t < 1$.

A consequence of Lemma 4.2 and Lemma 4.10 is the following:

COROLLARY 4.11 $m(y) = \tilde{m}(y)$ *for all* $y \in \partial\Omega$. *Consequently,* $\Sigma = \tilde{\Sigma}$.

PROOF OF LEMMA 4.10: We argue by contradiction. Suppose that $m(0) = (1-\epsilon)e_n \in \Sigma$ for some $0 < \epsilon < 1$. Clearly $1 - \epsilon > \bar{\epsilon} > 0$ for some $\bar{\epsilon}$ depending only on f and φ. Since $(1-\epsilon)e_n \in \Sigma$, there exist $X_i \to (1-\epsilon)e_n$, $z_i, \hat{z}_i \in \partial\Omega$, $z_i \ne \hat{z}_i$, such that

$$b_i := \text{dist}(\partial\Omega \text{ to } X_i) = \text{dist}(z_i \text{ to } X_i) = \text{dist}(\hat{z}_i \text{ to } X_i)\,.$$

After passing to a subsequence, we may assume that $z_i \to z$, $\hat{z}_i \to \hat{z}$, and $b_i \to b$. Clearly

$$b = \text{dist}(\partial\Omega \text{ to } (1-\epsilon)e_n) = 1 - \epsilon$$

and

$$\operatorname{dist}(z \text{ to } (1-\epsilon)e_n) = \operatorname{dist}(\hat{z} \text{ to } (1-\epsilon)e_n) = \operatorname{dist}(0 \text{ to } (1-\epsilon)e_n) = 1-\epsilon\,.$$

Since $\tilde{m}(0) = e_n$ and $1-\epsilon < 1$, there can only be one point on $\partial\Omega$ that realizes $\operatorname{dist}(\partial\Omega \text{ to } (1-\epsilon)e_n)$. So we must have $z = \hat{z} = 0$. Write

$$z_i = (x_i', f(x_i'))\,, \quad \hat{z}_i = (\hat{x}_i', f(\hat{x}_i'))\,,$$

and let ζ_i and $\hat{\zeta}_i$ be shortest geodesics, with unit speed, joining z_i and $\hat{z}_i$, respectively, to X_i. By Lemma 2.2, $\zeta_i \equiv \xi(x_i', \cdot)$ and $\hat{\zeta}_i \equiv \xi(\hat{x}_i', \cdot)$. So, $\zeta_i \to \xi(0', \cdot)$ and $\hat{\zeta}_i \to \xi(0', \cdot)$ in the C^1-norm. It follows that $\dot{\zeta}_i(b_i) \to e_n$ and $\dot{\hat{\zeta}}_i(b_i) \to e_n$. Therefore, there exist $\sigma_i', \hat{\sigma}_i' \to 0'$ such that

$$\zeta_i(t) \equiv \eta(\sigma_i', X_i, t+1-b_i)\,, \quad \hat{\zeta}_i(t) \equiv \eta(\hat{\sigma}_i', X_i, t+1-b_i)\,,$$

where $\eta(\sigma', X, t)$ are the spreading geodesics we have constructed. In particular,

$$\eta(\sigma_i', X_i, 1-b_i) = \zeta_i(0) = (x_i', f(x_i'))\,,$$
$$\eta(\hat{\sigma}_i', X_i, 1-b_i) = \hat{\zeta}_i(0) = (\hat{x}_i', f(\hat{x}_i'))\,.$$

Let $\tilde{f}^i$ denote the function whose graph is the parameter sphere given by $\eta(\,\cdot\,, X_i, 1-b_i)$. Then, by the previous arguments, $\tilde{f}^i$, $\nabla\tilde{f}^i$, and the Hessian converge to $\tilde{f}^{(\epsilon)}$, etc., in a fixed neighborhood of $0'$. Thus, by Lemma 4.9, for some $\delta' > 0$ independent of i,

$$(\tilde{f}^i - f)(x') \geq 0\,, \quad ((\tilde{f}^i - f)_{x_\alpha x_\beta}(x')) > 0\,, \quad \forall |x'| < \delta'\,,$$

for large i. On the other hand,

$$(\tilde{f}^i - f)(x_i') = (\tilde{f}^i - f)(\hat{x}_i') = 0\,, \quad x_i' \to 0',\ \hat{x}_i' \to 0',\ x_i' \neq \hat{x}_i'\,.$$

This is impossible. Lemma 4.10 is established. □

We assume that $m(0) = \tilde{m}(0) = e_n$. Let $\tilde{f}$ be the one given by Lemma 4.6. Recall that $\lambda \geq 0$ is the smallest eigenvalue of $(\tilde{f}_{x_\alpha x_\beta}(0') - f_{x_\alpha x_\beta}(0'))$.

LEMMA 4.12 *Suppose $m(0) = e_n$ and $\lambda > 0$. Then there is a point $Q \neq 0$ on $\partial\Omega$ whose distance to $e_n = 1$.*

PROOF: Since $m(0) = e_n$, there is a sequence of points $X_i \to e_n$, and $Q_i, \hat{Q}_i \in \partial\Omega$, $Q_i \neq \hat{Q}_i$, such that

$$b_i := \operatorname{dist}(\partial\Omega \text{ to } X_i) = \operatorname{dist}(Q_i \text{ to } X_i) = \operatorname{dist}(\hat{Q}_i \text{ to } X_i)\,.$$

Passing to a subsequence, $Q_i \to Q$, $\hat{Q}_i \to \hat{Q}$, $b_i \to \operatorname{dist}(\partial\Omega \text{ to } e_n) = 1$. Clearly $\operatorname{dist}(Q \text{ to } e_n) = \operatorname{dist}(\hat{Q} \text{ to } e_n) = 1$. If either Q or $\hat{Q}$ is not 0, we are done. Otherwise, $Q = \hat{Q} = 0$, and we write

$$Q_i = (x_i', f(x_i'))\,, \quad \hat{Q}_i = (\hat{x}_i', f(\hat{x}_i'))\,,$$

and let ζ_i and $\hat{\zeta}_i$ be shortest geodesics, with unit speed, joining Q_i and $\hat{Q}_i$, respectively, to X_i. By Lemma 2.2, $\zeta_i \equiv \xi(x_i', \cdot)$ and $\hat{\zeta}_i \equiv \xi(\hat{x}_i', \cdot)$. So $\zeta_i \to \xi(0', \cdot)$

and $\hat{\zeta}_i \to \xi(0', \cdot)$ in the C^1-norm. It follows that $\dot{\zeta}(b_i) \to e_n$ and $\dot{\hat{\zeta}}_i(b_i) \to e_n$. Therefore, there exists $\sigma_i', \hat{\sigma}_i' \to 0$ such that

$$\zeta_i(t) \equiv \eta(\sigma_i', X_i, t+1-b_i), \quad \hat{\eta}(\hat{\sigma}_i', X_i, t+1-b_i).$$

In particular,

$$\eta(\sigma_i', X_i, 1-b_i) = \zeta_i(0) = (x_i', f(x_i')),$$
$$\eta(\hat{\sigma}_i', X_i, 1-b_i) = \hat{\zeta}_i(0) = (\hat{x}_i', f(\hat{x}_i')).$$

Let $\tilde{f}^i$ denote the function whose graph is the parameter sphere given by $\eta(\cdot, X_i, 1-b_i)$; then the Hessian of $\tilde{f}^i$ converges to the Hessian of $\tilde{f}$ in a fixed neighborhood of $0'$. Thus, since $\lambda > 0$, there exists some $\delta' > 0$ independent of i such that

$$(\tilde{f}^i - f)(x') \geq 0, \quad ((\tilde{f}^i - f)_{x_\alpha x_\beta}(x')) > 0, \quad \forall |x'| < \delta',$$

for large i. On the other hand,

$$(\tilde{f}^i - f)(x_i') = (\tilde{f}^i - f)(\hat{x}_i') = 0, \quad x_i' \to 0,\ \hat{x}_i' \to 0,\ x_i' \neq \hat{x}_i'.$$

This is impossible. Lemma 4.12 is established. □

4.6

In this section we show that $m(0)$ is a conjugate point if and only if $\lambda = 0$. Since we never apply this result, the reader may choose to skip it.

LEMMA 4.13 *Suppose $m(0) = e_n$, and suppose $\lambda > 0$. Then e_n is not a conjugate point of* 0 *along the normal geodesic $\{te_n : 0 \leq t \leq 1\}$, as described in Section* 1.1.

PROOF: We first prove that

(4.14) 0 is an isolated point in $\{y \in \partial\Omega : \operatorname{dist}(y \text{ to } e_n) = \operatorname{dist}(\partial\Omega \text{ to } e_n)\}$.

We argue by contradiction. Suppose that for some $x_i' \to 0'$, $x_i' \neq 0'$, we have

$$\operatorname{dist}((x_i', f(x_i')) \text{ to } e_n) = \operatorname{dist}(\partial\Omega \text{ to } e_n) = 1.$$

Let ζ_i be a shortest geodesic, with unit speed, joining $(x_i', f(x_i'))$ to e_n; then, by Lemma 2.2, $\zeta_i \equiv \xi(x_i', \cdot)$. So $\zeta_i \to \xi(0', \cdot)$ in the C^1-norm, and in particular, $\dot{\zeta} \to e_n$ in the C^0-norm. Since $\lambda > 0$, $\tilde{f} > f$ near $0'$, and therefore, for some $t_i > 0$, $t_i \to 0$, we find $\zeta_i(t_i)$ on the graph of $\tilde{f}$. By Lemma 4.6, the graph of $\tilde{f}$ is the distance sphere near the origin, so $\operatorname{dist}(\zeta_i(t_i) \text{ to } e_n) = 1$. On the other hand, since ζ_i is a shortest geodesic with unit speed,

$$1 = t_i + (1 - t_i) = t_i + \operatorname{dist}(\zeta_i(t_i) \text{ to } e_n).$$

This leads to a contradiction. We have thus verified (4.14).

Property (4.14) implies that e_n cannot be a conjugate point. Indeed, if e_n is a conjugate point, then by (4.14) we may enlarge Ω without changing $\partial\Omega$ near the origin, so that $\operatorname{dist}(\partial\Omega \text{ to } e_n)$ is realized only at 0. For this larger Ω, e_n is still a conjugate point, and we still have $m(0) = e_n$ for the new Ω. In the following we still use Ω to denote the new one. Since e_n does not belong to G, there exist $X_i \to e_n$, $y_i \neq z_i$, $y_i, z_i \in \partial\Omega$, such that

$$b_i := \text{dist}(y_i \text{ to } X_i) = \text{dist}(z_i \text{ to } X_i) = \text{dist}(\partial\Omega \text{ to } X_i)\,.$$

Passing to a subsequence, $z_i \to z$, $y_i \to y$, and $b_i \to b = 1$. Since 0 is the only point on $\partial\Omega$ that realizes dist($\partial\Omega$ to e_n), we must have $y = z = 0$. Write

$$y_i = (x_i', f(x_i')), \quad z_i = (\hat{x}_i', f(\hat{x}_i'))\,,$$

then $x_i' \neq \hat{x}_i'$. As usual, $\xi(x_i', \cdot)$ is a shortest geodesic joining y_i to X_i, $\xi(x_i', 0) = y_i, \xi(x_i', b_i) = X_i$. Similarly, $\xi(\hat{x}_i', \cdot)$ is a shortest geodesic joining z_i to X_i, $\xi(\hat{x}_i', 0) = z_i, \xi(\hat{x}_i', b_i) = X_i$. We also know, as usual, for some σ_i' and $\hat{\sigma}_i'$,

$$\xi(x_i', t) \equiv \eta(\sigma_i', X_i, t + 1 - b_i)\,,$$
$$\xi(\hat{x}_i', t) \equiv \eta(\hat{\sigma}_i', X_i, t + 1 - b_i)\,.$$

Let $\tilde{f}^i$ be the function whose graph is $\eta(\,\cdot\,, X_i, 1 - b_i)$. We argue as before: the Hessian of $\tilde{f}^i$ converges to the Hessian of $\tilde{f}$ in a fixed neighborhood of $0'$. Since $\lambda > 0$, $(\tilde{f}^i - f)$ is strictly convex in a fixed neighborhood of $0'$, but we know that $\tilde{f}^i - f \geq 0$, $(\tilde{f}^i - f)(x_i') = (\tilde{f}^i - f)(\hat{x}_i') = 0$, $x_i' \neq \hat{x}_i'$, $x_i' \to 0$, and $\hat{x}_i' \to 0$. This is a contradiction. Lemma 4.13 is established. □

LEMMA 4.14 *Suppose $m(0) = e_n$, and suppose $\lambda = 0$. Then e_n is a conjugate point.*

A consequence of Lemmas 4.13 and 4.14 is the following:

COROLLARY 4.15 *Suppose $m(0) = e_n$. Then e_n is a conjugate point if and only if $\lambda = 0$.*

We present two proofs of Lemma 4.14; the second one is more traditional.

FIRST PROOF OF LEMMA 4.14: Let ζ be a unit eigenvector of $(\tilde{f}_{x_\alpha x_\beta} - f_{x_\alpha x_\beta})(0')$ associated with the least eigenvalue $\lambda = 0$, and let $x' \neq 0'$ be a multiple of ζ. Then

$$|(\tilde{f} - f)(x')| \leq C|x'|^3$$

and therefore

$$\text{dist}((x', f(x')) \text{ to } (x', \tilde{f}(x')) \leq C|x'|^3\,.$$

Let $s = \delta|x'|$ for some $\delta > 0$. We will fix some small $\delta > 0$, independent of x', and show, for small $|x'| > 0$, that

$$\text{dist}((x', f(x')) \text{ to } (1 + s)e_n) < 1 + s\,. \tag{4.15}$$

In fact, we will produce a curve joining $(x', f(x'))$ to $(1 + s)e_n$ in small neighborhood of $\{te_n : 0 \leq t \leq (1 + s)\}$ that has length less than $1 + s$. This means that e_n is a conjugate point.

By the triangle inequality, and using $\tilde{\xi}(x', 0) = (x', \tilde{f}(x'))$,

$$\begin{aligned}
&\operatorname{dist}((x', f(x')) \text{ to } (1+s)e_n) \\
&\quad \le \operatorname{dist}((x', f(x')) \text{ to } (x', \tilde{f}(x'))) + \operatorname{dist}((x', \tilde{f}(x')) \text{ to } \tilde{\xi}(x', 1-s)) \\
&\qquad + \operatorname{dist}(\tilde{\xi}(x', 1-s) \text{ to } (1+s)e_n) \\
&\quad \le C|x'|^3 + (1-s) + \operatorname{dist}(\tilde{\xi}(x', 1-s) \text{ to } (1+s)e_n)\,. \qquad (4.16)
\end{aligned}$$

Since $\tilde{\xi}(x', \cdot\,)$ and $\tilde{\xi}(0', \cdot\,)$ satisfy the same geodesic equations, we have

$$\begin{aligned}
|\tilde{\xi}(x', 0) - \tilde{\xi}(0', 0)| &\le C|\tilde{\xi}(x', 1) - \xi(0', 1)| + C|\dot{\tilde{\xi}}(x', 1) - \dot{\tilde{\xi}}(0', 1)| \\
&= C|\dot{\tilde{\xi}}(x', 1) - e_n|\,.
\end{aligned}$$

It follows, for some $c > 0$ depending only on f and φ, that

$$|\bar{e} - e_n| \ge c|x'| \quad \text{where } \bar{e} := \dot{\tilde{\xi}}(x', 1)\,. \qquad (4.17)$$

Now, a crucial point: since $\varphi(e_n; \bar{e}) = \varphi(\tilde{\xi}(x', 1); \dot{\tilde{\xi}}(x', 1)) = \varphi(e_n; e_n) = 1$, by the strict convexity hypothesis on ψ, we have for some $\hat{c}_0 > 0$ depending only on φ that

$$\varphi\left(e_n; \frac{e_n + \bar{e}}{2}\right) \le 1 - \hat{c}_0|e_n - \bar{e}|\,. \qquad (4.18)$$

Let

$$\eta(t) = (1-t)\tilde{\xi}(x', 1-s) + t(1+s)e_n\,, \quad 0 \le t \le 1\,,$$

be the straight segment joining $\tilde{\xi}(x', 1-s)$ to $(1+s)e_n$. Then, since $\tilde{\xi}(x', 1) = e_n$,

$$\eta(t) = e_n + O(s)\,.$$

Here and below $O(s)$ denotes some quantity that is bounded in absolute value by Cs for some constant C independent of x' and s.

Using $\ddot{\tilde{\xi}}(0', \cdot\,) \equiv 0$,

$$\begin{aligned}
\dot{\eta}(t) &= (1+s)e_n - \tilde{\xi}(x', 1-s) \\
&= (1+s)e_n - [\tilde{\xi}(x', 1) - \dot{\tilde{\xi}}(x', 1)s + O(|x'|s^2) \\
&= se_n + \dot{\tilde{\xi}}(x', 1)s + O(|x'|s^2) \\
&= s(e_n + \bar{e}) + O(|x'|s^2)\,.
\end{aligned}$$

It follows, using properties of our special coordinates and the homogeneity of φ in v, and making Taylor expansions, that

$$\begin{aligned}\operatorname{dist}(\tilde{\xi}(x', 1-s) \text{ to } (1+s)e_n) &\leq \int_0^1 \varphi(\eta; \dot{\eta})dt \\ &= s\int_0^1 \varphi(e_n + O(s); e_n + \bar{e} + O(|x'|s))dt \\ &= s(\varphi(e_n; e_n + \bar{e}) + O(s)),\end{aligned}$$

and therefore, by (4.17) and (4.18),

$$\operatorname{dist}(\tilde{\xi}(x', 1-s) \text{ to } (1+s)e_n) \leq 2s(1 - \hat{c}_0|x'| + O(s)),$$

where $\hat{c}_0 > 0$ is some constant independent of x' and s.

Back to (4.16), we find

$$\begin{aligned}\operatorname{dist}((x', f(x')) \text{ to } (1+s)e_n) &\leq C|x'|^3 + (1-s) + s(2 - \hat{c}_0|x'| + O(s)) \\ &= 1 + s - s(2\hat{c}_0|x'| + O(s)) + C|x'|^3.\end{aligned}$$

Now we fix some $\delta > 0$ from the beginning so that $2\hat{c}_0|x'| + O(s) \geq \hat{c}_0|x'|$, then for $|x'| > 0$ small, we obtain

$$\operatorname{dist}((x', f(x')) \text{ to } (1+s)e_n) \leq 1 + s - \hat{c}_0\delta|x'|^2 + C|x'|^3 < 1 + s,$$

estimate (4.15). It is clear that we have actually produced a curve in small neighborhood of $\{te_n : 0 \leq t \leq 1+s\}$ joining $(x', f(x'))$ to $(1+s)e_n$ with length less than $1+s$. Lemma 4.14 is established. □

SECOND PROOF OF LEMMA 4.14:

(i) Consider the spreading geodesics $\eta(\sigma', t)$. Since $\tilde{f} \geq f$, for small σ', there exists a unique $\bar{t}(\sigma') \leq 0$ such that $\eta(\sigma', \bar{t}(\sigma'))$ lies on $\partial\Omega$, i.e.,

$$\eta^n(\sigma', \bar{t}(\sigma')) = f(\eta'(\sigma', \bar{t}(\sigma'))). \tag{4.19}$$

The function $\bar{t}(\sigma')$ is a C^2-function in σ'. The curve $\{\eta(\sigma', t) : \bar{t}(\sigma') \leq t \leq 1\}$ has length

$$L(\sigma') = 1 - \bar{t}(\sigma'). \tag{4.20}$$

We also have

$$\eta^n(\sigma', 0) = \tilde{f}(\eta'(\sigma', 0)). \tag{4.21}$$

Differentiating (4.19) with respect to σ_α, we find

$$\eta^n_{\sigma_\alpha} + \dot{\eta}^n \bar{t}_{\sigma_\alpha} = f_{x_\gamma}\left(\eta^\gamma_{\sigma_\alpha} + \dot{\eta}^\gamma \bar{t}_{\sigma_\alpha}\right).$$

Differentiate with respect to σ_β, and set $\sigma' = 0'$. We get at $\sigma' = 0'$,

$$\eta^n_{\sigma_\alpha\sigma_\beta} + \bar{t}_{\sigma_\alpha\sigma_\beta} = f_{x_\gamma x_\delta}(0')\eta^\gamma_{\sigma_\alpha}\eta^\delta_{\sigma_\beta} \quad \text{at } (0', 0), \tag{4.22}$$

since, when $\sigma' = 0'$, $\eta^n_{\sigma_\alpha} = 0$ (following from $\tilde{f}_{x_\beta}(0') = 0$), and so, $\bar{t}_{\sigma_\alpha}(0') = 0$.

Similarly, from (4.21), we find

$$\eta^n_{\sigma_\alpha\sigma_\beta} = \tilde{f}_{x_\gamma x_\delta}\eta^\gamma_{\sigma_\alpha}\eta^\delta_{\sigma_\beta} \quad \text{at } (0', 0)\,, \tag{4.23}$$

so,

$$\bar{t}_{\sigma_\alpha\sigma_\beta}(0') = (f_{x_\gamma x_\delta}(0') - \tilde{f}_{x_\gamma x_\delta}(0'))\eta^\gamma_{\sigma_\alpha}(0', 0)\eta^\delta_{\sigma_\beta}(0', 0)\,. \tag{4.24}$$

Suppose, now, $\lambda = 0$. Then there is a unit vector $\hat{\zeta} = (\hat{\zeta}^1, \dots, \hat{\zeta}^{n-1})$ such that

$$(f_{x_\gamma x_\delta}(0') - \tilde{f}_{x_\gamma x_\delta}(0'))\hat{\zeta}^\delta = 0\,, \quad 1 \le \gamma \le n-1\,. \tag{4.25}$$

The matrix $\{\eta^\gamma_{\sigma_\alpha}(0', 0)\}$ is nonsingular. Choose $a = (a_1, \dots, a_{n-1})$ so that

$$a_\alpha \eta^\delta_\alpha(0', 0) = \hat{\zeta}^\delta\,. \tag{4.26}$$

Inserting this in (4.25), we find, by (4.24),

$$a_\alpha a_\beta \bar{t}_{\sigma_\alpha\sigma_\beta}(0') = 0\,. \tag{4.27}$$

(ii) Now the second variation. For $0 \le t \le 1$, te_n is the shortest connection from $\partial\Omega$ to e_n. For $\zeta(t)$ small, $0 \le t \le 1$, we consider the perturbation $te_n + \zeta(t)$. Here $\zeta(t) = 0$ and $\zeta(0) \in \partial\Omega$, i.e.,

$$\zeta^n(0) = f(\zeta'(0)) = \frac{1}{2} f_{x_\gamma x_\delta}(0')\zeta^\gamma(0)\zeta^\delta(0) + O(|\zeta'(0)|^3)\,. \tag{4.28}$$

The length of the curve $te_n + \zeta$ is, by the properties of our special coordinates,

$$\begin{aligned}
&\int_0^1 \varphi(te_n + \zeta; e_n + \dot{\zeta})dt \\
&\quad = 1 + \int_0^1 \left(\frac{1}{2}\varphi_{\eta^\alpha\eta^\beta}(te_n; e_n)\zeta^\alpha\zeta^\beta + \frac{1}{2}\varphi_{v^\alpha v^\beta}(te_n; e_n)\dot{\zeta}^\alpha\dot{\zeta}^\beta + \dot{\zeta}^n\right)dt \\
&\qquad + \text{higher order.}
\end{aligned}$$

So the second variation is, by (4.28),

$$\begin{aligned}
Q(\zeta') := \frac{1}{2}\int_0^1 &(\varphi_{\eta^\alpha\eta^\beta}(te_n; e_n)\zeta^\alpha\zeta^\beta + \varphi_{v^\alpha v^\beta}(te_n; e_n)\dot{\zeta}^\alpha\dot{\zeta}^\beta)dt \\
&- \frac{1}{2} f_{x_\alpha x_\beta}(0')\zeta^\alpha(0)\zeta^\beta(0)\,.
\end{aligned} \tag{4.29}$$

Now $\{\varphi_{v^\alpha v^\beta}(te_n; e_n)\}$ is positive definite, and the quadratic form $Q(\zeta')$ is positive semidefinite. If it vanishes for some $\zeta'(t)$ not identically zero, then e_n is a conjugate point by the usual argument: the second variation of the curve te_n for $0 \le t \le 1 + \epsilon$, for any $\epsilon > 0$, is not positive semidefinite.

(iii) Suppose $\lambda = 0$.

Claim 4.16 For $\zeta^\alpha(t) = a_\gamma \eta^\alpha_{\sigma_\gamma}(0', t)$, $Q(\zeta') = 0$.

This would then complete the proof of the lemma.

PROOF: From (4.20), we have

$$L_{\sigma_\alpha\sigma_\beta}(0') = -\bar{t}_{\sigma_\alpha\sigma_\beta}(0')\,. \tag{4.30}$$

Now

$$L(\sigma') = \int_{\bar{t}(\sigma')}^{1} \varphi(\eta(\sigma',t);\dot\eta(\sigma',t))dt\,,$$

and recall that $\varphi(\eta(\sigma',t);\dot\eta(\sigma',t)) \equiv 1$. So

$$L_{\sigma_\alpha} = -\bar{t}_{\sigma_\alpha} + \int_{\bar{t}}^{1} \big(\varphi_{\eta^i}\eta^i_{\sigma_\alpha} + \varphi_{v^i}\dot\eta^i_{\sigma_\alpha}\big)dt$$

and, at $\sigma' = 0'$, by properties of the special coordinates,

$$L_{\sigma_\alpha\sigma_\beta}(0') = -\bar{t}_{\sigma_\alpha\sigma_\beta} + \int_0^1 \big(\varphi_{\eta^i\eta^j}\eta^i_{\sigma_\alpha}\eta^j_{\sigma_\beta} + \varphi_{v^iv^j}\dot\eta^i_{\sigma_\alpha}\dot\eta^j_{\sigma_\beta} + \dot\eta^n_{\sigma_\alpha\sigma_\beta}\big)dt\,.$$

By (4.30), the last integral is 0. By properties of the special coordinates and by the homogeneity of φ in v, we have $\varphi_{\eta^i\eta^n}(te_n;e_n) \equiv \varphi_{v^iv^n}(te_n;e_n) \equiv 0$. Therefore we have

$$\int_0^1 (\varphi_{\eta^\gamma\eta^\delta}(te_n;e_n)\eta^\gamma_{\sigma_\alpha}(0',t)\eta^\delta_{\sigma_\beta}(0',t) \\ + \varphi_{v^\gamma v^\delta}(te_n;e_n)\dot\eta^\gamma_{\sigma_\alpha}(0',t)\dot\eta^\delta_{\sigma_\beta}(0',t) - \eta^n_{\sigma_\alpha\sigma_\beta}(0',0) = 0\,.$$

Multiplying the above by $a_\alpha a_\beta$ and summing, we find

$$\int_0^1 (\varphi_{\eta^\gamma\eta^\delta}(te_n;e_n)\zeta^\gamma\zeta^\delta + \varphi_{v^\gamma v^\delta}(te_n;e_n)\dot\zeta^\gamma\dot\zeta^\delta - \eta^n_{\sigma_\alpha\sigma_\beta}(0',0)a_\alpha a_\beta = 0\,. \tag{4.31}$$

From (4.22) and (4.27), we have

$$-\eta^n_{\sigma_\alpha\sigma_\beta}(0',0)a_\alpha a_\beta = -f_{x_\gamma x_\delta}(0')\zeta^\gamma\zeta^\delta\,.$$

Inserting this into (4.31), we obtain the claim. □

This completes the proof of the lemma. □

5 Main Estimates I

We now start the argument described in Section 1.5, with y as the origin. Without loss of generality, we may assume $\bar{s}(y) = \bar{s}(0) = 1$. Then we use our special coordinates of Section 3; near the origin Ω is given by $x_n > f(x')$ with

$$f(0') = 0\,, \quad \nabla f(0') = 0\,.$$

Then $m(y) = m(0) = e_n$. The "normal" geodesic from 0 lies along the x_n-axis.

For $|x'|$ small, as in Section 2, $\xi(x',\tau)$ is the geodesic, with τ as arc length, starting from $(x', f(x'))$ normal to $\partial\Omega$. We wish to find a point z on $\partial\Omega$ such that for $s = K|x'|$, with K a fixed large constant,

$$\operatorname{dist}(z \text{ to } \xi(x',1+s)) < 1+s\,. \tag{5.1}$$

To prove (5.1) we will follow the interior "normal" geodesic from z a distance $1-s$, then join its endpoint by a straight line segment $\eta(t)$, $0 \le t \le 1$, to $\xi(x', 1+s)$, and show that the Finsler length of η is less than $2s$.

To compute lengths we use expansions in x', s, etc.; the special coordinates make the computations easier. But things are not very easy.

For $\epsilon_0 > 0$, let $\Gamma := \{te_n : -\epsilon_0 \le t \le 1+\epsilon_0\}$ be the geodesic for $\varphi(\xi; v)$ satisfying, for $-\epsilon_0 \le t \le 1+\epsilon_0$, (4.1), (4.2), (4.3), (4.4), (4.5), and (4.6). We use notation $\xi(x', \tau)$ as in Section 4.

LEMMA 5.1 *Under the above hypotheses,*

$$\xi^n_{x_\alpha} = 0 \quad \text{at } (0', t) \quad \forall t,\ -\epsilon_0 \le t \le 1+\epsilon_0,\ 1 \le \alpha \le n-1\,, \tag{5.2}$$

and, for $|x'| \le \epsilon_1$, $-\epsilon_0 \le t \le 1+\epsilon_0$, $1 \le \alpha, \beta \le n-1$,

$$\left|\xi^n_{x_\alpha x_\beta}(x', t) + \varphi_{v^i v^j}(te_n; e_n)\dot{\xi}^i_{x_\alpha}(0', t)\xi^j_{x_\beta}(0', t)\right| \le C|x'|\,, \tag{5.3}$$

where C *depends only on* f *and* φ.

PROOF: By (2.14),

$$\xi^i_{x_\alpha}\varphi_{v^i}(\xi; \dot{\xi}) \equiv 0\,.$$

The first equality in the lemma follows easily from the above by the properties of the special coordinates. Applying ∂_{x_β} to the above, we have

$$\xi^n_{x_\alpha x_\beta}\varphi_{v^n} = -\xi^\gamma_{x_\alpha x_\beta}\varphi_{v^\gamma} - \varphi_{v^i\xi^j}\xi^i_{x_\alpha}\xi^j_{x_\beta} - \varphi_{v^i v^j}\xi^i_{x_\alpha}\dot{\xi}^j_{x_\beta}\,.$$

At $x' = 0$, using properties of the special coordinates, we have

$$\varphi_{v^n}(te_n; e_n) = 1\,, \quad \varphi_{v^\gamma}(te_n; e_n) = 0\,, \quad \varphi_{v^i\xi^j}(te_n; e_n) = 0\,,$$

and estimate (5.3) follows. □

We assume that

$$\tau = \operatorname{dist}(0 \text{ to } \tau e_n) = \min_{y\in\partial\Omega} \operatorname{dist}(y \text{ to } \tau e_n) \quad \forall \tau,\ 0 \le \tau \le 1\,,$$

and

$$\epsilon_0 = \operatorname{dist}((0', -\epsilon_0) \text{ to } 0)\,.$$

In particular,

$$\tau = \operatorname{dist}(0 \text{ to } \tau e_n) \le \operatorname{dist}((x', f(x')) \text{ to } \tau e_n) \quad \forall \tau,\ 0 \le \tau \le 1,\ |x'| \le \epsilon_1\,.$$

We now find z, for our program, in the simplest case.

PROPOSITION 5.2 *Assume that there exists* $Q \in \partial\Omega$, $|Q| \ge \hat{\epsilon} > 0$, *with*

$$\operatorname{dist}(0 \text{ to } e_n) = \operatorname{dist}(Q \text{ to } e_n)\,.$$

Then, we take $z = Q$; *i.e., there exist some large constant* $K \ge 1$ *and small constant* $0 < \hat{\delta} < \hat{\epsilon}$, *depending only on* $\hat{\epsilon}$, f, *and* φ *such that for all* $0 < |x'| \le \hat{\delta}$ *and* $s = K|x'|$ *we have*

$$\operatorname{dist}(Q \text{ to } \xi(x', 1+s)) < 1+s\,.$$

PROOF: Set $\bar{e} = \dot{\xi}(Q, 1)$. Since $\xi(Q, 1) = e_n$, and the fact that $\xi(Q, s)$ satisfies the geodesic equations, it follows that

$$|\bar{e}_n - e_n| = |\dot{\xi}(Q, 1) - e_n| \geq c_1|Q| \geq c_1\hat{\epsilon} \tag{5.4}$$

for some $c_1 > 0$ depending only on φ. We know that

$$\xi(0', 1) = e_n\,, \quad \xi(Q, 1) = e_n\,, \quad \text{and} \quad \dot{\xi}(0', 1) = e_n\,.$$

By Taylor expansion, since $|x'| = \frac{s}{K} \leq s$,

$$\begin{aligned}
&\xi(x', 1+s) = \xi(0', 1) + O(s) = e_n + O(s)\,,\\
&\xi(Q, 1-s) = \xi(Q, 1) - \dot{\xi}(Q, 1)s + O(s^2) = e_n - s\bar{e} + O(s^2)\,.
\end{aligned}$$

For the segment

$$\eta(t) := (1-t)\xi(Q, 1-s) + t\xi(x', 1+s) = e_n + O(s)\,,$$

we have

$$\begin{aligned}
\dot{\eta}(t) &= \xi(x', 1+s) - \xi(Q, 1-s)\\
&= \xi_{x_\alpha}(0', 1)x_\alpha + \dot{\xi}(0', 1)s + \dot{\xi}(Q, 1)s + O(s^2 + |x'|^2)\\
&= \xi_{x_\alpha}(0', 1)x_\alpha + (e_n + \bar{e})s + O(s^2 + |x'|^2)\,.
\end{aligned}$$

Using homogeneity, it follows that

$$\begin{aligned}
\int_0^1 \varphi(\eta(t); \dot{\eta}(t))dt = s\int_0^1 \varphi\biggl(e_n + O(s);\, &(e_n + \bar{e}) + \xi_{x_\alpha}(0', 1)\frac{x_\alpha}{s}\\
&+ O(s) + O\left(\frac{|x'|^2}{s}\right)\biggr)dt\\
&= s\varphi(e_n, e_n + \bar{e}) + O\left(\frac{s}{K} + s^2\right)\\
&= 2s\varphi\left(e_n, \frac{e_n + \bar{e}}{2}\right) + O\left(\frac{s}{K} + s^2\right).
\end{aligned}$$

Now comes a crucial point as in the proof of Lemma 4.14. Since $\varphi(e_n; e_n) = \varphi(e_n; \bar{e}_n) = 1$, by the strict convexity hypothesis on ψ, we have for some $\bar{c}_0$ depending only on φ that

$$\varphi\left(e_n; \frac{e_n + \bar{e}}{2}\right) \leq 1 - \bar{c}_0|e_n - \bar{e}_n| \leq 1 - c_0$$

with $c_0 > 0$ depending also on $\hat{\epsilon}$ by (5.4), from which we deduce, for some large K and small $\hat{\delta}$ (K chosen first and then $\hat{\delta}$), that

$$\int_0^1 \varphi(\eta(t); \dot{\eta}(t))dt \leq 2s(1 - c_0) + O\left(\frac{s}{K} + s^2\right) \leq 2s\left(1 - \frac{c_0}{2}\right) < 2s\,.$$

Consequently,

$$\begin{aligned}\operatorname{dist}(Q \text{ to } \xi(x', 1+s)) &\le \operatorname{dist}(Q \text{ to } \xi(Q, 1-s)) \\ &\quad + \operatorname{dist}(\xi(Q, 1-s) \text{ to } \xi(x', 1+s)) \\ &< 1+s\,.\end{aligned}$$

Proposition 5.2 is established. □

6 Main Estimates II

In the remaining cases we will take

$$z = (x' + q, f(x' + q))$$

for suitable choices of $q \in \mathbb{R}^{n-1}$, $|q| <$ small. In the following, the value of ϵ_0 is possibly smaller than the one appearing in Section 4.1.

We know that

$$\xi(0', \tau) = \tau e_n\,, \quad -\epsilon_0 \le \tau \le 1 + \epsilon_0\,.$$

For $x', x' + q \in \mathbb{R}^{n-1}$, $|x'|, |x' + q| \le \epsilon_1$, let $\eta(x', q, s; t)$, $0 \le t \le 1$, denote the straight segment going from $\xi(x' + q, 1 - s)$ to $\xi(x', 1 + s)$. We consider its length, $L(x', q, s)$, as a function of $2(n - 1) + 1$ variables, the x', q, and s being free variables (with small norms). Thus

$$\eta(x', q, s; t) = (1 - t)\xi(x' + q, 1 - s) + t\xi(x', 1 + s)\,, \quad 0 \le t \le 1\,, \tag{6.1}$$

and

$$L(x', q, s) = \int_0^1 \varphi(\eta(x', q, s; t); \dot\eta(x', q, s; t))dt \tag{6.2}$$

where the overdot denotes ∂_t.

For a suitable choice of q and with $s = K|x'|$, K large, we wish to show

$$L(x', q, s) < 2s\,.$$

The main term will be $L(0', q, s)$. Proposition 6.1 below presents a general estimate for the difference. This result is rather technical; it will be used for several cases. We stress that x', q, and s are free variables. The expression $O(|q|)$ is used to denote quantities bounded in absolute value by $C|q|$, where C depends only on f and φ.

The vector

$$A = e_n - \xi(q, 1) \tag{6.3}$$

plays an important role. Note that

$$A^j = \delta_n^j - \xi^j(q, 1) = -\xi^j_{x_\alpha}(0', 1)q_\alpha + O(|q|^2)\,. \tag{6.4}$$

PROPOSITION 6.1 *There exist $\bar\epsilon_1 \le \epsilon_1$ and $0 < \epsilon_3$, depending only on f and φ, such that $\forall x'$, q, and s satisfying $|x'|$, $|q|$, $|x'+q|$, $s \le \bar\epsilon_1$, $s > 0$, and if*

$$\frac{|A|}{s} < \frac{1}{4} \quad \text{and} \quad \frac{|x'|}{s} \le \epsilon_3\,, \tag{6.5}$$

then we have

$$\begin{aligned} J &:= L(x', q, s) - L(0', q, s) \\ &\le C|x'|^2 \left(|q| + s + \frac{|q|^2}{s}\right) + C|x'|\left(|A|\left(1 + \frac{|q|}{s}\right) + |q|^2 + s^2\right). \end{aligned} \tag{6.6}$$

PROOF: In formula (6.2), η is given by (6.1) and

$$\dot\eta = \xi(x', 1+s) - \xi(x'+q, 1-s)\,.$$

Clearly

$$\eta = e_n + O(|q| + |x'| + s)\,, \tag{6.7}$$

while

$$\begin{aligned} \dot\eta &= e_n(1+s) + O(|x'|) - \xi(q, 1-s) \\ &= e_n(1+s) - \xi(q,1) + \dot\xi(q,1)s + O(s^2) + O(|x'|)\,. \end{aligned}$$

Thus

$$\dot\eta = 2se_n + s\left(\frac{A}{s} + B\right), \tag{6.8}$$

where

$$|B| \le C_1\left(|q| + s + \frac{|x'|}{s}\right)$$

with C_1 depending only on f and φ. We now make $|B| \le \frac{1}{2}$ by choosing

$$\bar\epsilon_1 = \min\left(\epsilon_1, \frac{1}{8C_1}\right), \quad \epsilon_3 = \frac{1}{4C_1}\,. \tag{6.9}$$

In addition to (6.8) we have

$$D^k_{x'}\dot\eta = D^k_{x'}\xi(x', 1+s) - D^k_{x'}\xi(x'+q, 1-s) = O(|q|+s)\,, \quad 0 \le k \le 2\,. \tag{6.10}$$

Using Taylor expansion in x' about the origin, we have

$$\begin{aligned} J &= L(x', q, s) - L(0', q, s) \\ &= L_{x_\alpha}(0, q, s)x_\alpha + \int_0^1 \int_0^t L_{x_\alpha x_\beta}(\tau x', q, s)x_\alpha x_\beta \, d\tau\, dt\,. \end{aligned} \tag{6.11}$$

Now

$$L_{x_\alpha}(x', q, s) = \int_0^1 \left(\varphi_{\xi^i}\eta^i_{x_\alpha} + \varphi_{v^i}\dot\eta^i_{x_\alpha}\right)dt \tag{6.12}$$

and

$$L_{x_\alpha x_\beta}(x', q, s) = \int_0^1 \big[\varphi_{\xi^i}\eta^i_{x_\alpha x_\beta} + \varphi_{\xi^i\xi^j}\eta^i_{x_\alpha}\eta^j_{x_\beta} + \varphi_{\xi^i v^j}\eta^i_{x_\alpha}\dot\eta^j_{x_\beta} + \varphi_{v^i}\dot\eta^i_{x_\alpha x_\beta} + \varphi_{v^i\xi^j}\dot\eta^i_{x_\alpha}\eta^j_{x_\beta} + \varphi_{v^i v^j}\dot\eta^i_{x_\alpha}\dot\eta^j_{x_\beta}\big]dt\,.$$

By the properties of the special coordinates, at (e_n, e_n),

$$\varphi_{v^\alpha} = \varphi_{\xi^i} = \varphi_{\xi^i v^j} = \varphi_{v^n} - 1 = \varphi_{v^\alpha v^n} = \varphi_{\xi^j\xi^n} = 0\,.$$

Thus, by (6.7), (6.8), (6.10), and the homogeneity of φ in v, if we set

$$\{\quad\} = \frac{|A|}{s} + \frac{|x'|}{s} + |q| + s\,,$$

we find, at $(\eta, \dot\eta)$, that

$$|\varphi_{\xi^i}| + |\varphi_{\xi^i\xi^n}| \le Cs\{\quad\}\,,$$

$$|\varphi_{\xi^i v^j}| + |\varphi_{v^\alpha}| + |\varphi_{v^n} - 1| \le C\{\quad\}\,,$$

$$|\varphi_{v^\alpha v^n}| \le \frac{C}{s}\{\quad\}\,,$$

$$|\varphi_{\xi^i\xi^j}| + |\varphi_{\xi^i\xi^j\xi^k}| \le Cs\,,$$

$$|\varphi_{\xi^i\xi^j v^k}| \le C\,,$$

$$|\varphi_{\xi^i v^j v^k}| + |\varphi_{v^i v^j}| \le \frac{C}{s}\,,$$

$$|\varphi_{v^i v^j v^k}| \le \frac{C}{s^2}\,.$$

We deduce from the above, since $|\{\quad\}|$ is bounded, that

$$|L_{x_\alpha x_\beta}(\tau x', q, s)| \le C(|q| + s) + \frac{C}{s}(|q| + s)^2\,.$$

Consequently,

$$\left|\int_0^1\int_0^t L_{x_\alpha x_\beta}(\tau x', q, s)x_\alpha x_\beta\, d\tau\, dt\right| \le C|x'|^2\left(|q| + s + \frac{|q|^2}{s}\right). \tag{6.13}$$

Next, we estimate $L_{x_\alpha}(0', q, s)x_\alpha$. Here $x' = 0'$ in $(\eta, \dot\eta)$. By the estimates above,

$$\left|\int \varphi_{\xi^i}\eta^i_{x_\alpha}x_\alpha\right| \le C(|A| + s|q| + s^2)|x'|\,, \tag{6.14}$$

$$\left|\int \varphi_{v^\beta}\dot\eta^\beta_{x_\alpha}x_\alpha\right| \le C\left(\frac{|A|}{s} + |q| + s\right)(|q| + s)|x'|\,.$$

Write

$$\varphi_{v^n}\dot\eta^n_{x_\alpha} = \dot\eta^n_{x_\alpha} + (\varphi_{v^n} - 1)\dot\eta^n_{x_\alpha} .$$

Then, using the estimates on $(\varphi_{v^n} - 1)$ and on $|\dot\eta^n_{x_\alpha}|$, we find

$$\left| \int (\varphi_{v^n} - 1)\dot\eta^n_{x_\alpha} x_\alpha \right| \le C\left(\frac{|A|}{s} + |q| + s\right)(|q| + s)|x'| . \tag{6.15}$$

To complete the estimate of $L_{x_\alpha}(0', q, s)x_\alpha$, we need to estimate $|\dot\eta^n_{x_\alpha}(0', q, s)x_\alpha|$. Using Taylor expansion, we find

$$\begin{aligned} \dot\eta^n_{x_\alpha}(0', q, s) &= \xi^n_{x_\alpha}(0', 1) - \xi^n_{x_\alpha}(q, 1) + s\big(\dot\xi^n_{x_\alpha}(0', 1) + \dot\xi^n_{x_\alpha}(q, 1)\big) + O(s^2) \\ &= \xi^n_{x_\alpha}(0', 1) - \xi^n_{x_\alpha}(q, 1) + O(s|q| + s^2) , \end{aligned}$$

since $\dot\xi^n_{x_\alpha}(0', 1) = 0$, that follows from differentiating (5.2). Writing

$$\xi^n_{x_\alpha}(0', 1) - \xi^n_{x_\alpha}(q, 1) = -\int_0^1 \xi^n_{x_\alpha x_\beta}(\tau q, 1) q_\beta \, d\tau ,$$

we find, using (5.3), that

$$\dot\eta^n_{x_\alpha}(0', q, s) = \varphi_{v^i v^j}(e_n; e_n)\dot\xi^i_{x_\alpha}(0', 1)\xi^j_{x_\beta}(0', 1)q_\beta + O(|q|^2 + s^2) .$$

With the aid of (6.4), we see that

$$\dot\eta^n_{x_\alpha}(0', q, s) = O\left(\frac{|A|}{s}(|q| + s) + |q|^2 + s^2\right)$$

so that

$$\left| \int \varphi_{v^n}\dot\eta^n_{x_\alpha} x_\alpha \right| \le C\left(|A|\left(\frac{|q|}{s} + 1\right) + |q|^2 + s^2\right)|x'| . \tag{6.16}$$

Combining estimates (6.13), (6.14), (6.15), and (6.16), we obtain (6.6). □

7 Main Estimates III

Recalling $\bar\epsilon_1$ of Proposition 6.1, we now consider the case where there is a $\hat q$ satisfying the condition on q of Proposition 6.1 and in addition

$$1 = \text{dist}(0 \text{ to } e_n) = \text{dist}((\hat q, f(\hat q)) \text{ to } e_n) .$$

In this case the vector A of Section 6 is 0. We take

$$z = (\hat q + x', f(\hat q + x')) .$$

PROPOSITION 7.1 *Under the conditions above, there exist small positive constants $\bar\epsilon$ and $\bar\delta$ and a large constant $K > 1$, depending only on φ and f such that for $s = K|x'|$ and $0 < |x'| \le \min(\bar\delta, \bar\epsilon|\hat q|)$, we have*

$$L(x', \hat q, s) < 2s .$$

PROOF: We will apply Proposition 6.1 with $q = \hat{q}$. Since $A = 0$, we see that the conditions are satisfied provided $\frac{1}{K} \le \epsilon_3$. Then, from (6.6) we find

$$J = L(x', \hat{q}, s) - L(0', \hat{q}, s) \le Cs|\hat{q}|^2 \left(\frac{\bar{\epsilon}}{K} + \bar{\epsilon}^2 + \frac{1}{K^2} + \frac{1}{K} + \bar{\epsilon}^2 K \right). \tag{7.1}$$

We now consider the main term $L(0', \hat{q}, s)$. The estimate is technical. A crucial element, as in the proof of Proposition 5.2, is the strict convexity of $\{v : \varphi(e_n; v) = 1\}$, and the fact that

$$1 = \varphi(e_n; e_n) = \varphi(\xi(\hat{q}, 1); \dot{\xi}(\hat{q}, 1)) = \varphi(e_n; \dot{\xi}(\hat{q}, 1)) .$$

By the strict convexity it follows that for some $c_1 > 0$, depending only on φ,

$$\varphi(e_n; e_n + \dot{\xi}(\hat{q}, 1)) \le 2 - 2c_1 |\dot{\xi}(\hat{q}, 1) - e_n|^2 . \tag{7.2}$$

Since $\xi(\hat{q}, \cdot\,)$ satisfies the geodesic equations $\xi(\hat{q}, 1) = \xi(0', 1) = e_n$ and $|\xi(\hat{q}, 0) - \xi(0', 0)| = |\hat{q}|$, there are positive constants c_2 and c_3 so that

$$c_2 |\hat{q}| \le |\dot{\xi}(\hat{q}, 1) - e_n| = |\dot{\xi}(\hat{q}, 1) - \dot{\xi}(0', 1)| \le c_3 |\hat{q}| .$$

Inserting this in (7.2) we find, for some $c_0 > 0$ depending only on φ,

$$\varphi(e_n; e_n + \dot{\xi}(\hat{q}, 1)) \le 2 - 2c_0 |\hat{q}|^2 . \tag{7.3}$$

LEMMA 7.2 *There exist positive constants c_0 and C, depending only on φ such that for all $0 < s < \bar{\epsilon}_1$ and $0 < |\hat{q}| < \bar{\epsilon}_1$ above,*

$$L(0', \hat{q}, s) \le 2s(1 - c_0 |\hat{q}|^2) + C(s^4 + s^2 |\hat{q}|^2) . \tag{7.4}$$

PROOF: Let

$$\eta(t) = \eta(s, t) = (1 - t)\xi(\hat{q}, 1 - s) + t(1 + s)e_n .$$

Then

$$L(0', \hat{q}, s) = \int_0^1 \varphi(\eta(t); \dot{\eta}(t)) dt .$$

Since

$$\xi(\hat{q}, 1) = e_n \quad \text{and} \quad \dddot{\xi}(0', \tau) \equiv \frac{\partial^3}{\partial \tau^3} \xi(0', \tau) \equiv 0 ,$$

we have, by Taylor expansion, that

$$\begin{aligned} \xi(\hat{q}, 1 - s) &= e_n - \dot{\xi}(\hat{q}, 1)s + \frac{1}{2}\ddot{\xi}(\hat{q}, 1)s^2 + O(s^3 |\hat{q}|) , \\ \eta(t) &= e_n + s[te_n - (1 - t)\dot{\xi}(\hat{q}, 1)] + O(s^2 |\hat{q}|) , \\ \dot{\eta}(t) &= s[e_n + \dot{\xi}(\hat{q}, 1)] - \frac{1}{2}\ddot{\xi}(\hat{q}, 1)s^2 + O(s^3 |\hat{q}|) . \end{aligned}$$

It follows that

$$L(0', \hat{q}, s) = \int_0^1 \varphi\Big(\eta(t); se_n + s\dot{\xi}(\hat{q}, 1) - \frac{1}{2}\ddot{\xi}(\hat{q}, 1)s^2 + O(s^3|\hat{q}|)\Big)dt$$
$$= s\int_0^1 \varphi\Big(e_n + s[te_n - (1-t)\dot{\xi}(\hat{q}, 1)]; e_n + \dot{\xi}(\hat{q}, 1) - \frac{1}{2}\ddot{\xi}(\hat{q}, 1)s\Big)dt$$
$$+ O(s^3|\hat{q}|).$$

Since

$$\dot{\xi}(\hat{q}, 1) = \dot{\xi}(0', 1) + O(|\hat{q}|) = e_n + O(|\hat{q}|) \tag{7.5}$$

and

$$\ddot{\xi}(\hat{q}, \tau) = \ddot{\xi}(0', \tau) + O(|\hat{q}|) = O(|\hat{q}|) \quad \forall\tau,\ 0 \le \tau \le 1 + \epsilon_0,$$

we have, by properties of our special coordinates,

$$\varphi_{\xi^i}(e_n; e_n + \dot{\xi}(\hat{q}, 1)) = \varphi_{\xi^i}(e_n; 2e_n) + O(|\hat{q}|) = O(|\hat{q}|),$$
$$\varphi_{\xi^i v^j}(e_n; e_n + \dot{\xi}(\hat{q}, 1)) = \varphi_{\xi^i v^j}(e_n; 2e_n) + O(|\hat{q}|) = O(|\hat{q}|).$$

Making a Taylor expansion of φ about $(e_n, e_n + \dot{\xi}(\hat{q}, 1))$, we have

$$L(0', \hat{q}, s) = s\varphi(e_n; e_n + \dot{\xi}(\hat{q}, 1)) - \frac{1}{2}s^2\varphi_{v^i}(e_n; e_n + \dot{\xi}(\hat{q}, 1))\ddot{\xi}^i(\hat{q}, 1)$$
$$+ s^2\int_0^1 \varphi_{\xi^i}(e_n; e_n + \dot{\xi}(\hat{q}, 1))\big[t\delta_n^i - (1-t)\dot{\xi}^i(\hat{q}, 1)\big]dt$$
$$+ \frac{1}{2}s^3\int_0^1 \varphi_{\xi^i\xi^j}(e_n; e_n + \dot{\xi}(\hat{q}, 1))$$
$$\times \big[t\delta_n^i - (1-t)\dot{\xi}^i(\hat{q}, 1)\big]\big[t\delta_n^j - (1-t)\dot{\xi}^j(\hat{q}, 1)\big]dt$$
$$+ O(s^3|\hat{q}| + s^4)$$
$$= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + O(s^3|\hat{q}| + s^4).$$

First,

$$\mathrm{III} = \frac{1}{2}s^2\varphi_{\xi^i}(e_n; e_n + \dot{\xi}(\hat{q}, 1))\big[\delta_n^i - \dot{\xi}^i(\hat{q}, 1)\big] = O(s^2|\hat{q}|^2).$$

Using (4.6) and (7.5), we have

$$\varphi_{\xi^i\xi^j}(e_n; e_n + \dot{\xi}(\hat{q}, 1))\big[t\delta_n^i - (1-t)\dot{\xi}^i(\hat{q}, 1)\big]\big[t\delta_n^j - (1-t)\dot{\xi}^j(\hat{q}, 1)\big] = O(|\hat{q}|),$$

from which we deduce

$$\mathrm{IV} = O(s^3|\hat{q}|) = O(s^4 + s^2|\hat{q}|^2).$$

Differentiating $\varphi(\xi(\hat{q},\tau),\dot{\xi}(\hat{q},\tau)) \equiv 1$ in τ, we have, using $\xi(\hat{q},1)=e_n$,

$$\begin{aligned}
&\varphi_{v^i}(e_n;\dot{\xi}(\hat{q},1))\ddot{\xi}^i(\hat{q},1)\\
&\quad= -\varphi_{\xi^i}(e_n;\dot{\xi}(\hat{q},1))\dot{\xi}^i(\hat{q},1)\\
&\quad= -\varphi_{\xi^i}(e_n;e_n+[\dot{\xi}(\hat{q},1)-e_n])\dot{\xi}^i(\hat{q},1)\\
&\quad= -\varphi_{\xi^i}(e_n;e_n)\dot{\xi}^i(\hat{q},1) - \varphi_{\xi^i v^j}(e_n;e_n)\dot{\xi}^i(\hat{q},1)\big[\dot{\xi}^j(\hat{q},1)-\delta_n^j\big]\\
&\qquad + O(|\dot{\xi}(\hat{q},1)-e_n|^2)\\
&\quad= O(|\hat{q}|^2)\,.
\end{aligned}$$

Since $\varphi_{v^i}(e_n;e_n+\dot{\xi}(\hat{q},1)) = \varphi_{v^i}(e_n;\dot{\xi}(\hat{q},1)) + O(|\hat{q}|)$, we conclude that

$$\mathrm{II} = O(s^2|\hat{q}|^2)\,.$$

Based on the above, we have

$$L(0',\hat{q},s) = s\varphi(e_n;e_n+\dot{\xi}(\hat{q},1)) + O(s^4+s^2|\hat{q}|^2)\,.$$

Inserting (7.3), we obtain (7.4). □

We now complete the proof of Proposition 7.1. Combining (7.1) and (7.4), we obtain

$$L(x',\hat{q},s) \le 2s(1-c_0|\hat{q}|^2) + C(s^2|\hat{q}|^2+s^4) + Cs|\hat{q}|^2\left(\frac{\bar{\epsilon}}{K}+\bar{\epsilon}^2K+\frac{1}{K}\right).$$

Thus, by our conditions on x',

$$L(x',\hat{q},s) \le 2s(1-c_0|\hat{q}|^2) + Cs|\hat{q}|^2\left(\bar{\epsilon}^2K^3\bar{\delta}+K\bar{\delta}+\frac{\bar{\epsilon}}{K}+\bar{\epsilon}^2K+\frac{1}{K}\right).$$

Proposition 7.1 follows, if we choose first K large, then $\bar{\epsilon}$ small, and finally, $\bar{\delta}$ small. □

8 Main Estimates IV

We now take up another case for which we, again, choose z of the form

$$(x'+\bar{q},\, f(x'+\bar{q}))$$

with suitable $\bar{q}$. The choice of $\bar{q}$ is made so as to make $|A| = |e_n - \xi(\bar{q},1)|$ small.

Let $\zeta \in \mathbb{R}^{n-1}$ be a unit eigenvector of

$$(\tilde{f}_{x_\alpha x_\beta}(0') - f_{x_\alpha x_\beta}(0'))\zeta^\alpha = \lambda\zeta^\beta\,,\quad 1\le\beta\le n-1\,, \tag{8.1}$$

where we recall that $\lambda \ge 0$ is the smallest eigenvalue. We set

$$\bar{q} = \rho|x'|\zeta \tag{8.2}$$

with

$$\rho \ge K^{\frac{3}{4}}\,. \tag{8.3}$$

As before $L(0', \bar{q}, s)$ is the Finsler length of the segment joining $\xi(x' + \bar{q}, 1 - s)$ to $\xi(x', 1 + s)$.

PROPOSITION 8.1 *For any given positive constant $\epsilon' > 0$, there exist some large constant $K \geq 1$ and some small constant $\delta' > 0$, depending only on ϵ', f, and φ, such that for all $\epsilon'\lambda \leq |x'| \leq \delta'$, $s = K|x'|$, and $\bar{q}$ as above,*

$$L(x', \bar{q}, s) < 2s .$$

Consequently,

$$\text{dist}((x' + \bar{q}, f(x' + \bar{q})) \text{ to } \xi(x', 1 + s)) < 1 + s .$$

Remark 8.2. In proving the above proposition, K will be chosen first and then δ'.

We first establish the following:

LEMMA 8.3 *For some positive constants c_0, C, K, and δ', depending only on φ and f, we have, for x', $\bar{q}$, and s above, that*

$$L(0', \bar{q}, s) \leq 2s(1 - c_0|\bar{q}|^2) .$$

PROOF: Let $\tilde{\xi} = \tilde{\xi}(x', \tau)$, $\tau \geq 0$, denote the geodesics satisfying

$$\varphi(\tilde{\xi}; \dot{\tilde{\xi}}) \equiv 1 , \quad \tilde{\xi}(x', 0) = (x', \tilde{f}(x')) , \quad \dot{\tilde{\xi}}(x'; 0) = \tilde{V}(x') ,$$

where $\tilde{V}(x')$ is defined as $V(x')$ in Section 2, but for $\tilde{f}$ instead of for f. By the property of $\tilde{f}$,

$$\tilde{\xi}(x', 1) = e_n, \quad |x'| < \tilde{\epsilon}_1 .$$

For any $q \in \mathbb{R}^{n-1}$, $|q|$ small, let

$$\begin{aligned} \tilde{\eta}(x', q, s; t) &= (1 - t)\tilde{\xi}(x' + q, 1 - s) + t\tilde{\xi}(x', 1 + s) \\ &= \tilde{\xi}(x' + q, 1 - s) + t[\tilde{\xi}(x', 1 + s) - \tilde{\xi}(x' + q, 1 - s)] \end{aligned}$$

and

$$\tilde{L}(x', q, s) = \int_0^1 \varphi(\tilde{\eta}(x', q, s; t); \dot{\tilde{\eta}}(x', q, s; t))dt .$$

By Lemma 7.2, applied to $\tilde{f}$ with $\hat{q} = \bar{q}$, we have

$$\tilde{L}(0', \bar{q}, s) \leq 2s(1 - c_0|\bar{q}|^2) + O(s^4 + s^2|\bar{q}|^2) . \tag{8.4}$$

In the rest of the proof we mainly estimate $|L(0', \bar{q}, s) - \tilde{L}(0', \bar{q}, s)|$. Clearly, by our choice of the vector ζ,

$$|\tilde{\xi}(\bar{q}, 0) - \xi(\bar{q}, 0)| = \tilde{f}(\bar{q}) - f(\bar{q}) \leq C(\lambda|\bar{q}|^2 + |\bar{q}|^3)$$

and

$$|\dot{\tilde{\xi}}(\bar{q}, 0) - \dot{\xi}(\bar{q}, 0)| \leq C(\lambda|\bar{q}| + |\bar{q}|^2 .$$

Since both $\xi(\bar{q}, \cdot)$ and $\tilde{\xi}(\bar{q}, \cdot)$ satisfy the same ODE, we have

$$|\tilde{\xi}(\bar{q}, t) - \xi(\bar{q}, t)| + |\dot{\tilde{\xi}}(\bar{q}, t) - \dot{\xi}(\bar{q}, t)| \leq C(\lambda|\bar{q}| + |\bar{q}|^2) \quad \forall t . \tag{8.5}$$

Next, one verifies that

$$(8.6)\quad L(0',\bar q,s)= \\ s\int_0^1 \varphi\left(\tilde\eta(t)-(1-t)(\tilde\xi-\xi)(\bar q,1-s);\ \frac{1}{s}\dot{\tilde\eta}(t)+\frac{1}{s}(\tilde\xi-\xi)(\bar q,1-s)\right)dt\,,$$

where

$$\tilde\eta(t):=\tilde\eta(0',\bar q,s;t)$$

By (8.5), for $0\le t\le 1+\epsilon_0$,

$$(|\tilde\xi-\xi|+|\dot{\tilde\xi}-\dot\xi|)(\bar q,t)=O(\lambda|\bar q|+|\bar q|^2)\,.$$

We also have

$$\tilde\eta(t)=[1+(2t-1)s]e_n+O(|\bar q|)\,,\qquad \frac{1}{s}\dot{\tilde\eta}(t)=2e_n+O(|\bar q|)\,.$$

The last equality above needs some explanation: By Taylor expansion,

$$\begin{aligned}\dot{\tilde\eta}(t)&=\tilde\xi(0',1+s)-\tilde\xi(\bar q,1-s)\\&=(1+s)e_n-\tilde\xi(\bar q,1)+\dot{\tilde\xi}(\bar q,1)s-\frac{1}{2}\ddot{\tilde\xi}(\bar q,1-\theta s)s^2\,,\end{aligned}$$

where $0\le\theta\le 1$. Since $\tilde\xi(\bar q,1)=e_n$, $\dot{\tilde\xi}(0',1)=e_n$, and $\ddot{\tilde\xi}(0',t)=0$ for all $0\le t\le 1+\epsilon_0$, we have $\dot{\tilde\xi}(\bar q,1)=e_n+O(|\bar q|)$, $\ddot{\tilde\xi}(\bar q,1-\theta s)=O(|\bar q|)$, and therefore

$$\dot{\tilde\eta}(t)=2se_n+O(s|\bar q|)\,.$$

It is clear that

$$\xi^n(0',t)-t\equiv\xi^n_{x_\alpha}(0',t)\equiv 0\,,\quad \tilde\xi^n(0',t)-t\equiv\tilde\xi^n_{x_\alpha}(0',t)\equiv 0\,,\quad 0\le t\le 1+\epsilon_0\,.$$

It follows that

$$\begin{aligned}\tilde\xi^n(\bar q,1-s)&=\tilde\xi^n(0',1-s)+\tilde\xi^n_{x_\alpha}(0',1-s)\bar q_\alpha\\&\qquad+\int_0^1\int_0^t\tilde\xi^n_{x_\alpha x_\beta}(\tau\bar q,1-s)\bar q_\alpha\bar q_\beta d\tau dt\\&=(1-s)+\int_0^1\int_0^t\tilde\xi^n_{x_\alpha x_\beta}(\tau\bar q,1-s)\bar q_\alpha\bar q_\beta\,d\tau\,dt\,.\end{aligned}$$

Similarly,

$$\xi^n(\bar q,1-s)=(1-s)+\int_0^1\int_0^t\xi^n_{x_\alpha x_\beta}(\tau\bar q,1-s)\bar q_\alpha\bar q_\beta\,d\tau\,dt\,.$$

By (5.3), applied to both ξ and $\tilde\xi$, we deduce from the above that

$$(\tilde\xi^n-\xi^n)(\bar q,1-s)= \\ \frac{1}{2}\varphi_{v^iv^j}((1-s)e_n;e_n)\big(\dot\xi^i_{x_\alpha}\xi^j_{x_\beta}-\dot{\tilde\xi}^i_{x_\alpha}\tilde\xi^j_{x_\beta}\big)(0',1-s)\bar q_\alpha\bar q_\beta+O(|\bar q|^3)\,.$$

Thus, by (8.5), we have

$$|(\tilde{\xi}^n - \xi^n)(\bar{q}, 1-s)| \leq C(\lambda|\bar{q}| + |\bar{q}|^2)|\bar{q}|^2 + C|\bar{q}|^3 \leq C|\bar{q}|^3 . \tag{8.7}$$

Estimate (8.7) will be used below.

By Taylor expansion in (8.6), we have, using (8.5),

$$\begin{aligned} L(0', \bar{q}, s) = s \int_0^1 \varphi\left(\tilde{\eta}, \frac{1}{s}\dot{\tilde{\eta}}\right)dt \\ &- s\int_0^1 \varphi_{\xi^i}\left(\tilde{\eta}, \frac{1}{s}\dot{\tilde{\eta}}\right)(1-t)(\tilde{\xi}-\xi)^i(\bar{q}, 1-s)dt \\ &+ \int_0^1 \varphi_{v^i}\left(\tilde{\eta}, \frac{1}{s}\dot{\tilde{\eta}}\right)(\tilde{\xi}-\xi)^i(\bar{q}, 1-s)dt \\ &+ O(\lambda|\bar{q}|^2 + |\bar{q}|^3) + O\left(\frac{(\lambda|\bar{q}| + |\bar{q}|^2)^2}{s}\right). \end{aligned}$$

By the properties of special coordinates and the expressions of $\tilde{\eta}$ and $\frac{1}{s}\dot{\tilde{\eta}}$,

$$(|\varphi_{\xi^i}| + |\varphi_{v^\alpha}|)\left(\tilde{\eta}; \frac{1}{s}\dot{\tilde{\eta}}\right) \leq C|\bar{q}| \quad \forall\alpha,\ 1 \leq \alpha \leq n-1 , \qquad \left|\varphi_{v^n}\left(\tilde{\eta}; \frac{1}{s}\dot{\tilde{\eta}}\right)\right| \leq C .$$

It follows that

$$L(0', \bar{q}, s) = \tilde{L}(0', \bar{q}, s) + O\left((\lambda|\bar{q}|^2 + |\bar{q}|^3)\left(1 + \frac{\lambda + |\bar{q}|}{s}\right)\right).$$

Combining this with (8.7) and (8.4), we find

$$\begin{aligned} L(0', \bar{q}, s) &\leq 2s(1 - c_0|\bar{q}|^2) + C(s^4 + s^2|\bar{q}|^2) \\ &\qquad + C(\lambda|\bar{q}|^2 + |\bar{q}|^3)\left(1 + \frac{\lambda + |\bar{q}|}{s}\right) \\ &\leq 2s(1 - c_0|\bar{q}|^2) + Cs|\bar{q}|^2\left(\frac{s^3 + s|\bar{q}|^2}{|\bar{q}|^2}\right) \\ &\qquad + Cs|\bar{q}|^2\left(\frac{\lambda + |\bar{q}|}{s}\right)\left(1 + \frac{\lambda + |\bar{q}|}{s}\right). \end{aligned}$$

Since

$$\frac{s^3 + s|\bar{q}|^2}{|\bar{q}|^2} = \frac{K^3}{\rho^2}|x'| + K|x'| \leq (K^{\frac{3}{2}} + K)\delta'$$

and

$$\frac{\lambda + |\bar{q}|}{s} \leq \frac{1}{K\epsilon'} + \frac{\rho}{K} = \frac{1}{K\epsilon'} + \frac{1}{K^{1/4}} ,$$

we obtain the desired estimate by choosing first K large and then δ' small (recall that we also want $K\delta' < \bar{\epsilon}_1$ in Proposition 6.1). Lemma 8.3 is proven. □

PROOF OF PROPOSITION 8.1: We make use of Proposition 6.1, and for this we need an estimate of

$$A = e_n - \xi(\bar{q}, 1)\,.$$

In fact, since $\tilde{\xi}(\bar{q}, 1) = e_n$ we have from (8.5),

$$|A| \le C(\lambda|\bar{q}| + |\bar{q}|^2) \le C|\bar{q}|\left(\frac{|x'|}{\epsilon'} + |\bar{q}|\right).$$

We have to verify (6.5). We have that $|\bar{q}| < 1$ since $K\delta' \le \bar{\epsilon}_1 < 1$, so

$$\begin{aligned}\frac{|A| + |x'|}{s} &\le C\frac{|x'|}{s}\left(1 + \frac{|\bar{q}|}{\epsilon'}\right) + C\frac{|\bar{q}|}{s} \le C\left(\frac{1}{K} + \frac{1}{\epsilon' K} + \frac{1}{K^{\frac{1}{4}}}\right)\\ &< \epsilon_3 \quad \text{of Proposition 6.1}\end{aligned}$$

provided we increase K still further, which means decreasing δ'. We may thus apply Proposition 6.1 and conclude that

$$\begin{aligned}&L(x', q, s) - L(0', q, s) \le\\ &\qquad Cs|\bar{q}|^2\left(\frac{1}{K\rho} + \frac{1}{\rho^2} + \frac{\rho}{K^2} + \frac{1}{K\rho\epsilon'} + \frac{1}{K} + \frac{1}{K^2\epsilon'} + \frac{K}{\rho^2}\right).\end{aligned}$$

Recalling that $\rho = K^{3/4}$ and combining the above with Lemma 8.3, we obtain the desired result again, if necessary, by increasing K and decreasing δ'. □

9 Proof of Theorem 1.5

We consider Ω bounded. The proof for unbounded Ω goes the same. Following the notation in the introduction, we need to prove that $\bar{s}(y)$ is a Lipschitz function on $\partial\Omega$. Namely, we need to show that there exist some positive constants K and δ such that for any $\bar{y} \in \partial\Omega$,

$$\bar{s}(y) \le \bar{s}(\bar{y}) + K|y - \bar{y}| \quad \forall y \in \partial\Omega,\ |y - \bar{y}| \le \delta\,. \tag{9.1}$$

As before, by making a change of variables, we may assume without loss of generality that $\bar{y} = 0 \in \partial\Omega$, $\bar{s}(\bar{y}) = 1$, and, for some $\epsilon_0 > 0$, $\xi(\bar{y}, t) = te_n$ for all $-\epsilon_0 \le t \le 1 + \epsilon_0$. By our result in Section 3 on the existence of special coordinates, we may also assume, for all $-\epsilon_0 \le t \le 1 + \epsilon_0$, that (4.1) through (4.6) hold.

We may assume that for some $\epsilon_1 > 0$, $f(x')$ is a $C^{2,1}$-function defined in $|x'| < \epsilon_1$, $x' \in \mathbb{R}^{n-1}$, $f(0') = 0$, $\nabla f(0') = 0$, and $\{(x', f(x')) : |x'| < \epsilon_1\}$ is a local representation of $\partial\Omega$. In the following, as before, we use $\xi(x', t)$ to denote $\xi((x', f(x')), t)$. With this notation, we have

$$\xi(x', 0) = (x', f(x'))$$

and, by Lemma 2.2,

$$\dot{\xi}(x', 0) = V(x')\,,$$

where $V(x')$ is the vector field given in Section 2.

To prove (9.1), we only need to show that for some constants K and δ, depending only on $\partial\Omega$ and φ, we have

$$\operatorname{dist}(\partial\Omega, \xi(x', 1 + K|x'|)) < 1 + K|x'| \quad \forall |x'| < \delta\,. \tag{9.2}$$

We put together the results of Sections 4–7.

In the proof of Theorem 1.5 we distinguish two cases:

Case 1. There exists some $Q \in \partial\Omega \setminus \{0\}$ such that $\operatorname{dist}(0 \text{ to } e_n) = \operatorname{dist}(Q \text{ to } e_n)$.

Case 2. For all $y \in \partial\Omega \setminus \{0\}$, we have $\operatorname{dist}(0 \text{ to } e_n) < \operatorname{dist}(y \text{ to } e_n)$.

In Case 1, we may assume, because of Proposition 5.2, that $Q = (\hat{q}, f(\hat{q}))$ for some $\hat{q} \in \mathbb{R}^{n-1}$ satisfying $|\hat{q}| \le \bar{\epsilon}_1/9$, where $\bar{\epsilon}_1$ is that of Proposition 6.1.

Since

$$\operatorname{dist}(0 \text{ to } e_n) = \operatorname{dist}((x', \tilde{f}(x')) \text{ to } e_n) \quad \forall |x'| \le \tilde{\epsilon}_1\,,$$

we have

$$\tilde{f}(\hat{q}) = f(\hat{q})\,, \quad \tilde{f}(x') \ge f(x') \;\; \forall |x'| \le \tilde{\epsilon}_1\,,$$

and

$$(\tilde{f} - f)(x') = \frac{1}{2}(\tilde{f}_{x_\alpha x_\beta} - f_{x_\alpha x_\beta})(0')x_\alpha x_\beta + O(|x'|^3)\,.$$

Recall that $\lambda \ge 0$ is the least eigenvalue of $((\tilde{f}_{x_\alpha x_\beta} - f_{x_\alpha x_\beta})(0'))$. We thus have

$$0 = (\tilde{f} - f)(\hat{q}) \ge \frac{1}{2}\lambda|\hat{q}|^2 + O(|\hat{q}|^3)$$

and therefore

$$\lambda \le C|\hat{q}|\,,$$

where C depends only on the $C^{2,1}$-norm of f and $\tilde{f}$.

Let $\bar{\epsilon}$, $\bar{\delta}$, and K be the positive constants in Proposition 7.1; then, by Proposition 7.1,

$$\operatorname{dist}((x' + \hat{q}, f(x' + \hat{q})) \text{ to } \xi(x', 1 + K|x'|)) < 1 + K|x'|\,, \quad \forall |x'| \le \min\{\bar{\delta}, \bar{\epsilon}|\hat{q}|\}\,.$$

For $|x'| \ge \bar{\epsilon}|\hat{q}|$, we have, by the above,

$$|x'| \ge \frac{\bar{\epsilon}}{C}\lambda\,.$$

Let $\epsilon' = \bar{\epsilon}/C$; we have, by Proposition 8.1, for some K and δ' depending on $\bar{\epsilon}$,

$$\operatorname{dist}(\xi(x', 1 + K|x'|) \text{ to } \bar{q}) < 1 + K|x'| \quad \forall \epsilon'\lambda \le |x'| \le \delta'\,.$$

Thus we have established (9.2) for $\delta = \min\{\bar{\delta}, \delta'\}$ and some positive constant K.

In Case 2, we have, by Lemma 4.12, $\lambda = 0$. The desired estimate (9.2) then follows from Proposition 8.1.

10

In this section we consider the general case

$$H(x, u, \nabla u) = 1 \quad \text{in } \Omega\,, \tag{10.1}$$

a bounded domain in $\mathbb{R}^n$ with $\partial\Omega$ in $C^{2,1}$. In Theorem 10.5, under certain strict conditions, we find a positive viscosity solution u satisfying

$$u = 0 \quad \text{on } \partial\Omega\,, \tag{10.2}$$

and show that for its singular set Σ,

$$H^{n-1}(\Sigma) < \infty\,. \tag{10.3}$$

We then derive Propositions 1.8, 1.10, and 1.11 of Section 1.4.

We shall make two conditions. The first is Situation A of Section 1.4, which we repeat here as follows:

ASSUMPTION 10.1 The function $H(x, t, p)$, $t \in \mathbb{R}$, $p \in \mathbb{R}^n$, is assumed to satisfy: For every x in $\overline{\Omega}$ the set

$$V_x = \{(t, p) : H(x, t, p) < 1\}$$

is a convex set in $\mathbb{R} \times \mathbb{R}^{n+1}$ lying in a fixed downward cone

$$|p| \le k(C_1 - t)\,, \quad t < C_1 \text{ with } k, C_1 > 0\,; \tag{10.4}$$

see Figure 10.1 below. Thus t may be unbounded below in V_x. The boundary of V_x,

$$S_x = \{(t, p) : H(x, t, p) = 1\}\,,$$

is assumed to be a smooth, strictly convex hypersurface in (t, p)–space with positive principal curvature for t in the region

$$-1 \le t \le C_1 \tag{10.5}$$

uniformly for x in $\overline{\Omega}$. Furthermore, the origin in $\mathbb{R} \times \mathbb{R}^n$ lies in V_x and is bounded away from S_x by some number

$$r_0 > 0\,.$$

In addition, $H(x, t, p)$ is smooth in a neighborhood of $\bigcup_x S_x$. See Figure 10.1, for example.

Thus a common assumption that H is monotone in t does not necessarily hold here. Under another assumption on H we construct a viscosity solution; it need not, however, be unique. As we have said in Section 1.2, the function $H(x, t, p)$ is not so important; the important things are the sets V_x.

Because of Remark 1.7 we may take H to be homogeneous of degree 1 in (t, p). It is thus completely determined by the S_x. From now on we assume this homogeneity.

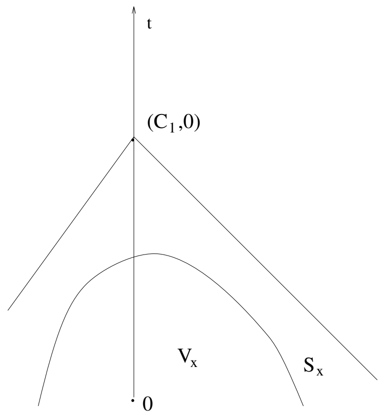

FIGURE 10.1

Our way of studying the problem is to set up a related problem in one higher dimension—in $\mathbb{R} \times \overline{\Omega}$: For $\tau \in \mathbb{R}$, given a function u in $\overline{\Omega}$, we define z in $\mathbb{R} \times \overline{\Omega}$ by

$$z(\tau, x) = e^{\tau} u(x) .$$

Multiplying equation (10.1) by e^{τ}—recall the homogeneity condition—we obtain

$$H(x, e^{\tau} u, e^{\tau} \nabla u) = e^{\tau} ,$$

that we rewrite as

$$e^{-\tau} H(x, z_{\tau}, \nabla_x z) = 1 . \tag{10.6}$$

This is a Hamilton-Jacobi equation in $\mathbb{R} \times \Omega$ for z, and we solve it under the boundary condition

$$z = 0 \quad \text{on } \mathbb{R} \times \partial\Omega . \tag{10.7}$$

As in Section 1, the solution involves the support function

$$\begin{aligned} \tilde{\varphi}(x, \tau; s, v) &= \sup_{e^{-\tau} H(x,t,p)=1} (st + v \cdot p) \\ &= e^{\tau} \sup_{H(x,t,p)=1} (st + v \cdot p) = e^{\tau} \varphi(x; s, v) \end{aligned}$$

where φ is the support function of S_x:

$$\varphi(x; s, v) = \sup_{H(x,t,p)=1} (st + v \cdot p) .$$

According to Section 1, which uses formula (55)$'$ of [12, p. 132], the viscosity solution of (10.6) and (10.7) is obtained using curves $(w(t), \xi(t))$ and $0 \leq t \leq T$ lying in $\mathbb{R} \times \overline{\Omega}$ with

$$w(0) = \tau, \quad \xi(0) = x ; \qquad w(T) = \mu, \quad \xi(T) = y \in \partial\Omega . \tag{10.8}$$

The solution is given by

$$z(\tau, x) = \inf_{\mu\in\mathbb{R}, y\in\partial\Omega} \inf_{(w,\xi)} \int_0^T e^{w(t)}\varphi(\xi(t); -\dot{w}, -\dot{\xi})dt\,. \tag{10.9}$$

Here $\inf_{w,\xi}$ means infimum over curves satisfying (10.8). By remark 5.5 in [12], z is a viscosity solution even though $\mathbb{R} \times \Omega$ is unbounded.

Note that

$$z(\tau, x) = e^{\tau} z(0, x)\,. \tag{10.10}$$

This follows from the following:

Remark 10.2. If $(w(t), \xi(t))$ is an eligible curve in (10.9) for $z(\tau, x)$, then $(w(t) - \tau, \xi(t))$ is one for $z(0, x)$.

We are really only interested in

$$u(x) := z(0, x) \tag{10.11}$$

because of the following:

CLAIM 10.3 Since $z(\tau, x)$ is a viscosity solution of (10.6)–(10.7), $u(x)$, given by (10.11), is a viscosity solution of (10.1)–(10.2).

PROOF: This is easily seen. For instance, to check that u is a viscosity subsolution we have to show that for any $\varphi \in C^1(\Omega)$ such that $u - \varphi$ has a local maximum $= 0$ at some point $x_0 \in \Omega$, necessarily,

$$H(x_0, u(x_0), \nabla\varphi(x_0)) \le 1\,. \tag{10.12}$$

To see this for such a φ, consider

$$\tilde{\varphi}(\tau, x) = e^{\tau}\varphi(x)\,.$$

Because of (10.10), $z - \tilde{\varphi}$ has a local maximum $= 0$ at $(0, x_0)$, and since z is a viscosity subsolution of (10.6),

$$H(x_0, \tilde{\varphi}_\tau(0, x_0), \nabla_x\tilde{\varphi}(0, x_0)) \le 1\,,$$

i.e., (10.12) holds. □

Turning now to the singular sets of z and u, we see from (10.10) that the singular set $\tilde{\Sigma}$ of z is a straight cylinder with generators parallel to the τ-axis lying over the singular set Σ of u. Thus if we know that

$$H^n(\tilde{\Sigma}) < \infty\,, \tag{10.13}$$

it follows that

$$H^{n-1}(\Sigma) < \infty\,,$$

our desired conclusion (10.3).

Indeed our main result, Theorem 1.5 of Section 1, yields exactly (10.13).

Wrong. We have to be more careful: the domain $\mathbb{R} \times \Omega$ is not bounded and we cannot apply our Lipschitz continuity result of Theorem 1.5 in Section 1; it holds for *compact subsets* of the boundary.

We are thus led to add a further restriction on the sets V_x relative to the domain Ω: Set

$$\overline{C} = \sup_{x\in\Omega} \inf_{y\in\partial\Omega} \inf_{\substack{\xi \\ \xi(0)=x \\ \xi(T)=y}} \int_0^T \varphi(\xi(t); 0, -\dot{\xi}(t))dt \, .$$

This is the shortest distance from x in Ω to $\partial\Omega$ in the restricted Finsler metric $\varphi(\xi(t); 0, -\dot{\xi}(t))dt$.

Next, consider the support function $\varphi(x; s, v)$. From its definition, we have

$$c_0(|s| + |v|) \leq \varphi(x; s, v) \leq C_0(|s| + |v|) \tag{10.14}$$

for suitable positive constants c_0 and C_0. Set

$$\sigma := \sup_{\substack{\varphi(x;s,v)=1 \\ x\in\overline{\Omega}}} s \, .$$

The additional condition we impose is as follows:

ASSUMPTION 10.4 $\sigma\overline{C} < 1$.

Assumption 10.4 may be expressed more directly in terms of the sets S_x: For any $x \in \overline{\Omega}$ denote by $\bar{t} = \bar{t}(x)$ the point $(\bar{t}, 0)$ on S_x with $\bar{t} > 0$. Since for every x, $H(x, t, p)$ is the support function of the convex hypersurface

$$\hat{S}_x = \{(s, v) : \varphi(x; s, v) \equiv 1\} \, ,$$

it follows that

$$1 = H(x, \bar{t}, 0) = \sup_{\varphi(x;s,v)=1} s\bar{t}$$

is achieved at a point where $s = \bar{s}$, the maximum value of s on $\hat{S}_x$. Thus $\bar{t} = 1/\bar{s}$ and so

$$\frac{1}{\sigma} = \min_{x\in\overline{\Omega}} \bar{t}(x) \, .$$

Hence Assumption 10.4 is equivalent to the condition

$$\min_{x\in\overline{\Omega}} \bar{t}(x) > \overline{C} \, .$$

We now state the main result of this section.

THEOREM 10.5 *Under Assumption* 10.1 *and Assumption* 10.4*, problem* (10.1)–(10.2) *possesses a positive viscosity solution and its singular set* Σ *satisfies*

$$H^{n-1}(\Sigma) < \infty \, .$$

The proof of Theorem 10.5 is based on Lemma 10.6 below; we assume Assumption 10.1 and Assumption 10.4.

For $x \in \Omega$ fixed and $0 < \epsilon$ fixed, so that

$$\sigma(\overline{C} + \epsilon) < \frac{1 + \sigma\overline{C}}{2}, \tag{10.15}$$

consider a competing curve $(w(t), \xi(t))$, $0 \le t \le T$, such that $w(0) = 0$, $\xi(0) = x$, $w(T) = \mu$, $\xi(T) = y \in \partial\Omega$, and such that

$$\int_0^T e^{w(t)} \varphi(\xi(t); -\dot{w}(t), -\dot{\xi}(t))dt < \overline{C} + \epsilon . \tag{10.16}$$

By our definition of $\overline{C}$, such a curve exists. Let us normalize the parameter t so that

$$\varphi(\xi(t); -\dot{w}(t), -\dot{\xi}(t)) \equiv 1 . \tag{10.17}$$

T is of course unknown.

LEMMA 10.6 *In the situation above,*

$$T, |w(t)| \le C(\overline{C}, \sigma) . \tag{10.18}$$

PROOF: By the definition of σ, because of (10.17),

$$-\dot{w}(t) \le \sigma .$$

Thus

$$w(t) \ge -\sigma t$$

and inserting this in (10.16), we obtain

$$\overline{C} + \epsilon > \int_0^T e^{-\sigma t}\, dt = \frac{1}{\sigma}(1 - e^{-\sigma T}) .$$

Using (10.15), we find

$$e^{-\sigma T} > \frac{1 - \sigma\overline{C}}{2} ,$$

from which a bound for T, as in (10.18), follows. By (10.14)

$$|\dot{w}(t)| \le \frac{1}{c_0} \quad \text{and thus} \quad |w(t)| \le \frac{T}{c_0} ,$$

completing (10.18). □

We have proven that if we consider competing curves for $z(0, x)$ with "lengths" close to $z(0, x)$ then, on them,

$$|w| \le C_1 \quad \text{uniformly in } x. \tag{10.19}$$

By Remark 10.2 it follows that for $|\tau| \le 1$ if we consider competing curves for $z(\tau, x)$ in (10.9) with lengths sufficiently close to $z(\tau, x)$, then on these

$$|w| \le C_2 \quad \text{uniformly in } x . \tag{10.20}$$

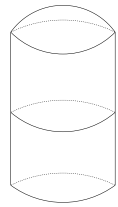

FIGURE 10.2

We are now in a position to give the following:

PROOF OF THEOREM 10.5: We change S_x to $\tilde{S}_x$ by making it bounded from below. We can do so with Assumption 10.4 unchanged for $\tilde{S}_x$. This can be done by taking $\delta > 0$ very small and changing S_x smoothly (also in x) so that it is unchanged for $t \geq -\delta$ but does not extend below $t = -2\delta$. It is clear that Assumption 10.4 still holds if we take $\delta > 0$ small and change S_x properly. By Remark 1.7 we may assume that S_x satisfies this additional property.

Finally, we construct a bounded domain D in $\mathbb{R}^{n+1}$, over Ω, with the $C^{2,1}$-boundary, which agrees with the cylinder when $|\tau| < 2C_2$, as pictured in Figure 10.2.

In D we solve (10.6)–(10.7) by the formula (10.9) where the curves $(w(t), \xi(t))$ go from (τ, x) to the boundary of D, obtaining function z.

As we indicated previously, $u(x) = z(0, x)$ is then a viscosity solution of (10.1)–(10.2). Applying our main result, Theorem 1.5, to z in D, we see that the singular set $\tilde{\Sigma}$ of z has

$$H^n(\tilde{\Sigma}) < \infty\,.$$

Now (10.10) holds for $|\tau| \leq 1$ and hence, for $|\tau| \leq 1$, the singular set of z is a finite cylinder over the singular set Σ of $u = z(0, x)$. Consequently,

$$H^{n-1}(\Sigma) < \infty\,,$$

and we are through. □

CONJECTURE 10.7 Theorem 10.5 holds merely under Assumption 10.1.

PROOF OF PROPOSITIONS 1.8 AND 1.10: From the conditions in these propositions it is clear that Assumption 10.4 is satisfied. Thus Theorem 10.5 applies, proving the propositions. □

PROOF OF PROPOSITION 1.11: For d_0 small we verify Assumption 10.4 by showing that $\bar{C}$ is small. Namely, from any point $x \in \Omega'$ we join it to y on $\partial\Omega'$ minimizing $|y - x|$ by a straight segment

$$\xi(t) = x + t(y - x)\,, \quad 0 \leq t \leq 1\,.$$

Then its Finsler length from y to x is

$$\int_0^1 \varphi(\xi(t); 0, x - y)dt \leq C_0\, d_0$$

by (10.14). Hence

$$\bar{C} \le C_0\, d_0 ;$$

it follows that for d_0 small, depending only on H, Assumption 10.4 holds and Theorem 10.5 applies. □

PROOF OF PROPOSITION 1.12: As usual, we may suppose that the set

$$V = \{(t, p) : H(t, p) = 1\}$$

is bounded and satisfies Assumption 10.1 as in the proof of Theorem 10.5, and that H is positive homogeneous of degree 1. As in the proof of Theorem 10.5 we consider the H-J equation (10.6) involving the extra variable τ:

$$e^{-\tau} H(z_\tau, \nabla_x z) = 1 ,$$

and consider the solution given by (10.9).

First we obtain a bound on

$$u(x) = z(0, x) .$$

To this end we consider a competing curve of the form

$$w(t) = -\lambda t , \quad \xi(t) = x - \lambda t V , \quad 0 \le t \le T ,$$

where V is a constant vector in $\mathbb{R}^n$ and

$$T = \frac{d_V(x)}{\lambda |V|} .$$

Here $d_V(x)$ is the length of the segment from x in the direction V until it hits $\partial\Omega$. The curve is an eligible one and its length

$$L = \int_0^T e^{-\lambda t} \varphi(\lambda, \lambda V) = \varphi(1, V)(1 - e^{-\lambda T}) .$$

We now choose V so as to minimize $\varphi(1, V)$.

Letting

$$\sigma = \max_{\varphi(s,v)=1} s ,$$

it's clear that

$$\varphi(\sigma, V) \ge 1 \ \ \forall V \quad \text{and} \quad \min_V \varphi(\sigma, V) = 1 ,$$

so

$$\min_V \varphi(1, V) = \frac{1}{\sigma} .$$

Now fix V so that

$$\varphi(1, V) = \frac{1}{\sigma}.$$

Since $\bar{t} < \hat{t}$, $V \ne 0$. Recall that $\sigma = 1/\bar{t}$. Thus

$$L = \bar{t}\big(1 - e^{-\frac{d_V(x)}{|V|}}\big) = \bar{t} - a , \quad a > 0 ,$$

and hence

$$u(x) = z(0, x) \le \bar{t} - a .$$

We now follow the proof of Theorem 10.5. Consider a competing curve $(w(t), \xi(t))$, $0 \le t \le T$, satisfying $w(0) = 0$, $\xi(0) = x$, $w(T) = \mu$, $\xi(T) = y \in \partial\Omega$ and such that

$$\int_0^T e^{w(t)} \varphi(-\dot{w}; -\dot{\xi}) dt \le \bar{t} - \frac{a}{2} . \tag{10.21}$$

As usual, we normalize the parameter t so that

$$\varphi(-\dot{w}; -\dot{\xi}) \equiv 1 .$$

LEMMA 10.8 *In the situation above,*

$$T, |w(t)| \le C \quad \textit{independent of } x . \tag{10.22}$$

PROOF: It is the same as that of Lemma 10.6. Namely, we have

$$-\dot{w} \le \sigma = \frac{1}{\bar{t}} .$$

Thus

$$w \ge -\sigma t .$$

Inserting this into (10.21) we find

$$\bar{t} - \frac{a}{2} \ge \int_0^T e^{w(t)} dt \ge \int_0^T e^{-\sigma t} dt = \frac{1}{\sigma}(1 - e^{-\sigma T}) ,$$

i.e.,

$$e^{-\sigma T} \ge \frac{\bar{t} a}{2} .$$

The bound for T in (10.22) follows. Then, as before, we have $|\dot{w}(t)| \le 1/c_0$, so $|w(t)| \le T/c_0$. Lemma 10.8 is proven. □

The proof of Proposition 1.12 then proceeds as in the proof of Theorem 10.5. □

The assumption $\bar{t} < \hat{t}$ in Proposition 1.12 seems strange. However, in the case $\bar{t} = \hat{t}$, our method of proof must fail. Indeed, if we take

$$H(t, p) = (t^2 + |p|^2)^{\frac{1}{2}} \tag{10.23}$$

the corresponding Finsler metric is

$$e^w \varphi(-\dot{w}; -\dot{\xi}) = e^w (\dot{w}^2 + |\dot{\xi}|^2)^{\frac{1}{2}}$$

in $\mathbb{R} \times \Omega$ and is, in fact, an incomplete Riemannian metric. In the case $n = 1$ and $\Omega = (-R, R)$, then, for $R > \pi$, *there is no geodesic* $(w(t), \xi(t))$ *starting at* $(0, 0)$ *going to the boundary of the strip* $\mathbb{R} \times \Omega$. Nonetheless, for a bounded domain Ω in $\mathbb{R}^n$, and for H of (10.23), the function

$$u(x) = \begin{cases} 1 & \text{if } d(x) \ge \frac{\pi}{2} \\ \sin(d(x)) & \text{if } d(x) \le \frac{\pi}{2} \end{cases}$$

where $d(x)$ is the Euclidean distance from x to $\partial\Omega$, is a viscosity solution of (1.1), (1.3). In addition, for its singular set Σ,

$$H^{n-1}(\Sigma) < \infty . \tag{10.24}$$

Indeed,

$$\Sigma = \Sigma_1 \cup \Sigma_2$$

where $\Sigma_1 = \{x \in \Omega : d(x) = \frac{\pi}{2}\}$ and $\Sigma_2 =$ singular set of the distance function to $\partial\Omega$. Since Σ_1 is contained in the set of the points in Ω of all straight segments going normal to the boundary and having length $\frac{\pi}{2}$,

$$H^{n-1}(\Sigma_1) < \infty .$$

And by Theorem 1.1 in the introduction, rather, Corollary 1.3,

$$H^{n-1}(\Sigma_2) < \infty .$$

We plan to take up the general case $\bar{t} = \hat{t}$ in a later work.

Appendix A: About Remark 1.2; Examples with $C^{2,\alpha}$ Boundary

We start with $n = 2$. Essentially the same examples work for $n \geq 3$. For $0 < \alpha < \alpha + 3\epsilon \leq 1$, let

$$f(x) = 1 - \sqrt{1 - x^2} - g(x) , \quad x \in \mathbb{R} ,$$

where

$$g(x) = |x|^{2+\alpha+3\epsilon}(2 + \sin(|x|^{-\epsilon})) .$$

Clearly f is smooth in $(-1, 0) \cup (0, 1)$ and

$$\begin{aligned} f'(x) &= x - g'(x) + O(|x|^3) = O(|x|) , \\ f''(x) &= 1 - g''(x) + O(x^2) = O(1) , \\ g'(x) &= O(|x|^{1+\alpha+2\epsilon}) , \\ g''(x) &= -\epsilon^2 |x|^{\alpha+\epsilon} \sin(|x|^{-\epsilon}) + O(|x|^{\alpha+2\epsilon}) , \\ g'''(x) &= O(|x|^{\alpha-1}) . \end{aligned}$$

It follows from the above that for any $0 < x < y \leq \frac{1}{2}$,

$$|g''(x) - g''(y)| \leq \int_0^1 |g'''(t)| dt \leq C \int_0^1 t^{\alpha-1} dt \leq C|x - y|^\alpha .$$

So $g'' \in C^\alpha(-\frac{1}{2}, \frac{1}{2})$ and $f \in C^{2,\alpha}(-\frac{1}{2}, \frac{1}{2})$. We also see that for some $0 < \delta < \frac{1}{2}$,

$$f''(x) \geq \frac{1}{2} \quad \forall |x| < \delta .$$

Since $1 - \sqrt{1 - x^2}$ is a part of the graph of the unit circle centered at $(0, 1)$ and $f(x) \leq 1 - \sqrt{1 - x^2}$ with equality holds only at $x = 0$, we can construct a strictly

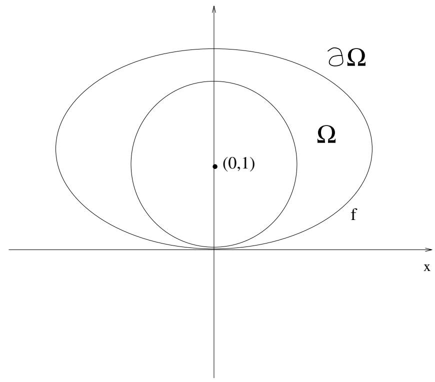

FIGURE A.1

convex $C^{2,\alpha}$ domain Ω that has $\{(x, f(x)) : |x| < \delta\}$ as a part of its boundary $\partial\Omega$, and

$$\mathrm{dist}((0,1), Q) > 1 \quad \forall Q \in \partial\Omega \setminus \{(0,0)\}.$$

See Figure A.1.

Clearly, $m(0,0) = (0,1)$. We will show that there exists some positive constant $c > 0$ such that for any $0 < x < \delta$ satisfying $\cos(x^{-\epsilon}) = 0$ and $\sin(x^{-\epsilon}) = 1$, we have

$$|m(x, f(x)) - (x, f(x))| \le 1 - c|x|^{\alpha+\epsilon}. \tag{A.1}$$

This implies that m is not in C^β for any $\beta > \alpha + \epsilon$. Indeed, for $x_k = (2k\pi + \frac{\pi}{2})^{-1/\epsilon} \to 0$ as $k \to \infty$, we have, for large k,

$$\begin{aligned}
|m(x_k, f(x_k)) - m(0,0)| &\ge |m(0,0)| - |m(x_k, f(x_k)) - (x_k, f(x_k))| \\
&\quad - |(x_k, f(x_k))| \\
&\ge 1 - (1 - c|x_k|^{\alpha+\epsilon}) - C|x_k| \\
&= c|x_k|^{\alpha+\epsilon} - C|x_k| \\
&\ge \frac{c}{2}|x_k|^{\alpha+\epsilon} \\
&\ge \frac{c}{4}|(x_k, f(x_k))|^{\alpha+\epsilon}.
\end{aligned}$$

In the following we establish (A.1). The curvature of the graph of f is given by

$$k(x) = \frac{f''(x)}{\sqrt{1 + f'(x)^2}}.$$

Thus

$$k(x) = f''(x) + O(x^2) = 1 - g''(x) + O(x^2).$$

Since $\cos(x_k^{-\epsilon}) = 0$ and $\sin(x_k^{-\epsilon}) = 1$, we have

$$k(x) = 1 - g''(x) + O(x^2) = 1 + \epsilon^2 x^{\alpha+\epsilon} + O(x^{\alpha+2\epsilon}).$$

This implies that

$$|m(x, f(x)) - (x, f(x))| \leq 1 - \epsilon^2 x^{\alpha+\epsilon} + O(x^{\alpha+2\epsilon}),$$

from which (A.1) follows.

For $n \geq 3$,

$$f(x) = 1 - \sqrt{1 - |x|^2} - g(x), \quad x \in \mathbb{R}^{n-1},$$

where

$$g(x) = |x|^{2+\alpha+3\epsilon}(2 + \sin(|x|^{-\epsilon})).$$

We still have $f \in C^{2,\alpha}$, and we can still construct Ω essentially the same way. For $x = (x_1, \dots, x_n)$, considering the curve, $((x_1, 0, \dots, 0), f(x_1, 0, \dots, 0))$, we already know that for $x_1 > 0$, $\cos(x_1^\epsilon) = 1$, and $\sin(x_1^\epsilon) = 0$, the curvature of the curve is $\geq 1 + c|x_1|^{\alpha+\epsilon}$ for some constant $c > 0$, and therefore, for such x_1,

$$|m((x_1, 0, \dots, 0), f(x_1, 0, \dots, 0)) - ((x_1, 0, \dots, 0), f(x_1, 0, \dots, 0))| \geq \frac{c}{5}|x_1|^{\alpha+\epsilon}.$$

So m is not in C^β for any $\beta > \alpha + \epsilon$.

Appendix B

LEMMA B.1 *Let $\mathcal{X}$ be the set of $k \times k$ real matrices. For $A \in \mathcal{X}$, A positive definite, consider the following linear equations for $X \in \mathcal{X}$:*

$$AX = X^{\mathsf{T}} A.$$

The dimension of the space of solutions is $\frac{k(k+1)}{2}$.

PROOF: Let $Y = AX$. Then the equation takes the form $Y^{\mathsf{T}} = Y$, i.e., Y is symmetric. The dimension of the space of real symmetric matrices is $\frac{k(k+1)}{2}$. □

LEMMA B.2 *Let A be a $k \times k$ real, symmetric, positive definite matrix, and let D be a $k \times k$ real, antisymmetric matrix, i.e., $D^{\mathsf{T}} = -D$. Then the dimension of the space of solutions to the following linear equations*

$$X^{\mathsf{T}} A - AX = D, \quad X \in \mathcal{X},$$

is $\frac{k(k+1)}{2}$.

PROOF: Both sides of equations are antisymmetric, so the number of equations is $\frac{k(k-1)}{2}$. By Lemma B.1, the dimension of the kernel is $\frac{k(k+1)}{2}$. The lemma follows since $\dim \mathcal{X} = k^2 = \frac{k(k-1)}{2} + \frac{k(k+1)}{2}$. □

Appendix C: Path-Connectedness of Σ

In this appendix we give a proof of the path-connectedness of the singular set Σ, as mentioned in the introduction.

PROOF: The proof is based on Lemma 4.1, the continuity of the map $y \to m(y)$ for $y \in \partial\Omega$. Suppose X and Y are points in Σ. Connect them by a smooth curve lying in Ω. It suffices to show that if we have a smooth arc $x(t)$ lying in G except for its endpoints, X_0 and X_1, which lie in Σ, then X_0 can be joined to X_1 by a continuous arc lying in Σ.

Consider the smooth arc $x(t)$, $0 \le t \le 1$, with $X(0) = X_0$, $X(1) = X_1$. For every t in $(0, 1)$ there is a unique point $y(t)$ on $\partial\Omega$ that is the closest point on $\partial\Omega$ to $x(t)$. Clearly, $y(t)$ is a continuous curve for $0 < t < 1$.

As $t \to 0$, $y(t)$ need not have a unique limit. Choose a sequence $t_i \to 0$, $t_{i+1} < t_i$, so that $y(t_i)$ converge to some point y_0. We have

$$m(y(t_i)) = x(t_i) \to X_0\,.$$

For $i \ge k$ large, replace the curve $y(t)$ for $t_{i+1} \le t \le t_i$ by the shortest arc on $\partial\Omega$ from $y(t_i)$ to $y(t_{i+1})$. Continuing this for all $i \ge k$ we get a new curve $\bar{y}(t)$ tending to y_0 as $t \to 0$, and $m(\bar{y}(t)) \to X_0$ as $t \to 0$ by the continuity of the map $y \to m(y)$. Doing the same near the other endpoint, for $t \to 1$, we obtain the desired arc $m(\bar{y}(t))$ in Σ connecting X_0 to X_1. □

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