On Some Conformally Invariant Fully Nonlinear Equations

AOBING LI
AND
YANYAN LI
Rutgers–The State University of New Jersey

1 Introduction

1.1 Statement of Problem

For $n \geq 3$, consider

$$-\Delta u = \frac{n-2}{2} u^{\frac{n+2}{n-2}} \quad \text{on } \mathbb{R}^n. \tag{1.1}$$

The celebrated Liouville-type theorem of Caffarelli, Gidas, and Spruck [1] asserts that positive C^2 solutions of (1.1) are of the form

$$u(x) = (2n)^{\frac{n-2}{4}} \left(\frac{a}{1 + a^2 |x - \bar{x}|^2} \right)^{\frac{n-2}{2}},$$

where $a > 0$ and $\bar{x} \in \mathbb{R}^n$. Under an additional decay hypothesis $u(x) = O(|x|^{2-n})$, the result was proven by Obata [27] and Gidas, Ni, and Nirenberg [10].

Let ψ be a Möbius transformation, i.e., a transformation generated by translations, multiplications by nonzero constants, and the inversion $x \to x/|x|^2$. Then for any positive C^2 function u on $\mathbb{R}^n$,

$$\left(u_\psi^{-\frac{n+2}{n-2}} \Delta u_\psi \right) = \left(u^{-\frac{n+2}{n-2}} \Delta u \right) \circ \psi \quad \text{on } \mathbb{R}^n,$$

where $u_\psi := |J_\psi|^{(n-2)/(2n)}(u \circ \psi)$ and J_ψ denotes the Jacobian of ψ. In particular, if u is a positive solution of (1.1), so is u_ψ.

Let $\mathcal{S}^{n\times n}$ denote the set of $n \times n$ real symmetric matrices.

DEFINITION 1.1 Let $H \in C^0(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^{n\times n})$. $H(\,\cdot\,, u, \nabla u, \nabla^2 u)$ is *conformally invariant on* $\mathbb{R}^n$ if for any Möbius transformation ψ and any positive function $u \in C^2(\mathbb{R}^n)$, it holds that

$$H\left(\,\cdot\,, u_\psi, \nabla u_\psi, \nabla^2 u_\psi\right) \equiv H(\cdot, u, \nabla u, \nabla^2 u) \circ \psi \quad \text{on } \mathbb{R}^n. \tag{1.2}$$

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Let $\mathcal{S}_+^{n\times n} \subset \mathcal{S}^{n\times n}$ denote the set of positive definite matrices, and let $O(n)$ denote the set of $n \times n$ real orthogonal matrices. The following result gives a characterization of conformally invariant operators.

THEOREM 1.2 *Let $H(\cdot, u, \nabla u, \nabla^2 u)$ be conformally invariant on $\mathbb{R}^n$. Then*

$$H(\cdot, u, \nabla u, \nabla^2 u) \equiv H\left(0, 1, 0, -\frac{n-2}{2}A^u\right),$$

where

$$(1.3)\quad A^u := -\frac{2}{n-2}u^{-\frac{n+2}{n-2}}\nabla^2 u + \frac{2n}{(n-2)^2}u^{-\frac{2n}{n-2}}\nabla u \otimes \nabla u - \frac{2}{(n-2)^2}u^{-\frac{2n}{n-2}}|\nabla u|^2 I$$

and I is the $n \times n$ identity matrix. Moreover, $H(0, 1, 0, \cdot)$ is invariant under orthogonal conjugation, i.e.,

$$H\left(0, 1, 0, -\frac{n-2}{2}O^{-1}AO\right) = H\left(0, 1, 0, -\frac{n-2}{2}A\right) \quad \forall A \in \mathcal{S}^{n\times n},\ O \in O(n).$$

Remark 1.3. If F is invariant under orthogonal conjugation, then it is elementary to check that $F(A^u)$ is conformally invariant.

Let $U \subset \mathcal{S}^{n\times n}$ be an open set satisfying

$$(1.4)\qquad O^{-1}UO = U \quad \forall O \in O(n)$$

and

$$(1.5)\qquad U \cap \{M + tN : 0 < t < \infty\} \quad \text{convex } \forall M \in \mathcal{S}^{n\times n},\ N \in \mathcal{S}_+^{n\times n}.$$

Let $F \in C^1(U)$ satisfy

$$(1.6)\qquad F(O^{-1}MO) = F(M) \quad \forall M \in U,\ O \in O(n),$$

and

$$(1.7)\qquad (F_{ij}(M)) > 0 \quad \forall M \in U \text{ where } F_{ij}(M) := \frac{\partial F}{\partial M_{ij}}(M).$$

The following theorem extends the result of Obata and Gidas, Ni, and Nirenberg to all conformally invariant operators of elliptic type (see Corollary 1.6 and Remark 1.9).

THEOREM 1.4 *For $n \geq 3$, let $U \subset \mathcal{S}^{n\times n}$ be open and satisfy* (1.4) *and* (1.5)*, and let $F \in C^1(U)$ satisfy* (1.6) *and* (1.7)*. Assume that $u \in C^2(\mathbb{R}^n)$ is a positive function satisfying*

$$F(A^u) = 1 \quad \text{on } \mathbb{R}^n$$

and

$$(A^u) \in U \quad on\ \mathbb{R}^n. \tag{1.8}$$

In addition, we assume, for $u_{0,1}(x) := |x|^{2-n}u(x/|x|^2)$, *that*

(1.9) $u_{0,1}$ *can be extended to a positive continuous function near the origin*,

and $u_{0,1}$ *satisfies*

$$\limsup_{x\to 0}(x \cdot \nabla u_{0,1}(x)) < \frac{n-2}{2}u_{0,1}(0) \tag{1.10}$$

and

$$\lim_{x\to 0}\left(|x|^2|\nabla u_{0,1}(x)|\right) = 0\,. \tag{1.11}$$

Then for some $\bar{x} \in \mathbb{R}^n$ *and some positive constants* a *and* b *satisfying* $2b^2a^{-2}I \in U$ *and* $F(2b^2a^{-2}I) = 1$,

$$u(x) \equiv \left(\frac{a}{1+b^2|x-\bar{x}|^2}\right)^{\frac{n-2}{2}} \quad \forall x \in \mathbb{R}^n. \tag{1.12}$$

Remark 1.5. If $\operatorname{trace}(M) \geq 0$ for all $M \in U$, then hypotheses (1.9), (1.10), and (1.11) follow from the following two hypotheses:

$$\limsup_{x\to\infty}(|x|^{n-2}u(x)) < \infty \tag{1.13}$$

and

$$\lim_{x\to 0}(|x||\nabla u_{0,1}(x)|) = 0\,. \tag{1.14}$$

Indeed, $\operatorname{trace}(A^u) \geq 0$ implies the superharmonicity of u, and therefore by the maximum principle $\liminf_{x\to\infty}(|x|^{n-2}u(x)) > 0$. This, together with (1.13), implies that $u_{0,1}$ is bounded from below and above by positive constants near the origin. By (1.8) and (1.4), $A^{u_{0,1}} \in U$, and therefore $u_{0,1}$ is also superharmonic. Condition (1.14) implies that $\lim_{r\to 0}\operatorname{osc}_{\partial B_r} u_{0,1} = 0$. On the other hand, the superharmonicity of $u_{0,1}$ implies that $\min_{r_1\leq|x|\leq r_2} u_{0,1}(x) \geq \min_{\partial B_{r_1}\cup\partial B_{r_2}} u_{0,1}$. It follows that $\lim_{x\to 0} u_{0,1}(x)$ exists, and therefore (1.9) is satisfied.

Let (M, g) be an n-dimensional, smooth Riemannian manifold without boundary; consider the Schouten tensor

$$A_g = \frac{1}{n-2}\left(\operatorname{Ric}_g - \frac{R_g}{2(n-1)}g\right),$$

where Ric_g and R_g denote, respectively, the Ricci tensor and the scalar curvature associated with g.

Similar to Definition 1.1, we have the following:

DEFINITION 1.1′ Let $(\mathbb{S}^n, g_0)$ be the standard sphere. $G(\cdot, w, \nabla_{g_0} w, \nabla^2_{g_0} w)$ is *conformally invariant on* $\mathbb{S}^n$ if G depends continuously on its arguments and for any conformal diffeomorphism $\varphi : \mathbb{S}^n \to \mathbb{S}^n$ and for any positive function $w \in C^2(\mathbb{S}^n)$, it holds that

$$G\big(\cdot, w_\varphi, \nabla_{g_0} w_\varphi, \nabla^2_{g_0} w_\varphi\big) \equiv G\big(\cdot, w, \nabla_{g_0} w, \nabla^2_{g_0} w\big) \circ \varphi ,$$

where $w_\varphi = |J_\varphi|^{(n-2)/(2n)}(w \circ \varphi)$, and J_φ denotes the Jacobian of φ.

It is elementary to deduce from Theorem 1.2 the following via a stereographic projection:

THEOREM 1.2′ *Let* $G(\cdot, w, \nabla_{g_0} w, \nabla^2_{g_0} w)$ *be conformally invariant on* $\mathbb{S}^n$. *Then*

$$G\big(\cdot, w, \nabla_{g_0} w, \nabla^2_{g_0} w\big) \equiv G\left(S, 1, 0, -\frac{n-2}{2} A_g\right) \quad \forall \text{ positive } w \in C^2(\mathbb{S}^n) ,$$

where $g = w^{4/(n-2)} g_0$ *and* S *is the south pole of* $\mathbb{S}^n$. *Moreover,* $G(S, 1, 0, -\frac{n-2}{2} A_g)$ *depends only on* $\lambda(A_g)$, *the eigenvalues of* A_g *with respect to* g.

A consequence of Theorem 1.4 is the following:

COROLLARY 1.6 *For* $n \geq 3$, *let* $U \subset \mathcal{S}^{n\times n}$ *satisfy* (1.4) *and* (1.5), *and let* $F \in C^1(U)$ *satisfy* (1.6) *and* (1.7). *Assume that* $w \in C^2(\mathbb{S}^n)$ *is a positive function satisfying*

$$F(A_g) = 1 \ \text{ on } \mathbb{S}^n \quad \text{and} \quad A_g \in U \ \text{ on } \mathbb{S}^n \quad \text{where } g = w^{\frac{4}{n-2}} g_0 .$$

Then

$$w \equiv a|J_\varphi|^{\frac{n-2}{2n}}$$

where $a > 0$ *is some constant and* $\varphi : \mathbb{S}^n \to \mathbb{S}^n$ *is some conformal diffeomorphism.*

Remark 1.7. In the above corollary, $A_g \in U$ means that at every point $x \in \mathbb{S}^n$, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_x(\mathbb{S}^n, g)$, $(A_g(e_i, e_j)) \in U$. Similarly, $F(A_g) = 1$ means that $F(A_g(e_i, e_j)) = 1$.

Remark 1.8. In fact, the regularity assumption on w in Corollary 1.6 can be weakened (as in Theorem 1.4). For instance, we only need to assume, for some $\bar{P} \in \mathbb{S}^n$, that w is a positive function in $C^2(\mathbb{S}^n \setminus \{\bar{P}\}) \cap C^0(\mathbb{S}^n)$ satisfying

$$\limsup_{P \to \bar{P}} \big(\text{dist}_{g_0}(P, \bar{P})|\nabla_{g_0} w(P)|\big) < \frac{n-2}{2} w(\bar{P}) .$$

Remark 1.9. Corollary 1.6 in the special case $F = \text{trace}$ and $U = \{M \in \mathcal{S}^{n\times n} : \text{trace}(M) > 0\}$ is the previously mentioned result of Obata and Gidas, Ni, and Nirenberg.

For $1 \le k \le n$, let

$$\sigma_k(\lambda) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1}, \ldots, \lambda_{i_k}, \quad \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n,$$

denote the k^{th} symmetric function, and let Γ_k denote the connected component of $\{\lambda \in \mathbb{R}^n : \sigma_k(\lambda) > 0\}$ containing the positive cone $\{\lambda \in \mathbb{R}^n : \lambda_1, \ldots, \lambda_n > 0\}$.

Clearly,

$$\Gamma_n = \left\{\lambda \in \mathbb{R}^n : \lambda_1, \ldots, \lambda_n > 0\right\}, \quad \Gamma_1 = \left\{\lambda \in \mathbb{R}^n : \lambda_1 + \cdots + \lambda_n > 0\right\}.$$

It is known (see, e.g., [2]) that Γ_k is a convex cone with its vertex at the origin,

$$\Gamma_n \subset \cdots \subset \Gamma_2 \subset \Gamma_1\,, \tag{1.15}$$

$$\frac{\partial \sigma_k}{\partial \lambda_i} > 0 \quad \text{in } \Gamma_k\,,\ 1 \le i \le n\,, \tag{1.16}$$

$$\sigma_k^{\frac{1}{k}} \text{ is concave in } \Gamma_k\,,$$

and, for some $\delta_1 = \delta_1(n) > 0$, that

$$\sum_{i=1}^{n} \lambda_i \ge \delta_1 \quad \text{in the set } \{\lambda \in \Gamma_k : \sigma_k(\lambda) \ge 1\} \quad \forall\, 1 \le k \le n\,. \tag{1.17}$$

Fully nonlinear elliptic equations involving $\sigma_k((\lambda(D^2u))$, as well as for more general f instead of σ_k, have been investigated in the classical and pioneering paper of Caffarelli, Gidas, and Nirenberg [2]. For extensive studies and outstanding results on such equations, see, for example, Guan and Spruck [11], Trudinger [33], Trudinger and Wang [34], and the references therein. On Riemannian manifolds of nonnegative curvature, Li studied in [23] equations

$$\sigma_k^{\frac{1}{k}}\left(\lambda(\nabla_g^2 u + g)\right) = \psi(x, u)\,, \tag{1.18}$$

where $\lambda(\nabla_g^2 u + g)$ denotes eigenvalues of $\nabla_g^2 u + g$ with respect to g. On general Riemannian manifolds, Viaclovsky [35, 37] introduced and systematically studied equations

$$\sigma_k^{\frac{1}{k}}(\lambda(A_g)) = \psi(x, u)\,, \tag{1.19}$$

where $\lambda(A_g)$ denotes the eigenvalues of A_g with respect to g. On 4-dimensional general Riemannian manifolds, remarkable results on (1.19) for $k = 2$ were obtained by Chang, Gursky, and Yang in [3, 4], which include Liouville-type theorems, existence and compactness of solutions, as well as applications to topology. On the other hand, works on the Yamabe equation by Caffarelli, Gidas, and Spruck [1], Schoen [30, 31], Li and Zhu [26], and Li and Zhang [25] have played an important role in our approach to the study of (1.19) as developed in the present paper.

A large part of this paper concerns

$$\sigma_k(\lambda(A_g)) = 1 \tag{1.20}$$

together with

$$\lambda(A_g) \in \Gamma_k\,. \tag{1.21}$$

Let $g_1 = u^{4/(n-2)} g_0$ be a conformal change of metrics; then (see, e.g., [35]),

$$\begin{aligned} A_{g_1} = &-\frac{2}{n-2} u^{-1} \nabla^2_{g_0} u + \frac{2n}{(n-2)^2} u^{-2} \nabla_{g_0} u \otimes \nabla_{g_0} u \\ &- \frac{2}{(n-2)^2} u^{-2} |\nabla_{g_0} u|^2_{g_0} g_0 + A_{g_0}\,. \end{aligned}$$

Let $g = u^{4/(n-2)} g_{\mathrm{flat}}$, where g_{flat} denotes the Euclidean metric on $\mathbb{R}^n$. Then by the above transformation formula,

$$A_g = u^{\frac{4}{n-2}} A^u_{ij}\, dx^i\, dx^j$$

where A^u is given by (1.3). Equations (1.20) and (1.21) take the form

$$\sigma_k(\lambda(A^u)) = 1 \quad \text{on } \mathbb{R}^n \tag{1.22}$$

and

$$\lambda(A^u) \in \Gamma_k \quad \text{on } \mathbb{R}^n. \tag{1.23}$$

By (1.15), (1.23) implies that

$$-\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \Delta u = \sum_{i=1}^{n} A^u_{ii} = \operatorname{trace}(A^u) > 0\,;$$

i.e., u is superharmonic.

Our next result extends the Liouville-type theorem of Caffarelli, Gidas, and Spruck to all σ_k, $1 \le k \le n$. For $k = 1$, equation (1.22) is (1.1).

THEOREM 1.10 *For $n \ge 3$ and $1 \le k \le n$, let $u \in C^2(\mathbb{R}^n)$ be a positive solution of* (1.22) *satisfying* (1.23). *Then for some $a > 0$ and $\bar{x} \in \mathbb{R}^n$,*

$$u(x) = c(n,k) \left(\frac{a}{1 + a^2 |x - \bar{x}|^2} \right)^{\frac{n-2}{2}} \quad \forall x \in \mathbb{R}^n \tag{1.24}$$

where

$$c(n,k) = 2^{\frac{n-2}{4}} \binom{n}{k}^{\frac{n-2}{4k}}.$$

Remark 1.11. The case $k = 2$ and $n = 4$ was obtained by Chang, Gursky, and Yang [3]. More recently they [5] independently established the result for $k = 2$ and $n = 5$, and they also established the result for $k = 2$ and $n \ge 6$ under the additional hypothesis $\int_{\mathbb{R}^n} u^{2n/(n-2)} < \infty$. Under the additional hypothesis that $(1/|x|^{n-2}) u(x/|x|^2)$ can be extended to a C^2 positive function near $x = 0$, the case $2 \le k \le n$ was obtained by Viaclovsky [35, 36]. As mentioned above, the case $k = 1$ was obtained by Caffarelli, Gidas, and Spruck, while under an additional hypothesis that $(1/|x|^{n-2})u(x/|x|^2)$ is bounded near $x = 0$, the case $k = 1$ was obtained by Obata and by Gidas, Ni, and Nirenberg.

Remark 1.12. By the ellipticity and the smoothness of $\sigma_k^{1/k}$ in Γ_k (see [2]), a C^2 solution of (1.22) satisfying (1.23) is C^∞.

The methods of Chang, Gursky, and Yang in [3, 5] include an ingenious way of using the Obata technique that, as they pointed out, can be generalized to establish the uniqueness of solutions on general Einstein manifolds. Our proof of Theorem 1.10 is very different from that of [3, 5]. A crucial ingredient in our proof is the following Harnack-type inequality.

THEOREM 1.13 *For $n \geq 3$, $1 \leq k \leq n$, and $R > 0$, let $B_{3R} \subset \mathbb{R}^n$ be a ball of radius $3R$ and let $u \in C^2(B_{3R})$ be a positive solution of*

$$\sigma_k(A^u) = 1 \quad \text{in } B_{3R} \tag{1.25}$$

satisfying

$$\lambda(A^u) \in \Gamma_k \quad \text{in } B_{3R}\,. \tag{1.26}$$

Then

$$\Big(\max_{\overline{B}_R} u\Big)\Big(\min_{\overline{B}_{2R}} u\Big) \leq C(n)R^{2-n}. \tag{1.27}$$

Remark 1.14. The above Harnack-type inequality for $k = 1$ was obtained by Schoen [30] based on the Liouville-type theorem of Caffarelli, Gidas, and Spruck. An important step toward our proof of Theorem 1.13 was taken in an earlier work of Li and Zhang [25], where they gave a different proof of Schoen's Harnack-type inequality without using the Liouville-type theorem.

Our next result concerns existence and compactness of solutions to a fully nonlinear version of the Yamabe problem.

THEOREM 1.15 *For $n \geq 3$ and $1 \leq k \leq n$, let (M, g) be an n-dimensional, smooth, compact, locally conformally flat Riemannian manifold without boundary satisfying*

$$\lambda(A_g) \in \Gamma_k \quad \text{on } M.$$

Then there exists some smooth positive function u on M such that $\hat{g} = u^{4/(n-2)}g$ satisfies

$$\lambda(A_{\hat{g}}) \in \Gamma_k\,, \quad \sigma_k(\lambda(A_{\hat{g}})) = 1\,, \quad \text{on } M. \tag{1.28}$$

Moreover, if (M, g) is not conformally diffeomorphic to the standard n-sphere, all solutions of the above satisfy, for all $m \geq 0$, that

$$\|u\|_{C^m(M,g)} + \|u^{-1}\|_{C^m(M,g)} \leq C$$

where C depends only on (M, g) and m.

Remark 1.16. For $k = 1$, it is the Yamabe problem for locally conformally flat manifolds with positive Yamabe invariants, and the result is due to Schoen [29, 31]. The Yamabe problem was solved through the work of Yamabe, Trudinger, Aubin, and Schoen. For $k = 2$ and $n = 4$, the result was proven without the local conformal flatness hypothesis by Chang, Gursky, and Yang [3]. For $k = n$, the existence result was established by Viaclovsky [37] for a class of manifolds that are not necessarily locally conformally flat. For $k \neq \frac{n}{2}$, the existence part was independently obtained by Guan and Wang in [13] using a heat flow method. More recently, Guan, Viaclovsky, and Wang [12] have proven the algebraic fact that $\lambda(A_g) \in \Gamma_k$ for $k \geq \frac{n}{2}$ implies the positivity of the Ricci tensor, and therefore (M, g) is conformally covered by $\mathbb{S}^n$, and both the existence and compactness results in this case follow from known results.

Theorem 1.10, Theorem 1.13, and Theorem 1.15 have been announced, with a rather detailed outline of proofs, in [17]. Our methods, except in a part of the proof of the $2 \leq k \leq \frac{n}{2}$ case of Theorem 1.10, do not rely on any variational structures of the equation and therefore, as developed in the present paper, can be extended to more general, fully nonlinear equations. Also, we present a new proof of the existence and compactness results that does not rely on Liouville-type theorems. As a result, the existence and compactness results are established for more general, fully nonlinear equations including those for which Liouville-type theorems are not available. In the following we present these more general results, as well as some new results (e.g., Theorem 1.31). Theorems 1.20, 1.25, 1.27, 1.28, and 1.31 include $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ as special cases.

Let $U \subset \mathcal{S}^{n\times n}$ be an open set satisfying (1.4), and let $F \in C^1(U)$ satisfy (1.6). Then

$$F(M) = f(\lambda(M)), \quad M \in U,$$

for some C^1 function f defined on

$$\Gamma = \Gamma_U = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \lambda_1, \dots, \lambda_n \text{ are eigenvalues of some } M \in U\}.$$

From now on, let

(1.29) $\Gamma \subset \mathbb{R}^n$ be an open convex cone with vertex at the origin

satisfying

$$\Gamma_n \subset \Gamma \subset \Gamma_1 \quad \text{and} \quad \Gamma \text{ is symmetric in the } \lambda_i, \tag{1.30}$$

and let

(1.31) f be a smooth, symmetric, concave function on Γ,

which satisfies, for some constant $\delta > 0$,

$$\min_{1\leq i\leq n} f_{\lambda_i} > 0, \quad \sum_i f_{\lambda_i} \geq \delta \quad \text{on } \Gamma, \tag{1.32}$$

and

$$\liminf_{\lambda\in\Gamma,\lambda\to 0} f(\lambda) > -\frac{1}{\delta}\,. \tag{1.33}$$

If Γ satisfies (1.29) and (1.30), and $f \in C^1(\Gamma)$ satisfies $f_{\lambda_i} > 0$ in Γ for $1 \le i \le n$, then it is easy to see that (F, U) satisfies (1.4), (1.5), (1.6), and (1.7), where $F(M) := f(\lambda(M))$ and $U = \{M \in \mathcal{S}^{n\times n} \mid \lambda(M) \in \Gamma\}$.

DEFINITION 1.17 Let (f, Γ) satisfy (1.29), (1.30), (1.31), (1.32), and (1.33). We say that (f, Γ) *satisfies condition* (H_α) for some $\alpha > 0$, if there exist some positive constants ϵ_1 and c_1 such that for any $(\lambda, \xi) \in \Gamma \times \mathbb{R}^n$ satisfying $f(\lambda) \le \alpha$, $|\xi| \ge 1/\epsilon_1$, and $|\xi_i(\lambda_i - \frac{1}{2}|\xi|^2)| \le \epsilon_1 |\xi|^3$, $1 \le i \le n$, we have

$$\sum_i f_{\lambda_i}(\lambda)\left\{\left(\lambda_i + \frac{1}{2}|\xi|^2 - |\xi_i|^2\right)^2 + |\xi_i|^2\big(|\xi|^2 - |\xi_i|^2\big)\right\} \ge c_1|\xi|^4 \sum_i f_{\lambda_i}(\lambda)\,. \tag{1.34}$$

Remark 1.18. For $1 \le k \le n$, $(\sigma_k^{1/k}, \Gamma_k)$ satisfy condition (H_α) for all $\alpha > 0$. When f is homogeneous of degree 1, condition (H_α) is the condition under which Guan and Wang established in an earlier paper [14] local gradient estimates of positive solutions of (1.37) (e.g., (1.34) above is (19) in their paper). In fact, the most crucial and delicate part of their local gradient estimates amounts to establishing (1.34) for all $(\sigma_k^{1/k}, \Gamma_k)$. When f is not homogeneous of degree 1, equation (1.37) is not the same as the one studied in [14]; nevertheless, condition (H_α) is still sufficient for local gradient estimates of positive solutions of (1.37).

Let (f, Γ) satisfy condition (H_α). In order to establish existence results, we introduce a useful homotopy: for $0 \le t \le 1$, let

$$f_t(\lambda) = f(t\lambda + (1-t)\sigma_1(\lambda)e) \tag{1.35}$$

be defined on

$$\Gamma_t := \{\lambda \in \mathbb{R}^n : t\lambda + (1-t)\sigma_1(\lambda)e \in \Gamma\} \quad \text{where } e = (1, 1, \dots, 1). \tag{1.36}$$

PROPOSITION 1.19 *Condition* (H_α) *has the following useful properties*:

(i) *For* $n \ge 3$, *if* (f, Γ) *satisfies condition* (H_α), *then* (f_t, Γ_t), *as defined in* (1.35) *and* (1.36), *satisfies condition* (H_α) *with some* δ', ϵ_1', *and* c_1' *depending only on* δ, ϵ_1, c_1, *and an upper bound of* α. *In particular, these constants are independent of* $0 \le t \le 1$.

(ii) *Assume that* (f, Γ) *satisfies condition* (H_α) *for some positive constant* α. *For some* $0 < s_0 \le 1$, *let* $f^{(s)}(\lambda) := s^{-1} f(s\lambda)$, $s \ge s_0 > 0$. *Then* $(f^{(s)}, \Gamma)$ *satisfies condition* $(H_{\alpha/s})$ *with the constants* δ *and* ϵ_1 *replaced by* $s_0\delta$ *and* $\sqrt{s_0}\epsilon_1$.

(iii) *Let* $\beta \in C^\infty(0, \infty)$ *satisfy, for some* $\delta > 0$, $\liminf_{s\to 0} \beta(s) > \delta^{-1}$, $\beta' \ge \delta$, *and* $\beta'' \le 0$ *on* $(0, \infty)$, *and let* $f(\lambda) := \beta(\sigma_1(\lambda))$. *Then* (f, Γ_1) *satisfies condition* (H_α) *for all* $\alpha > 0$.

Our next theorem, a slightly more general version of a result of Guan and Wang [14], provides local gradient and second derivative estimates for positive solutions of

$$f\big(\lambda(A_{u^{4/(n-2)}g})\big) = h\,, \quad \lambda\big(A_{u^{4/(n-2)}g}\big) \in \Gamma\,. \tag{1.37}$$

THEOREM 1.20 *Let (M, g) be a smooth, complete Riemannian manifold of dimension $n \geq 3$. Assume that (M, g) has a positive injectivity radius i_0 and that the curvature tensor R_{ijkl} together with their covariant derivatives up to second order are bounded. For a geodesic ball B_{3r} in M of radius $3r \leq \frac{1}{2} i_0$, assume that, for some $\alpha > 0$, (f, Γ) satisfies condition (H_α), and assume that $h \in C^2(B_{3r})$ satisfies $0 < h \leq \alpha$ on B_{3r}. Then for any positive C^2 solution u of (1.37) in B_{3r}, we have, on B_r,*

$$\|\nabla_g(\log u)(x)\|_g \leq C + C\Big(\sup_{B_{2r}} u^{\frac{2}{n-2}}\Big)\sqrt{1 + \sup_{B_{2r}}|\nabla h|} \tag{1.38}$$

and

$$\begin{aligned}\big\|\nabla_g^2(\log u)(x)\big\|_g \leq \\ C + C\Big(\sup_{B_{2r}} u^{\frac{4}{n-2}}\Big)\Big(1 + \sup_{B_{2r}}|\nabla h|\Big) + C\sup_{B_{2r}}(|\nabla h|^2 + |\nabla^2 h|)\,,\end{aligned} \tag{1.39}$$

where C is some positive constant depending only on δ, ϵ_1, c_1, an upper bound of α, r, i_0, and a bound of R_{ijkl} together with their covariant derivatives up to second order.

Remark 1.21. In Section 3, we introduce a condition (H'_α) that is weaker than condition (H_α). When substituting condition (H'_α) for condition (H_α), (1.38) still holds after replacing the right-hand side by $C(1 + \sup_{B_{2r}} u^{2/(n-2)})$, and (1.38) still holds after replacing the right-hand side by $C(1+\sup_{B_{2r}} u^{4/(n-2)})$, where C depends only on (f, Γ), an upper bound of α, $\sup_{B_{2r}}(|\nabla h| + |\nabla^2 h|)$, i_0, r, and a bound of R_{ijkl} together with their covariant derivatives up to second order. This, as well as a more explicit dependence of C, can be obtained by modifying the proof of Theorem 1.20 in Section 3. Similarly, Theorem 1.27, Theorem 1.28, and Theorem 1.31, as well as estimate (1.44) in Theorem 1.25, hold when substituting the weaker condition (H'_1) for condition (H_1).

Remark 1.22. In Theorem 1.20, Γ does not need to be a convex cone; an open subset will be enough, C^2 regularity of f is enough, and $\min_{1\leq i\leq n} f_{\lambda_i} > 0$ in (1.32) can be weakened to $\min_{1\leq i\leq n} f_{\lambda_i} \geq 0$.

Remark 1.23. Guan and Wang [14] obtained local gradient and second derivative estimates for $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$, $1 \leq k \leq n$. Our proof of (1.38) follows closely that of [14], while our proof of (1.39) differs in that we estimate the full Hessian of u instead of only the trace of the Hessian. An estimate of the trace would not be enough in this generality. Global gradient and second derivative estimates for $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ were obtained by Viaclovsky [37]. For the related equation

(1.18) on manifolds of nonnegative curvature, global gradient and second derivative estimates were obtained by Li in [23]. It remains open whether such estimates hold for (1.18) without any curvature hypothesis. Another question is, as raised to the second author by Ivochkina, whether the estimates hold for the σ_k equations under some weaker σ_j curvature hypothesis.

Remark 1.24. Let (f, Γ) be as in Theorem 1.20, and let (f_t, Γ_t) be as in (1.35) and (1.36). Then estimates (1.38) and (1.39) hold for some constant C independent of $0 \le t \le 1$.

Next we give an extension of Theorem 1.15, for which we also assume

$$\limsup_{\lambda \to \bar{\lambda}} f(\lambda) \le 0 \quad \forall \bar{\lambda} \in \partial\Gamma \tag{1.40}$$

and

$$\liminf_{s \to \infty} \min_{\lambda \in K} f(s\lambda) = \infty \quad \forall \text{ compact } K \subset \Gamma. \tag{1.41}$$

THEOREM 1.25 *For $n \ge 3$, let (f, Γ) satisfy condition* (H_1), (1.40), *and* (1.41), *and let (M, g) be an n-dimensional, smooth, compact, locally conformally flat Riemannian manifold without boundary that satisfies*

$$\lambda(A_g) \in \Gamma \quad \text{on } M. \tag{1.42}$$

Then there exists some smooth positive function u on M such that $\hat{g} = u^{4/(n-2)} g$ satisfies

$$\lambda(A_{\hat{g}}) \in \Gamma\,, \quad f(\lambda(A_{\hat{g}})) = 1\,, \quad \text{on } M. \tag{1.43}$$

Moreover, if (M, g) is not conformally diffeomorphic to the standard n-sphere, all solutions of the above satisfy, for all $m \ge 0$, that

$$\|u\|_{C^m(M,g)} + \|u^{-1}\|_{C^m(M,g)} \le C \tag{1.44}$$

where C depends only on (M, g), (f, Γ), and m.

Remark 1.26. For $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$, the above theorem is Theorem 1.15. For general (f, Γ), the problem does not have a variational formulation, and therefore the heat flow method in [13] does not seem to apply.

The following is an extension of Theorem 1.13.

THEOREM 1.27 *For $n \ge 3$, assume that (f, Γ) satisfies condition* (H_1) *and*

$$0 \text{ does not belong to } \overline{f^{-1}(1)}\,. \tag{1.45}$$

Let $B_{3R} \subset \mathbb{R}^n$, $R > 0$, be a ball of radius $3R$, and let $u \in C^2(B_{3R})$ be a positive solution of

$$f(\lambda(A^u)) = 1 \quad \text{in } B_{3R} \tag{1.46}$$

satisfying

$$\lambda(A^u) \in \Gamma \quad \text{in } B_{3R}\,. \tag{1.47}$$

Then

$$\Big(\max_{\overline{B}_R} u\Big)\Big(\min_{\overline{B}_{2R}} u\Big) \le C R^{2-n} \tag{1.48}$$

where C depends only on (f, Γ).

For our next theorem, we also assume, for some constant $\delta_2 > 0$,

$$\sum_{i=1}^{n} \lambda_i \ge \delta_2 \quad \text{in the set } \{\lambda \in \Gamma : f(\lambda) = 1\}\,. \tag{1.49}$$

THEOREM 1.28 *For $n \ge 3$, assume that (f, Γ) satisfies condition* (H_1), (1.45)*, and* (1.49)*. Let $u \in C^2(\mathbb{R}^n)$ be a positive solution of*

$$f(\lambda(A^u)) = 1 \quad \text{on } \mathbb{R}^n \tag{1.50}$$

satisfying

$$\lambda(A^u) \in \Gamma \quad \text{on } \mathbb{R}^n. \tag{1.51}$$

Then

$$0 < \liminf_{|x|\to\infty}(|x|^{n-2}u(x)) \le \limsup_{|x|\to\infty}(|x|^{n-2}u(x)) < \infty \tag{1.52}$$

and

$$\limsup_{|x|\to\infty} \Big(|x|^{n-1}|\nabla u(x)| + |x|^n|\nabla^2 u(x)|\Big) < \infty\,. \tag{1.53}$$

Moreover,

$$\int_{\mathbb{R}^n} u^{\frac{2n}{n-2}} + \int_{\mathbb{R}^n} R_g u^{\frac{2n}{n-2}} + \int_{\mathbb{R}^n} R_g u^{\frac{n+2}{n-2}} < \infty\,, \tag{1.54}$$

where R_g denotes the scalar curvature of $g = u^{4/(n-2)} g_{\text{flat}}$.

We make the following conjecture:

CONJECTURE 1.29 *Under the hypothesis of Theorem* 1.28*, u is of the form* (1.12).

Remark 1.30. Let u be as in Theorem 1.28; then, by (1.52) and (1.53), the Kelvin transformation $u_{0,1}(x) = (1/|x|^{n-2})u(x/|x|^2)$ satisfies, for some positive constant C,

$$C^{-1} \le u_{0,1}(x) \le C\,, \quad |x||\nabla u_{0,1}(x)| + |x|^2\big|\nabla^2 u_{0,1}(x)\big| \le C\,, \quad \forall x \in B_1 \setminus \{0\}\,.$$

In view of Theorem 1.4, to prove Conjecture 1.29, we only need to show that $\lim_{x\to 0} |x||\nabla u_{0,1}(x)| = 0$. By interpolation, a sufficient way to establish this is to show that $u_{0,1}$ can be extended to a Hölder-continuous function near the origin. As in the proof of Theorem 1.10 for $k > \frac{n}{2}$, if $\Gamma \subset \Gamma_k$ for some $k > \frac{n}{2}$, then indeed $u_{0,1}$ can be extended to a Hölder-continuous function. Therefore Conjecture 1.29 holds under an additional hypothesis that $\Gamma \subset \Gamma_k$ for some $k > \frac{n}{2}$.

A problem related to Theorem 1.10 is to classify positive $C^{1,1}$ viscosity solutions of

$$\begin{cases} \sigma_k(\lambda(A^u)) = 0 & \text{in } \mathbb{R}^n \\ \lambda(A^u) \in \overline{\Gamma}_k \text{ a.e.} & \text{in } \mathbb{R}^n. \end{cases} \tag{1.55}$$

For $k = 2$ and $n = 4$, Chang, Yang, and Gursky proved in [3] that solutions of (1.55) must be constants.

Our last theorem follows:

THEOREM 1.31 *For $n \geq 3$, assume that (f, Γ) satisfies condition* (H_1) *and assume in addition that f is homogeneous of degree $\beta \in (0, 1]$. Let u be a positive continuous function on $\mathbb{R}^n$ having the following property: for any $R > 1$, there exists a sequence of positive numbers $\epsilon_i \to 0$ and a sequence of positive functions $u_i \in C^2(B_{2R})$ satisfying, on B_{2R},*

$$f(\lambda(A^{u_i})) = \epsilon_i, \quad \lambda(A^{u_j}) \in \Gamma, \quad \text{on } B_{2R},$$

and $u_i \to u$ in $C^0(\bar{B}_R)$. Then u must be a constant on $\mathbb{R}^n$.

We make the following conjecture.

CONJECTURE 1.32 *For $n \geq 3$ and $1 \leq k \leq n$, all positive $C^{1,1}$ viscosity solutions of* (1.55) *must be constants.*

1.2 Organization of the Paper

Our paper is organized as follows: In Section 2, we establish Theorem 1.4. Our proof, different from the ones in [1, 10, 27, 35, 36], is in the spirit of the new proof of the Liouville-type theorem of Caffarelli, Gidas, and Spruck given by Li and Zhu [26] (see also a related work [8]). We also make use of the substantial simplifications of Li and Zhang in [25] to the proof in [26]. Our proof includes some new ingredients as well. The proof is along the line of the pioneering work of Gidas, Ni, and Nirenberg [10], which in particular does not need the kind of divergence structure needed for the method of Obata [27] and therefore can be applied in much more generality.

In Section 3, we prove Proposition 1.19 and also establish Theorem 1.20 under a weaker condition (H'_α). In our paper, blowup arguments are used, which require local derivative estimates of solutions. For (σ_1, Γ_1) (the Yamabe problem), such estimates follow from standard elliptic theories, while it is not the case even for $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$, $k \geq 2$. These very useful local gradient and second derivative estimates for $(\sigma_k^{1/k}, \Gamma_k)$ were obtained by Guan and Wang [14]. By the concavity of f, $C^{2,\alpha}$ estimates hold due to the classical work of Evans [9] and Krylov [16]. Our proof of the existence part of Theorem 1.25, even for $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$, requires that local gradient and second derivative estimates hold in more generality. Theorem 1.20, together with (i) in Proposition 1.19, provides this generality. More specifically, for $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$, we extend the local estimates in [14] for $(\sigma_k^{1/k}, \Gamma_k)$ to $((\sigma_k^{1/k})_t, (\Gamma_k)_t)$, with estimates uniform in $0 \le t \le 1$.

In Section 4, we prove Theorem 1.25. We first establish the compactness result (1.44). Although our proof of (1.44), as in Schoen [31] for the Yamabe problem (i.e., $(f, \Gamma) = (\sigma_1, \Gamma_1)$), makes use of the deep result of Schoen and Yau [32] on developing maps, the proof is different from that of [31] even in the case of the Yamabe problem. The proof of (1.44) for $(f, \Gamma) = (\sigma_1, \Gamma_1)$ in [31] relies on the Liouville-type theorem of Caffarelli, Gidas, and Spruck, but our new proof does not. As a result, our new argument allows us to establish (1.44) even though Liouville-type theorems are not available in this generality. To prove the existence result, we only need to consider the case that (M, g) is not conformally diffeomorphic to standard spheres. We introduce a homotopy (f_t, Γ_t), $0 \le t \le 1$, as defined in (1.35) and (1.36), which, together with another homotopy $tf_0 + (1-t)\sigma_1$, establishes a natural link between (1.43) and the Yamabe equation. In fact, estimate (1.44) holds for (f_t, Γ_t) with a bound C independent of t. Then we make use of the degree for second-order, fully nonlinear elliptic operators [22] to show that the total degree of solutions to f_1 $(= f)$ is the same as that to f_0. Making another homotopy, $H_t = tf_0 + (1-t)\sigma_1$, we show that the total degree for H_1 $(= f_0)$ is equal to the total degree for H_0 $(= \sigma_1)$. By the result in [31] for the Yamabe problem, the total degree for σ_1 is equal to -1; thus the total degree for f is nonzero, and the existence result follows.

In Section 5, we establish Theorem 1.27, the Harnack-type inequality. This is an important step in our proof of the Liouville-type theorems Theorem 1.28 and Theorem 1.10. The Harnack-type inequality in the case $(f, \Gamma) = (\sigma_1, \Gamma_1)$ was established by Schoen in [30]. The proof in [30] relies on the Liouville-type theorem of Caffarelli, Gidas, and Spruck. Li and Zhang gave in [25] a different proof of Schoen's Harnack-type inequality that does not make use of the Liouville-type theorem. Our proof of Theorem 1.27 is an adaptation of the arguments in [25], with the help of Theorem 1.20 and the classical $C^{2,\alpha}$ estimates of Evans [9] and Krylov [16].

In Section 6, we first use Theorem 1.27 to establish Theorem 1.28, which gives, for a large class of (f, Γ), sharp asymptotic behavior at infinity of an entire solution. We point out that in our proof of Theorems 1.27, 1.28, and 1.31 and the compactness part of Theorem 1.25 for $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$, the use of Theorem 1.20 can be replaced by the use of theorem 1 in [14]. Then we establish Theorem 1.10 by distinguishing two cases: In the case $k > \frac{n}{2}$, Theorem 1.10 is proven by the sharp asymptotic behavior of an entire solution together with a result of Trudinger and Wang [34]. In the case $1 \le k \le \frac{n}{2}$, Theorem 1.10 is proven by the sharp asymptotic behavior of an entire solution together with the Obata-type integral formula of Viaclovsky [35].

In Section 7 we establish Theorem 1.2, a characterization of conformally invariant operators on $\mathbb{R}^n$ and $\mathbb{S}^n$. In Section 8 we prove Theorem 1.31.

At the end of this introduction, we draw the attention of readers to some very recent work of Gursky and Viaclovsky [15] on fully nonlinear equations with negative curvature ($-A_g \in \Gamma_k$ instead of $A_g \in \Gamma_k$). We also draw attention to a forthcoming paper of Gursky and Viaclovsky where they have introduced some new invariants in the positive curvature case and have used them to obtain new existence results for (1.28) on Riemannian manifolds that are not necessarily locally conformally flat. We are delighted to be informed that Theorem 1.10 has been useful in defining the invariants.

1.3 Note Added in 2003

(i) After submitting the present paper, we removed technical assumption (1.34) in condition (H_1) in Theorem 1.25 and Theorem 1.27; see [18, 21]. Neither the gradient estimates in Theorem 1.20 nor that in [14, theorem 1] were used in establishing these more general results. The proof of the more general results require only one more ingredient: the following quantitative version of a calculus lemma [23, lemma 11.2] used in the proofs of Theorem 1.25 and Theorem 1.27.

LEMMA 1.33 *Let $a > 0$ be a constant and let $B_{8a} \subset \mathbb{R}^n$ be the ball of radius $8a$ and centered at the origin, $n \geq 3$. Assume that $u \in C^1(B_{8a})$ is a nonnegative function satisfying*

$$u_{x,\lambda}(y) \leq u(y) \quad \forall x \in B_{4a},\ y \in B_{8a},\ 0 < \lambda < 2a,\ \lambda < |y-x|\,,$$

where

$$u_{x,\lambda}(y) := \left(\frac{\lambda}{|y-x|}\right)^{n-2} u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right).$$

Then

$$|\nabla u(x)| \leq \frac{n-2}{2a} u(x) \quad \forall |x| < a\,.$$

With this lemma, the C^0 and C^1 estimates needed for the proofs of Theorem 1.25 and Theorem 1.27 follow immediately from the arguments in the proofs and hold for very general nonlinearity (f, Γ). In addition, the concavity assumption on F in Theorem 1.27 can be removed. This requires another argument based on the maximum principle. We did not see the elementary proof of Lemma 1.33 at the time of submitting the present paper, but proved it soon afterwards. These more general results, as well as our C^0 and C^1 estimates based on the method of moving planes (or spheres), were presented by the second author in his 45-minute invited talk at ICM 2002 in August 2002 in Beijing.

(ii) We have also extended the Liouville-type theorem, Theorem 1.10, to general, conformally invariant, fully nonlinear equations. Two proofs are given in [19, 20]; see also [21]. The proofs are different from that of Theorem 1.10. In particular, neither the gradient estimates in Theorem 1.20 nor that in [14, theorem 1] are used in the new proofs.

(iii) Similar results on certain manifolds with boundary are established in [21] under natural geometric boundary conditions.

2 Proof of Theorem 1.4

In this section we prove Theorem 1.4, using ideas from [8, 25, 26].

For $x \in \mathbb{R}^n$ and $\lambda > 0$, consider the Kelvin transformation of u:

$$u_{x,\lambda}(y) = \frac{\lambda^{n-2}}{|y-x|^{n-2}} u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right), \quad y \in \mathbb{R}^n \setminus \{x\}.$$

A straightforward calculation yields that for any x, λ, and y, there exists some $O \in O(n)$ (depending on x, λ, and y), such that

$$A^{u_{x,\lambda}}(y) = O^{-1} A^u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) O.$$

It follows, by (1.4) and (1.6), that

$$F(A^{u_{x,\lambda}}(y)) = F\left(A^u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right)\right) \quad \forall y \in \mathbb{R}^n \setminus \{x\}.$$

We mainly need to establish, under the hypotheses of Theorem 1.4, the following:

LEMMA 2.1 *For every $x \in \mathbb{R}^n$, there exists $0 < \bar{\lambda}(x) < \infty$ such that*

$$u_{x,\bar{\lambda}(x)}(y) = u(y) \quad \forall y \in \mathbb{R}^n \setminus \{x\}.$$

PROOF: It is elementary to check that for every $x \in \mathbb{R}^n$, $u(x + \cdot)$ satisfies the same hypothesis of u. Therefore we only need to prove the lemma for $x = 0$. Let $w := u_{0,1}$. For $r > 0$ and $\theta \in \mathbb{S}^{n-1}$,

$$\frac{\partial}{\partial r}\left(r^{\frac{n-2}{2}} w(r,\theta)\right) = r^{\frac{n-4}{2}}\left(\frac{n-2}{2} w(r,\theta) - r\frac{\partial w}{\partial r}(r,\theta)\right).$$

It follows, by (1.10), that there exists $r_0 > 0$ such that

$$\frac{\partial}{\partial r}\left(r^{\frac{n-2}{2}} w(r,\theta)\right) > 0 \quad \forall 0 < r < r_0, \ \forall \theta \in \mathbb{S}^{n-1},$$

which implies

$$w_\lambda(y) := \frac{\lambda^{n-2}}{|y|^{n-2}} w\left(\frac{\lambda^2 y}{|y|^2}\right) < w(y) \quad \forall 0 < \lambda < |y| < r_0. \tag{2.1}$$

On the other hand, by the positivity of u, there exists some constant $c > 0$ such that

$$w(y) \geq c|y|^{2-n}, \quad |y| \geq r_0.$$

It follows that

$$w_\lambda(y) \leq w(y) \quad \forall 0 < \lambda \leq \lambda_1, \ \forall |y| \geq r_0, \tag{2.2}$$

where

$$\lambda_1 = \min\left\{r_0, \left(\frac{c}{\max_{\bar{B}_{r_0}} w}\right)^{\frac{1}{n-2}}\right\}.$$

Combining (2.1) and (2.2), we have

$$w_\lambda(y) \le w(y) \quad \forall\, 0 < \lambda \le \lambda_1,\ \forall |y| \ge \lambda\,,$$

i.e., after setting $y = \lambda^2 x/|x|^2$,

$$w_\lambda(x) \ge w(x) \quad \forall\, 0 < \lambda \le \lambda_1,\ \forall\, 0 < |x| \le \lambda\,.$$

By (1.9), $\lim_{y\to\infty}(|y|^{n-2}u(y)) > 0$, so we can apply the above argument to u to show that there exists some $\lambda_2 > 0$ such that

$$u_\lambda(y) \ge u(y) \quad \forall\, 0 < \lambda \le \frac{1}{\lambda_2},\ \forall\, 0 < |y| \le \lambda\,,$$

i.e.,

$$w_\lambda(x) \le w(x) \quad \forall \lambda \ge \lambda_2,\ \forall\, 0 < |x| \le \lambda\,.$$

Set

$$\underline{\lambda} = \sup\{\mu : w_\lambda(x) \ge w(x)\ \forall\, 0 < |x| \le \lambda \le \mu\}$$

and

$$\bar{\lambda} = \inf\{\mu : w_\lambda(x) \le w(x)\ \forall \lambda \ge \mu, 0 < |x| \le \lambda\}\,.$$

Since

$$w_\lambda(x) = \frac{1}{\lambda^{n-2}} u\left(\frac{x}{\lambda^2}\right),$$

we have

$$\lim_{x\to 0} w_\lambda(x) = \frac{1}{\lambda^{n-2}} u(0) > 0 \quad \forall \lambda > 0\,. \tag{2.3}$$

By the definition of $\underline{\lambda}$ and $\bar{\lambda}$,

$$w_{\lambda_1}(x) \ge w(x) \ge w_{\lambda_2}(x) \quad \forall\, 0 < |x| \le \lambda_1 < \underline{\lambda},\ \bar{\lambda} < \lambda_2\,.$$

In particular,

$$w_{\underline{\lambda}}(0) \ge w(0) \ge w_{\bar{\lambda}}(0)\,,$$

from which we deduce

$$0 < \underline{\lambda} \le \bar{\lambda} < \infty\,.$$

To prove the lemma, we only need to show that $\underline{\lambda} = \bar{\lambda}$. In view of (2.3), this amounts to proving

$$w_{\bar{\lambda}}(0) = w(0) = w_{\underline{\lambda}}(0)\,. \tag{2.4}$$

Suppose $w_{\underline{\lambda}}(0) > w(0)$; then we will show that

$$(w_{\underline{\lambda}} - w)(y) > 0 \quad \forall y \in B_{\underline{\lambda}} \tag{2.5}$$

and

$$\left.\frac{\partial((w_{\underline{\lambda}} - w)}{\partial r}\right|_{\partial B_{\underline{\lambda}}} < 0\,. \tag{2.6}$$

By (1.6),

$$F(A^{w_{\underline{\lambda}}}) = 1 = F(A^w) \quad \text{in } \mathbb{R}^n \setminus \{0\}\,.$$

Let $w_t = t w_{\underline{\lambda}} + (1-t)w$; we have

$$0 = F(A^{w_{\underline{\lambda}}}) - F(A^w) = \int_0^1 \left(\frac{d}{dt} F(A^{w_t})\right) dt = L(w_{\underline{\lambda}} - w)$$

where

$$L = a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x)\,, \quad a_{ij} = -\frac{2}{n-2}\int_0^1 w_t^{-\frac{n+2}{n-2}} F_{ij}(A^{w_t})dt\,,$$

and b_i and c are continuous functions.

If $(w_{\underline{\lambda}} - w)(\bar{y}) = 0$ for some $0 < |\bar{y}| < \underline{\lambda}$, then, since $w_{\underline{\lambda}} \geq w$ in $B_{\underline{\lambda}}$, we have $\nabla w_{\underline{\lambda}}(\bar{y}) = \nabla w(\bar{y})$ and $(\nabla^2 w_{\underline{\lambda}}(\bar{y})) \geq (\nabla^2 w(\bar{y}))$. This implies $A^{w_{\underline{\lambda}}}(\bar{y}) \leq A^w(\bar{y})$, and

$$A^{w_t}(\bar{y}) = tA^{w_{\underline{\lambda}}}(\bar{y}) + (1-t)A^w(\bar{y}) = M + (1-t)N\,,$$

where $M := A^{w_{\underline{\lambda}}}(\bar{y}) - \epsilon(1-t)I$ and $N := A^w(\bar{y}) - A^{w_{\underline{\lambda}}}(\bar{y}) + \epsilon I$. For $\epsilon > 0$ small, $N \in \mathcal{S}_+^{n\times n}$ and both M and $M+N$ belong to U, since $A^{w_{\underline{\lambda}}}(\bar{y})$ and $A^w(\bar{y})$ belong to the open set U. Thus, by (1.5), $A^{w_t}(\bar{y}) \in U$ for all $0 \leq t \leq 1$. By (1.7), $(F_{ij}(A^{w_t}(\bar{y}))) > 0$. By continuity, $(F_{ij}(A^{w_t}(y))) > 0$ for y close to $\bar{y}$. So L is elliptic near $\bar{y}$. By the strong maximum principle, $w_{\underline{\lambda}} - w \equiv 0$ near $\bar{y}$. Clearly we end up with $w_{\underline{\lambda}} - w \equiv 0$ in $B_{\underline{\lambda}}$, contradicting $w_{\underline{\lambda}}(0) > w(0)$. We have proven (2.5). Estimate (2.6) can be established in a similar way.

With (2.5), (2.6), and $w_{\underline{\lambda}}(0) > w(0)$, we can easily see that the definition of $\underline{\lambda}$ is violated. So we have proven $w_{\underline{\lambda}}(0) = w(0)$. The other equality in (2.4) can be established in a similar way. We deduce from (2.4) that $\bar{\lambda} = \underline{\lambda}$. Lemma 2.1 follows immediately. □

PROOF OF THEOREM 1.4: It follows from Lemma 2.1 and a calculus lemma (see, e.g., [25, lemma 11.1]) that u is of the form (1.12) for some $\bar{x} \in \mathbb{R}^n$ and some positive constants a and b. Since $A^u(0) = 2b^2a^{-2}I$, we know that $2b^2a^{-2}I \in U$ and $F(2b^2a^{-2}I) = 1$. By (1.7), b is uniquely determined once a is fixed. □

3 Condition (H_α) and the Proof of Theorem 1.20

In this section we first prove Proposition 1.19. Then we introduce condition (H'_α), a weaker hypothesis than (H_α) and prove Theorem 1.20 under this weaker hypothesis condition.

To prove Proposition 1.19, we need the following:

LEMMA 3.1 *For $n \geq 3$, there exists some positive constant ϵ_0 such that for any $(\lambda, \xi) \in \bar{\Gamma}_1 \times \mathbb{R}^n$ satisfying $|\xi_i(\lambda_i - \frac{1}{2}|\xi|^2)| \leq \epsilon_0|\xi|^3$ for $1 \leq i \leq n$, we have*

$$\sum_i \left\{\left(\lambda_i + \frac{1}{2}|\xi|^2 - |\xi_i|^2\right)^2 + |\xi_i|^2\big(|\xi|^2 - |\xi_i|^2\big)\right\} \geq \epsilon_0\big(|\xi|^4 + \sigma_1^2(\lambda)\big)\,.$$

PROOF: Replacing ξ by $\xi/|\xi|$ and λ by $\lambda/|\xi|^2$, we may assume without loss of generality that $|\xi| = 1$. We prove the lemma with an argument by contradiction. Suppose the contrary; then there exists a sequence $\{(\lambda^{(j)}, \xi^{(j)})\}$ in $\bar{\Gamma}_1 \times \mathbb{R}^n$ such that $|\xi^{(j)}| = 1$,

$$\left|\xi_i^{(j)}\left(\lambda_i^{(j)} - \frac{1}{2}\right)\right| < \frac{1}{j} \quad \forall\, 1 \le i \le n\,,$$

and

$$\sum_i \left\{\left[\lambda_i^{(j)} + \frac{1}{2} - \big(\xi_i^{(j)}\big)^2\right]^2 + \big(\xi_i^{(j)}\big)^2\big(1 - (\xi_i^{(j)})^2\big)\right\} \le \frac{1}{j}\big(1 + \sigma_1^2(\lambda^{(j)})\big)\,.$$

It is easy to see from the above that $\{|\lambda^{(j)}|\}$ is bounded. Passing to a subsequence,

$$\lambda^{(j)} \to \bar{\lambda} \in \bar{\Gamma}_1\,, \quad \xi^{(j)} \to \bar{\xi} \text{ with } |\bar{\xi}| = 1\,.$$

Clearly, $\bar{\lambda}$ and $\bar{\xi}$ satisfy

$$\bar{\xi}_i\left(\bar{\lambda}_i - \frac{1}{2}\right) = \bar{\lambda}_i + \frac{1}{2} - |\bar{\xi}_i|^2 = |\bar{\xi}_i|^2\big(1 - |\bar{\xi}_i|^2\big) = 0 \quad \forall\, 1 \le i \le n\,.$$

It follows that either $(\bar{\lambda}_i, \bar{\xi}_i) = (-\frac{1}{2}, 0)$ or $(\bar{\lambda}_i, \bar{\xi}_i) = (\frac{1}{2}, \pm 1)$. Since $|\bar{\xi}| = 1$, we have, after permutation, $\bar{\lambda} = (-\frac{1}{2}, \dots, -\frac{1}{2}, \frac{1}{2})$, contradicting $\bar{\lambda} \in \bar{\Gamma}_1$. □

Remark 3.2. In Lemma 3.1, the assumption $\lambda \in \bar{\Gamma}_1$ cannot be dropped. This can be seen by taking $\lambda = (-\frac{1}{2}, \dots, -\frac{1}{2}, \frac{1}{2})$ and $\xi = (0, \dots, 0, 1)$.

Next we point out an elementary fact: if (f, Γ) satisfies (1.29), (1.31), and (1.33), then for any $f(\lambda) \le \alpha$, we have

$$\sum_i f_{\lambda_i}(\lambda)\lambda_i \le 2\left(\alpha + \frac{1}{\delta}\right). \tag{3.1}$$

Indeed, by the concavity of f and the cone structure of Γ, we have

$$f(s\lambda) \le f(\lambda) + (s - 1)\sum_i f_{\lambda_i}(\lambda)\lambda_i \quad \forall\, 0 < s \le 1\,.$$

Fix $s \in (0, \frac{1}{2})$ small so that

$$f(s\lambda) \ge -\frac{1}{\delta}\,.$$

It follows that

$$\sum_i f_{\lambda_i}(\lambda)\lambda_i \le \frac{1}{1 - s}[f(\lambda) - f(s\lambda)] \le 2\left(\alpha + \frac{1}{\delta}\right).$$

PROOF OF PROPOSITION 1.19: We first prove (i). It is easy to see that (f_t, Γ_t) satisfies (1.29), (1.30), (1.31), (1.32), and (1.33). Let $\mu = t\lambda + (1-t)\sigma_1(\lambda)e$. For $(\lambda, \xi) \in \Gamma_t \times \mathbb{R}^n$, we have $\sigma_1(\lambda) = [t + n(1-t)]^{-1}\sigma_1(\mu) > 0$ and

$$\begin{aligned} A &:= \sum_i (f_t)_{\lambda_i}(\lambda)\left\{\left(\lambda_i + \frac{1}{2}|\xi|^2 - |\xi_i|^2\right)^2 + |\xi_i|^2\big(|\xi|^2 - |\xi_i|^2\big)\right\} \\ &= t\sum_i f_{\mu_i}(\mu)\left\{\left(\lambda_i + \frac{1}{2}|\xi|^2 - |\xi_i|^2\right)^2 + |\xi_i|^2\big(|\xi|^2 - |\xi_i|^2\big)\right\} \\ &\quad + (1-t)\bigg(\sum_j f_{\mu_j}(\mu)\bigg)\sum_i \left\{\left(\lambda_i + \frac{1}{2}|\xi|^2 - |\xi_i|^2\right)^2 + |\xi_i|^2\big(|\xi|^2 - |\xi_i|^2\big)\right\}. \end{aligned}$$

It is easy to see that

$$\sum_i (f_t)_{\lambda_i}(\lambda) = [t + n(1-t)]\sum_j f_{\mu_j}(\mu)\,.$$

Let ϵ_1 and c_1 be the constants in condition (H_α) for (f, Γ), and take any pair $(\lambda, \xi) \in \Gamma_t \times \mathbb{R}^n$ satisfying

$$f_t(\lambda) \le \alpha\,, \quad |\xi| > \frac{1}{\epsilon_1'}\,, \quad \left|\xi_i\left(\lambda_i - \frac{1}{2}|\xi|^2\right)\right| \le \epsilon_1'|\xi|^3\,,$$

where the positive constant $\epsilon_1' \le \min\{\frac{1}{2}, \epsilon_0\}$ will be chosen below. Here ϵ_0 is the number in Lemma 3.1.

If

$$(1-t)\big[\sigma_1^2(\lambda) + |\xi|^4\big] \ge (\epsilon_1')^9|\xi|^4\,,$$

then, by Lemma 3.1,

$$A \ge \epsilon_0(1-t)\big[\sigma_1^2(\lambda) + |\xi|^4\big]\sum_j f_{\mu_j}(\mu) \ge \frac{1}{n}\epsilon_0(\epsilon_1')^9|\xi|^4\sum_i (f_t)_{\lambda_i}(\lambda)\,,$$

and, by taking $c_1' = \frac{1}{n}\epsilon_0(\epsilon_1')^9$, we are done.

So in the rest of the proof, we assume that

$$(1-t)\big[\sigma_1^2(\lambda) + |\xi|^4\big] \le (\epsilon_1')^9|\xi|^4\,. \tag{3.2}$$

Since $\epsilon_1' \le \frac{1}{2}$, we have $t \ge \frac{1}{2}$, and therefore

$$A \ge \frac{1}{2}\sum_i f_{\mu_i}(\mu)\left\{\left(\lambda_i + \frac{1}{2}|\xi|^2 - |\xi_i|^2\right)^2 + |\xi_i|^2\big(|\xi|^2 - |\xi_i|^2\big)\right\} =: B\,.$$

Recall that $\mu = t\lambda + (1-t)\sigma_1(\lambda)e$; we have

$$\lambda_i = \frac{\mu_i}{t} - \frac{1-t}{t}\sigma_1(\lambda)\,,$$

so

$$
\begin{aligned}
2B = {} & \sum_i f_{\mu_i}(\mu)\left\{\left(\frac{\mu_i}{t}\right)^2 - 2\xi_i^2\frac{\mu_i}{t} + \frac{\mu_i}{t}|\xi|^2 + \frac{|\xi|^4}{4}\right\} \\
& + \sum_i f_{\mu_i}(\mu)\left\{-2\frac{\mu_i}{t}\frac{1-t}{t}\sigma_1(\lambda) + \left(\frac{1-t}{t}\right)^2\sigma_1^2(\lambda)\right. \\
& \qquad \left. + 2\xi_i^2\frac{1-t}{t}\sigma_1(\lambda) - \frac{1-t}{t}\sigma_1(\lambda)|\xi|^2\right\} \\
=: {} & \mathrm{I} + \mathrm{II}\,.
\end{aligned}
$$

By (3.2),

$$
(1-t)\sigma_1(\lambda) \le \sqrt{1-t}(\epsilon_1')^4|\xi|^2\,. \tag{3.3}
$$

It follows that

$$
\begin{aligned}
\left|\sqrt{t}\xi_i\left(\mu_i - \frac{1}{2}|\sqrt{t}\xi|^2\right)\right| & \le \left|\xi_i\left(t\lambda_i + (1-t)\sigma_1(\lambda) - \frac{1}{2}|\sqrt{t}\xi|^2\right)\right| \\
& \le \left|\xi_i\left(t\lambda_i - \frac{1}{2}t|\xi|^2\right)\right| + (1-t)\sigma_1(\lambda)|\xi| \\
& \le \epsilon_1'|\xi|^3 + (\epsilon_1')^4|\xi|^3 \le 9\epsilon_1'|\xi|^3 \le \epsilon_1|\xi|^3
\end{aligned}
$$

and

$$
|\sqrt{t}\xi| \ge \frac{1}{2}|\xi| \ge \frac{1}{2\epsilon_1'} \ge \frac{1}{\epsilon_1}\,.
$$

Here we require that $9\epsilon_1' \le \epsilon_1$.

Since (f, Γ) satisfies (H_α), we have, by applying (H_α) to the pair $(\mu, \sqrt{t}\xi)$,

$$
\begin{aligned}
\mathrm{I} & = \frac{1}{t^2}\sum_i f_{\mu_i}(\mu)\left\{\mu_i^2 - 2|(\sqrt{t}\xi)_i|^2\mu_i + \mu_i|\sqrt{t}\xi|^2 + \frac{|\sqrt{t}\xi|^4}{4}\right\} \\
& = \frac{1}{t^2}\sum_i f_{\mu_i}(\mu)\left\{\left(\mu_i + \frac{1}{2}|\sqrt{t}\xi|^2 - |(\sqrt{t}\xi_i|)^2\right)^2\right. \\
& \qquad \left. + |(\sqrt{t}\xi)_i|^2\big(|\sqrt{t}\xi|^2 - |(\sqrt{t}\xi)_i|^2\big)\right\} \\
& \ge c_1|\sqrt{t}\xi|^4\sum_i f_{\mu_i}(\mu) \ge \frac{c_1}{4n}|\xi|^4\sum_i (f_t)_{\lambda_i}(\lambda)\,.
\end{aligned}
$$

By (3.1) and (1.32), $\sum_i f_{\mu_i}(\mu)\mu_i \le C\sum_i f_{\mu_i}(\mu)$; thus, by (3.3),

$$\begin{aligned}\mathrm{II} &\ge \sum_i f_{\mu_i}(\mu)\left\{-2\frac{\mu_i}{t}\frac{1-t}{t}\sigma_1(\lambda) - \frac{1-t}{t}\sigma_1(\lambda)|\xi|^2\right\}\\ &= -\frac{2(1-t)}{t^2}\sigma_1(\lambda)\sum_i f_{\mu_i}(\mu)\mu_i - \frac{1-t}{t}\sigma_1(\lambda)|\xi|^2\sum_i f_{\mu_i}(\mu)\\ &\ge -C(1-t)\sigma_1(\lambda)|\xi|^2\sum_i f_{\mu_i}(\mu) \ge -C(\epsilon_1')^4|\xi|^4\sum_i f_{\mu_i}(\mu).\end{aligned}$$

Notice that the constants C and c_1 are independent of ϵ_1'; taking ϵ_1' small enough, we have

$$A \ge B = \frac{1}{2}(\mathrm{I}+\mathrm{II}) \ge \frac{c_1}{8n}|\xi|^4\sum_i (f_t)_{\lambda_i}(\lambda)\,.$$

We have established (i) in Proposition 1.19.

To prove (ii), let $(\lambda,\xi)\in\Gamma\times\mathbb{R}^n$ satisfy $f^s(\lambda)\le\frac{\alpha}{s}$, $|\xi|\ge 1/(\sqrt{s_0}\,\epsilon_1)$, and $|\xi_i(\lambda_i-\frac{1}{2}|\xi|^2)|\le\sqrt{s_0}\,\epsilon_1|\xi|^3$, $1\le i\le n$. Let $\tilde\lambda = s\lambda$ and $\tilde\xi=\sqrt{s}\,\xi$. Then $f(\tilde\lambda)\le\alpha$, $|\tilde\xi|\ge 1/\epsilon_1$, and $|\tilde\xi_i(\tilde\lambda_i-\frac{1}{2}|\tilde\xi|^2)|\le\epsilon_1|\tilde\xi|^3$, $1\le i\le n$. It follows that

$$\liminf_{\lambda\in\Gamma,\lambda\to 0} f^{(s)}(\lambda) \ge -\frac{1}{s_0\delta}$$

and

$$\begin{aligned}&\sum_i f^{(s)}_{\lambda_i}(\lambda)\left\{\left(\lambda_i+\frac{1}{2}|\xi|^2-|\xi_i|^2\right)^2 + |\xi_i|^2\big(|\xi|^2-|\xi_i|^2\big)\right\}\\ &\quad= \frac{1}{s^2}\sum_i f_{\lambda_i}(\tilde\lambda)\left\{\left(\tilde\lambda_i+\frac{1}{2}|\tilde\xi|^2-|\tilde\xi_i|^2\right)^2 + |\tilde\xi_i|^2\big(|\tilde\xi|^2-|\tilde\xi_i|^2\big)\right\}\\ &\quad\ge \frac{c_1}{s^2}|\tilde\xi|^4\sum_i f_{\lambda_i}(\tilde\lambda) = c_1|\xi|^4\sum_i f^{(s)}_{\lambda_i}(\lambda)\,.\end{aligned}$$

(iii) is obvious. □

Now we introduce condition (H'_α), a weaker hypothesis than (H_α).

DEFINITION 3.3 Let (f,Γ) satisfy (1.29), (1.30), (1.31),

$$\min_{1\le i\le n} f_{\lambda_i} > 0 \quad \text{in } \Gamma, \tag{3.4}$$

and, for some $\delta>0$,

$$\inf_{\substack{\lambda\in\Gamma\\ |\lambda|\ge\frac{1}{\delta}}}\left(|\lambda|\sum_i f_{\lambda_i}(\lambda)\right) \ge \delta\,. \tag{3.5}$$

We say that (f, Γ) *satisfies condition* (H'_α) *for some* $\alpha > 0$ if there exist some positive constants $a < \frac{1}{4}$, ϵ_1, and c_1 such that for any $(\lambda, \xi) \in \Gamma \times \mathbb{R}^n$ satisfying $f(\lambda) \le \alpha$, $|\xi| \ge \frac{1}{\epsilon_1}$, and $|\xi_i(\lambda_i - \frac{1}{2}|\xi|^2)| \le \epsilon_1 |\xi|^3$, $1 \le i \le n$, we have

$$|\xi|^2 \sum_i f_{\lambda_i}(\lambda) \ge c_1 , \quad \sum_i f_{\lambda_i}(\lambda)\lambda_i \le a c_1 |\xi|^2 \sum_i f_{\lambda_i}(\lambda) ,$$

and

$$\sum_i f_{\lambda_i}(\lambda) \left\{ \left(\lambda_i + \frac{1}{2}|\xi|^2 - |\xi_i|^2 \right)^2 + |\xi_i|^2 \big(|\xi|^2 - |\xi_i|^2 \big) \right\} \ge c_1 |\xi|^4 \sum_i f_{\lambda_i}(\lambda) .$$

The rest of this section is devoted to proving Theorem 1.20.

PROOF OF THEOREM 1.20: Our proof of (1.38) follows closely that of [14], while our proof of (1.39) differs in that we estimate the full Hessian of u instead of only the trace of the Hessian; i.e., we adapt the arguments of [14] with a different test function that is similar to the one used in [23].

We first establish (1.38). We may assume without loss of generality that $r = 1$. Let $u^{4/(n-2)} g = e^{-2v} g$, i.e., $v = -\frac{2}{n-2} \log u$; then the equation takes the form

$$F(\widetilde{W}) := f(\lambda_g(\widetilde{W})) = h , \quad \lambda_g(\widetilde{W}) \in \Gamma, \quad \text{on } B_3 , \tag{3.6}$$

where

$$\widetilde{W} := e^{2v} \left(\nabla^2 v + \nabla v \otimes \nabla v - \frac{|\nabla v|^2}{2} g + A_g \right) ,$$

and $\lambda_g(\widetilde{W})$ denotes eigenvalues of $\widetilde{W}$ with respect to g.

Let $H = \rho |\nabla v|^2$, where ρ is a smooth, nonnegative cutoff function that is equal to 1 on B_1, equal to 0 outside B_2, and $|\nabla \rho| \le C\sqrt{\rho} \le C$ in B_3. Here and in the following, C denotes various positive constants having the dependence specified in the theorem. Let $H(x_0) = \max_{\bar{B}_1} H$ for some $x_0 \in B_2$. Taking some g-geodesic normal coordinates centered at x_0 so that $\tilde{W}$ is diagonal at x_0, let $\tilde{w}_{ij}$ be the (ij) entry of $\widetilde{W}$. At x_0,

$$\begin{cases} \tilde{w}_{ii} = e^{2v}(v_{ii} + v_i^2 - \frac{1}{2}|\nabla v|^2 + (A_g)_{ii}) & \forall i \\ v_{ij} = -v_i v_j - (A_g)_{ij} & \forall i \ne j. \end{cases} \tag{3.7}$$

We use the notation

$$F^{ij} := \frac{\partial F}{\partial \tilde{w}_{ij}} , \quad \bar{v}_{ij} := v_{ij} + (A_g)_{ij} , \quad \lambda_i := \tilde{w}_{ii}(x_0) , \quad \lambda = (\lambda_1, \dots, \lambda_n) .$$

Since $(\tilde{w}_{ij}(x_0))$ is diagonal, $F^{ij}(\tilde{w}_{ij}(x_0)) = f_{\lambda_i}(\lambda)\delta_{ij}$.

Since H has a maximum point at x_0, we have, at x_0,

$$H_i = \rho_i |\nabla v|^2 + 2\rho v_{li} v_l = 0 \tag{3.8}$$

and

$$(H_{ij}) = \left(\left(-\frac{2\rho_i\rho_j}{\rho} + \rho_{ij}\right)|\nabla v|^2 + 2\rho v_{lij}v_l + 2\rho v_{li}v_{lj}\right) \le 0\,. \tag{3.9}$$

Here and in the following, subindices denote covariant differentiation with respect to g. It follows that at x_0

$$\left|\sum_l v_{il}v_l\right| = \frac{|\rho_i|}{2\rho}|\nabla v|^2 \le \frac{C}{\sqrt{\rho}}|\nabla v|^2 = \frac{C}{\sqrt{H}}|\nabla v|^3 \quad \forall i \tag{3.10}$$

from which we deduce, by using (3.7), that

$$\left|v_i\left(e^{-2v}\tilde{w}_{ii} - \frac{1}{2}|\nabla v|^2\right)\right| = \left|\sum_l v_{il}v_l + \sum_l (A_g)_{il}v_l\right| \le \frac{C}{\sqrt{H}}|\nabla v|^3 \quad \forall i\,. \tag{3.11}$$

At x_0, we have, by (3.9) and the fact that $(\tilde{w}_{ij}(x_0))$ is diagonal, that

$$\begin{aligned}
0 &\ge \sum_{ij} F^{ij}H_{ij} \\
&= \sum_i F^{ii}\left(-\frac{2\rho_i^2}{\rho} + \rho_{ii}\right)|\nabla v|^2 + 2\rho\sum_{i,l}F^{ii}v_{lii}v_l + 2\rho\sum_{i,l}F^{ii}v_{il}^2 \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}\,,
\end{aligned}$$

$$\mathrm{I} = \sum_i F^{ii}\left(-2\frac{\rho_i^2}{\rho} + \rho_{ii}\right)|\nabla v|^2 \ge -C\sum_i f_{\lambda_i}(\lambda)|\nabla v|^2\,.$$

Commuting covariant differentiation,

$$\mathrm{II} = 2\rho\sum_{i,l}F^{ii}(v_{iil} + R^r{}_{ili}v_r)v_l \ge 2\rho\sum_{i,l}F^{ii}v_{iil}v_l - C\rho\sum_i F^{ii}|\nabla v|^2\,.$$

Differentiating the equation of v, we have

$$F^{ii}\tilde{w}_{iil} = h_l \quad \forall l\,.$$

Thus, using (3.10) and $H = \rho|\nabla v|^2 \ge 1$, we have

$$\begin{aligned}
\mathrm{II} &\ge 2\rho\sum_{i,l}F^{ii}(e^{-2v}\tilde{w}_{ii})_l v_l - 2\rho\sum_{i,l}F^{ii}\left(v_i^2 - \frac{|\nabla v|^2}{2}g_{ii}\right)_l v_l - C\rho\sum_i F^{ii}|\nabla v|^2 \\
&= 2e^{-2v}\rho\sum_l h_l v_l - 4\rho e^{-2v}|\nabla v|^2\sum_i f_{\lambda_i}(\lambda)\lambda_i - 4\rho\sum_{i,l}F^{ii}v_{il}v_iv_l \\
&\quad + 2\rho\sum_{i,k,l}F^{ii}v_{kl}v_kv_l - C\rho\sum_i F^{ii}|\nabla v|^2 \\
&\ge -2e^{-2v}\rho|\nabla h||\nabla v| - 4e^{-2v}\rho|\nabla v|^2\sum_i f_{\lambda_i}(\lambda)\lambda_i - \frac{C\rho}{\sqrt{H}}\sum_i f_{\lambda_i}(\lambda)|\nabla v|^4\,,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{III} &= 2\rho \sum_{i,l} F^{ii} (v_{il})^2 \\
&= 2\rho \sum_{i,l} F^{ii} (\bar{v}_{il})^2 - 4\rho \sum_{il} F^{ii} (A_g)_{il} \bar{v}_{il} + 2\rho \sum_{il} F^{ii} ((A_g)_{il})^2 \\
&\geq 2\rho \sum_{i,l} F^{ii} (\bar{v}_{il})^2 - 4\rho \sum_{il} F^{ii} \left\{ ((A_g)_{il})^2 + \frac{1}{4} \bar{v}_{il}^2 \right\} \\
&\geq \rho \sum_{i,l} F^{ii} (\bar{v}_{il})^2 - C\rho \sum_{i} F^{ii} \\
&= \rho \sum_{i} f_{\lambda_i}(\lambda) \left\{ \left(e^{-2v} \lambda_i + \frac{1}{2} |\nabla v|^2 - v_i^2 \right)^2 + v_i^2 \big(|\nabla v|^2 - v_i^2 \big) \right\} \\
&\quad - C\rho \sum_{i} f_{\lambda_i}(\lambda) .
\end{aligned}$$

Combining all the above estimates and using $H = \rho |\nabla v|^2 \geq 1$, we have

$$\sum_i f_{\lambda_i}(\lambda) \left\{ \left(e^{-2v} \lambda_i + \frac{1}{2} |\nabla v|^2 - v_i^2 \right)^2 + v_i^2 (|\nabla v|^2 - v_i^2) \right\} \leq \\
\frac{C}{\sqrt{H}} e^{-2v} |\nabla h| |\nabla v|^2 + 4 e^{-2v} \sum_i f_{\lambda_i}(\lambda) \lambda_i |\nabla v|^2 + \frac{C}{\sqrt{H}} |\nabla v|^4 \sum_i f_{\lambda_i}(\lambda) ,$$

i.e.,

$$\sum_i f_{\lambda_i}(\lambda) \left\{ \lambda_i + \frac{1}{2} |\xi|^2 - (\xi_i)^2)^2 + (\xi_i)^2 \big(|\xi|^2 - (\xi_i)^2 \big) \right\} \leq \\
\frac{C |\nabla h| |\xi|^2}{\sqrt{H}} + 4 \sum_i f_{\lambda_i}(\lambda) \lambda_i |\xi|^2 + \frac{C}{\sqrt{H}} |\xi|^4 \sum_i f_{\lambda_i}(\lambda) , \tag{3.12}$$

where $\xi = e^v \nabla v$. By (3.11), we know

$$\left| \xi_i \left(\lambda_i - \frac{1}{2} |\xi|^2 \right) \right| \leq \frac{C}{\sqrt{H}} |\xi|^3 .$$

We may assume that (at x_0), $|\xi| \geq 1/\epsilon_1$ and $C/\sqrt{H} \leq \epsilon_1$; otherwise we are done. Now we deduce from (3.12), by (1.32) and condition (H_α), that

$$\left(c_1 - \frac{C}{\sqrt{H}} \right) |\xi|^2 \leq \frac{C |\nabla h|}{\delta} + \frac{4}{c_1} .$$

It follows that

$$H \leq C + C e^{-2v} (1 + |\nabla h|) ,$$

from which (1.38) follows.

Next we prove (1.39). Again we may assume $r = 1$. Let

$$G(x) := \rho \max_{|\xi|_g=1} \xi^i(v_{ij} + v_i v_j)\xi^j = \rho \max_{\xi\in\mathbb{R}^n\setminus\{0\}} \frac{\xi^i(v_{ij} + v_i v_j)\xi^j}{g_{ij}\xi^i\xi^j},$$

where ρ is the cutoff function defined earlier. Clearly $\max_{|\xi|_g=1} \xi^i(v_{ij} + v_i v_j)\xi^j$ is the largest eigenvalue of $\nabla^2 v + \nabla v \otimes \nabla v$ with respect to g. Assume $G(y_0) = \max_{\bar{B}_2} G$ for some $y_0 \in B_2$. We work with some geodesic normal coordinates at y_0 with the property that $(v_{ij} + v_i v_j)$ is diagonal at y_0 and $G(y_0) = \rho(v_{11} + v_1^2)$. In particular, at y_0,

$$v_{ii} + v_i^2 \le v_{11} + v_1^2 \quad \forall\, 1 \le i \le n.$$

Since $\lambda(\widetilde{W}) \in \Gamma \subset \Gamma_1$ implies that, at y_0, $-v_{ii} \le (n-1)v_{11} + C(1 + |\nabla v|^2)$, we have $|v_{ii}| \le (n-1)v_{11} + C(1 + |\nabla v|^2)$, $1 \le i \le n$. We may assume that at y_0,

$$(1 + |\nabla v|^2) \le C v_{11}\,; \tag{3.13}$$

otherwise $\max_{\bar{B}_2} G = G(y_0) \le C(1 + |\nabla v|^2)$, and we are done by (1.38). So we have, at y_0,

$$|v_{ij}| \le C v_{11}\,, \quad \forall\, 1 \le i, j \le n\,. \tag{3.14}$$

Now consider a new function

$$\tilde{G} := \frac{\rho(v_{11} + v_1^2)}{g_{11}}\,.$$

It is easy to see that y_0 is a local maximum point of $\tilde{G}$ and $\tilde{G}(y_0) = G(y_0)$. Recall that v_{11} denotes the covariant differentiation. In the following, we use $(v_{11})_i$ and $(v_{11})_{ij}$ to denote covariant differentiations of the local function v_{11}. Similar notation applies to $(v_1)_i$, $(v_1)_{ij}$, $(g_{11})_i$, and $(g_{11})_{ij}$. At y_0, $\forall\, 1 \le i, j \le n$,

$$(v_{11})_i = v_{11i}\,, \quad |v_{1ij} - v_{ij1}| + |v_{11i} - v_{i11}| \le C|\nabla v|\,,$$

$$(v_1)_i = v_{1i}\,, \quad |(v_1)_{ij} - v_{1ij}| \le C|\nabla v|\,,$$

$$|(v_{11})_{ij} - v_{11ij}| + |v_{11ij} - v_{ij11}| \le C\sum_{i,j} |v_{ij}| \le C v_{11}\,.$$

The last inequality above follows from (3.14). These, as well as (3.13) and (3.14), will be frequently used below without being mentioned.

Since y_0 is an interior maximum point of $\tilde{G}$, we have

$$0 = \tilde{G}_i(y_0) = \frac{\rho_i}{\rho}\tilde{G} + \rho[2v_1 v_{1i} + (v_{11})_i] \quad \forall\, 1 \le i \le n \tag{3.15}$$

and

(3.16) $\quad 0 \ge (\tilde{G}_{ij}(y_0)) =$

$$\left(\frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2}\tilde{G} + \rho[(v_{11})_{ij} + 2v_1(v_1)_{ij} + 2v_{1i}v_{1j}] + \rho\big(v_{11} + v_1^2\big)\left(\frac{1}{g_{11}}\right)_{ij}\right).$$

By (3.4), (F^{ij}) is positive definite at y_0. So, at y_0,

$$
\begin{aligned}
0 \geq F^{ij}\tilde{G}_{ij} &\geq \sum_{i,j} F^{ij}\frac{\rho\rho_{ij}-2\rho_i\rho_j}{\rho^2}\tilde{G} + 2\rho\sum_{i,j}F^{ij}v_{1i}v_{1j} \\
&\quad + 2\rho\sum_{i,j}F^{ij}v_{ij1}v_1 + \rho\sum_{i,j}F^{ij}v_{ij11} - C\rho\sum_i F^{ii}v_{11} \qquad (3.17)\\
&=: \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}\,.
\end{aligned}
$$

$$
\mathrm{I} \geq -C\sum_i F^{ii}\frac{\tilde{G}}{\rho} = -C\sum_i F^{ii}\big(v_1^2+v_{11}\big) \geq -C\sum_i F^{ii}v_{11}\,.
$$

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be eigenvalues of $(\tilde{w}_{ij}(y_0))$.

Differentiating (3.6) and using the concavity of F, we have, at y_0, that

$$
\sum_{i,j}F^{ij}\tilde{w}_{ijl} = h_l\,, \quad \sum_{i,j}F^{ij}\tilde{w}_{ijlm} \geq h_{lm}\,, \quad \forall\, 1 \leq l, m \leq n\,.
$$

By computation,

$$
\begin{aligned}
\mathrm{III} &\geq 2\rho\sum_{i,j}F^{ij}\big(e^{-2v}\tilde{w}_{ij}\big)_1 v_1 - 2\rho\sum_{i,j}F^{ij}\left(v_iv_j - \frac{|\nabla v|^2}{2}g_{ij}\right)_1 v_1 \\
&\quad - C\rho\sum_i F^{ii}v_{11} \\
&= 2\rho e^{-2v}h_1v_1 - 4\rho e^{-2v}v_1^2\sum_{i,j}F^{ij}\tilde{w}_{ij} - 4\rho\sum_{ij}F^{ij}v_{i1}v_j \\
&\quad + 2\rho\sum_{i,k}F^{ii}v_kv_{k1}v_1 - C\rho\sum_i F^{ii}v_{11} \\
&\geq -2\rho e^{-2v}|\nabla h||\nabla v| - C\rho(1+|\nabla v|^2)\sum_i F^{ii}v_{11} - 4\rho e^{-2v}v_1^2\sum_{i,j}F^{ij}\tilde{w}_{ij}\,.
\end{aligned}
$$

Using (3.1) and (1.32), we have

$$
\sum_{i,j}F^{ij}\tilde{w}_{ij} = \sum_i f_{\lambda_i}\lambda_i \leq C\sum_i f_{\lambda_i} = C\sum_i F^{ii}\,.
$$

Thus

$$
\begin{aligned}
\mathrm{IV} &= \rho \sum_{i,j} F^{ij}\big(e^{-2v}\tilde{w}_{ij}\big)_{11} - \rho \sum_{i,j} F^{ij}\left(v_i v_j - \frac{|\nabla v|^2}{2} g_{ij}\right)_{11} \\
&\geq -4\rho e^{-2v} v_1 \sum_{i,j} F^{ij}\tilde{w}_{ij1} + 4\rho e^{-2v} v_1^2 \sum_{i,j} F^{ij}\tilde{w}_{ij} \\
&\quad + \rho e^{-2v} \sum_{i,j} F^{ij}\tilde{w}_{ij11} - 2\rho \sum_{i,j} F^{ij}(v_{i1}v_{j1} + v_{i11}v_j) \\
&\quad + \rho \sum_{i,k} F^{ii}\big(v_{k1}^2 + v_k v_{k11}\big) - C\rho(1+e^{-2v}) \sum_i F^{ii} v_{11} \\
&\geq -4\rho e^{-2v}|\nabla h||\nabla v| + 4\rho e^{-2v} v_1^2 \sum_{i,j} F^{ij}\tilde{w}_{ij} \\
&\quad - \rho e^{-2v}|\nabla^2 h| - 2\rho \sum_{i,j} F^{ij}(v_{i1}v_{j1} + v_{i11}v_j) \\
&\quad + \rho \sum_{i,k} F^{ii}\big(v_{k1}^2 + v_k v_{k11}\big) - C\rho(1+e^{-2v}) \sum_i F^{ii} v_{11} \\
&\geq -2\rho \sum_{i,j} F^{ij}(v_{i1}v_{j1} + v_{11i}v_j) + \rho \sum_{i,k} F^{ii}\big(v_{k1}^2 + v_k v_{11k}\big) - \rho e^{-2v}|\nabla^2 h| \\
&\quad - 4\rho e^{-2v}|\nabla h||\nabla v| - C\rho(1+e^{-2v}+|\nabla v|^2) \sum_i F^{ii} v_{11} \\
&\quad + 4\rho e^{-2v} v_1^2 \sum_{i,j} F^{ij}\tilde{w}_{ij}\,.
\end{aligned}
$$

Using (3.15), we have

$$
\begin{aligned}
\mathrm{IV} &\geq -2\rho \sum_{i,j} F^{ij} v_{i1}v_{j1} + 2\sum_{i,j} F^{ij}\left(\rho_i \frac{\tilde{G}}{\rho} + 2\rho v_1 v_{1i}\right) v_j + \rho \sum_{i,k} F^{ii} v_{k1}^2 \\
&\quad - \sum_{i,k} F^{ii} v_k \left(\rho_k \frac{\tilde{G}}{\rho} + 2\rho v_1 v_{1k}\right) + 4\rho e^{-2v} v_1^2 \sum_{i,j} F^{ij}\tilde{w}_{ij} - \rho e^{-2v}|\nabla^2 h| \\
&\quad - 4\rho e^{-2v}|\nabla h||\nabla v| - C\rho(1+e^{-2v}+|\nabla v|^2) \sum_i F^{ii} v_{11} \\
&\geq -2\rho \sum_{i,j} F^{ij} v_{i1}v_{j1} + \rho \sum_i F^{ii} v_{11}^2 \\
&\quad + 4\rho e^{-2v} v_1^2 \sum_{i,j} F^{ij}\tilde{w}_{ij} - 4\rho e^{-2v}|\nabla h||\nabla v| \\
&\quad - C\rho(1+e^{-2v}+|\nabla v|^2) \sum_i F^{ii} v_{11} - \rho e^{-2v}|\nabla^2 h|\,.
\end{aligned}
$$

Returning to (3.17), we have

$$0 \geq \sum_{i,j} F^{ij}\tilde{G}_{ij} \geq \rho \sum_i F^{ii} v_{11}^2 - C(1 + e^{-2v} + |\nabla v|^2) \sum_i F^{ii} v_{11} - C\rho e^{-2v}(|\nabla h||\nabla v| + |\nabla^2 h|)\,,$$

from which we deduce that

$$\sum_i F^{ii}(\rho v_{11})^2 \leq C(1 + e^{-2v} + |\nabla v|^2) \sum_i F^{ii}(\rho v_{11}) + C\rho^2 e^{-2v}(|\nabla h||\nabla v| + |\nabla^2 h|)\,. \tag{3.18}$$

Using (3.13) and (3.14), we have

$$\frac{1}{C} e^{2v} v_{11} \leq |\lambda| \leq C e^{2v} v_{11}\,.$$

If

$$C(1 + e^{-2v} + |\nabla v|^2) \geq \frac{1}{2}(\rho v_{11})\,,$$

we are done because of (1.38); otherwise we deduce from (3.18) that

$$\sum_i F^{ii}(v_{11})^2 \leq C e^{-2v}(|\nabla h||\nabla v| + |\nabla^2 h|)\,.$$

Thus we have

$$v_{11}\left(|\lambda| \sum_i f_{\lambda_i}(\lambda)\right) \leq C(|\nabla h||\nabla v| + |\nabla^2 h|)\,,$$

from which we deduce (1.39) by using (3.5). Theorem 1.20 is established. □

4 Proof of Theorem 1.25

In this section we prove Theorem 1.25. We will first establish estimate (1.44).

PROOF OF (1.44): Let (M, g) be an n-dimensional ($n \geq 3$), locally conformally flat, compact, smooth Riemannian manifold without boundary, and let $(\widetilde{M}, \widetilde{g})$ denote the universal cover with $i : \widetilde{M} \to M$ a covering map and $\widetilde{g} = i^* g$. It is well-known that there exists a conformal immersion

$$\Phi : (\widetilde{M}, \widetilde{g}) \to (\mathbb{S}^n, g_0)$$

where g_0 denotes the standard metric on $\mathbb{S}^n$.

By (1.42) and the fact that $\Gamma \subset \Gamma_1$, we have $R_g > 0$. So, by a theorem of Schoen and Yau [32], Φ is injective. Let

$$\Omega = \Phi(\widetilde{M})\,.$$

We first prove that for some positive constant C, depending only on (M, g) and (f, Γ), such that

$$u \leq C \quad \text{on } M \tag{4.1}$$

where $u \in C^2(M)$ is any positive solution of (1.43) with $\hat{g} = u^{4/(n-2)}g$.

Suppose the contrary of (4.1); then there exists a sequence of positive functions $u_j \in C^2(M)$, $\max_M u_j \to \infty$, such that $\hat{g}_j = u_j^{4/(n-2)}g$ satisfies(1.43). For convenience, we let

$$U = \{A \in \mathcal{S}^{n\times n} : \lambda(A) \in \Gamma\} \quad \mathit{and} \quad F(A) = f(\lambda(A)),\ A \in U.$$

It is easy to see that U satisfies (1.4) and (1.5) and F satisfies (1.6) and (1.7). We distinguish two cases:

$$\text{Case 1: } \Omega = \mathbb{S}^n$$
$$\text{Case 2: } \Omega \neq \mathbb{S}^n.$$

In case 1 $(\Phi^{-1})^*\widetilde{g} = \eta^{4/(n-2)}g_0$ on $\mathbb{S}^n$. Let $\tilde{u}_j = u_j \circ i$. Since $F(A_{\tilde{u}_j^{4/(n-2)}\widetilde{g}}) = 1$ on $\widetilde{M}$, we have

$$F\big(A_{[(\tilde{u}_j\circ\Phi^{-1})\eta]^{\frac{4}{n-2}}g_0}\big) = 1 \quad \text{on } \mathbb{S}^n\,.$$

By Corollary 1.6, $(\tilde{u}_j \circ \Phi^{-1})\eta = a_j|J_{\varphi_j}|^{(n-2)/(2n)}$ for some positive constants a_j and some conformal diffeomorphism $\varphi_j : \mathbb{S}^n \to \mathbb{S}^n$. Since $\varphi_j^* g_0 = |J_{\varphi_j}|^{2/n}g_0$, we have, by the above equation, that

$$f\big(a_j^{-\frac{4}{n-2}}(n-1)e\big) = f\big(a_j^{-\frac{4}{n-2}}\lambda(A_{g_0})\big) = 1\,.$$

By (1.40), $\{a_j\}$ is bounded from above by some positive constant. Thus, after passing to a subsequence, $\{\tilde{u}_j \circ \Phi^{-1}\eta\}$ blows up at precisely one point. On the other hand, since (M, g) is not conformally diffeomorphic to $(\mathbb{S}^n, g_0)$, $\pi_1(M)$ is nontrivial. So $\{\tilde{u}_j \circ \Phi^{-1}\eta\}$ blows up at more than one point. A contradiction.

In case 2, Ω is, by [32], an open dense subset of $\mathbb{S}^n$. We may assume that $u_j(x_j) = \max_M u_j \to \infty$, $x_j \in M$, $x_j \to x \in M$. Fixing $\tilde{x} \in \widetilde{M}$ with $i(\tilde{x}) = x$, we may assume, by composing another conformal diffeomorphism of $\mathbb{S}^n$, that $\Phi(\tilde{x}) = S$, the south pole of $\mathbb{S}^n$, and the north pole N belongs to Ω. Let $P : \mathbb{S}^n \to \mathbb{R}^n$ be the stereographic projection, and let v be the positive function on the open subset $P(\Omega)$ of $\mathbb{R}^n$ determined by $(P^{-1})^*(\eta^{4/(n-2)}g_0) = v^{4/(n-2)}g_{\text{flat}}$. Then $P(\Omega)$ contains an open neighborhood of the origin 0, and, by the Harnack inequality of Cheng and Yau [7] (see also discussions on [32, p. 57]) $v(y) \to \infty$ as y tends to any boundary point of $P(\Omega)$. On $P(\Omega)$, we have

$$F(A^{\hat{u}_j}) = 1 \ \text{ on } P(\Omega) \quad \text{where } \hat{u}_j = \big(\tilde{u}_j \circ \Phi^{-1} \circ P^{-1}\big)v\,.$$

We also know that

$$\lambda(A^{\hat{u}_j}) \in \Gamma \quad \text{on } P(\Omega)\,.$$

Let $\tilde{x}_j \to \tilde{x}$ and $i(\tilde{x}_j) = x_j$, and let $y_j = P(\Phi(\tilde{x}_j)) \in \mathbb{R}^n$. Clearly $y_j \to 0$. Consider

$$w_j(y) = \frac{1}{\hat{u}_j(y_j)}\hat{u}_j\left(\frac{y}{\hat{u}_j(y_j)^{\frac{2}{n-2}}} + y_j\right), \quad y \in O_j\,,$$

where $O_j := \{y \in \mathbb{R}^n : \hat{u}_j(y_j)^{-2/(n-2)}(y + y_j) \in P(\Omega)\}$. It follows that w_j satisfies

$$F(A^{w_j}) = 1 \text{ on } O_j \quad \text{and} \quad \lambda(A^{w_j}) \in \Gamma \text{ on } O_j\,.$$

For any compact subset K of $\mathbb{R}^n$, since $\tilde{x}_j$ is a maximum point of $\tilde{u}_j$, there exists some constant $C(K)$ such that

$$w_j \le C(K) \text{ on } K \quad \text{for large } j.$$

By Theorem 1.20,

$$|\nabla(\log w_j)| + |\nabla^2(\log w_j)| \le C(K) \quad \text{on } K\,.$$

Since we also know that $w_j(0) = 1$, we deduce from the above that

$$w_j \ge \frac{1}{C(K)} \quad \text{on } K\,.$$

Due to the above estimates, $\{\lambda(A^{w_j}(x)) : x \in K\}$ stays in a compact subset of Γ, and therefore, by (1.31) and (1.32), the equation satisfied by w_j on K is uniformly elliptic and concave in its second derivatives. By the interior $C^{2,\alpha}$ estimates of Evans and Krylov, $\|w_j\|_{C^{2,\alpha}(K)} \le C(K, \alpha)$ for some $0 < \alpha < 1$ independent of j. Thus, after passing to a subsequence,

$$w_j \to w \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n)\,,$$

where $w \in C^2(\mathbb{R}^n)$ is a positive solution of

$$F(A^w) = 1 \quad \text{in } \mathbb{R}^n \tag{4.2}$$

and

$$\lambda(A^w) \in \Gamma \quad \text{in } \mathbb{R}^n. \tag{4.3}$$

For fixed j, since $w_j(y)$ tends to infinity as y goes to a boundary point of O_j, and since infinity is a regular point of w_j, we can argue as in Section 2 to show that for any $x \in \mathbb{R}^n$, we have

$$(w_j)_{x,\lambda}(y) \le w_j(y) \quad \forall \lambda > 0,\ \forall |y - x| \ge \lambda,\ y \in O_j,\ B_\lambda(x) \subset O_j\,.$$

Passing to limit, we have

$$w_\lambda(y) \le w(y) \quad \forall \lambda > 0,\ \forall |y| \ge \lambda\,.$$

Similarly, for any $x \in \mathbb{R}^n$, we can show that

$$(w_j)_{x,\lambda}(y) \le w_j(y) \quad \forall \lambda > 0,\ \forall |y - x| \ge \lambda\,,$$

from which we deduce

$$w_{x,\lambda}(y) \le w(y) \quad \forall \lambda > 0,\ \forall |y - x| \ge \lambda\,.$$

Then by a calculus lemma (see, e.g., [25, lemma 11.2]), w must be a positive constant and therefore $A^w \equiv 0$. This violates (1.40) in view of (4.2), and we have a contradiction.

Next we prove that for some positive constant C, depending only on (M, g) and (f, Γ), we have

$$\max_M u \geq \frac{1}{C} \quad \text{on } M. \tag{4.4}$$

Our proof of (4.4) adapts an argument in [37] for $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$. Let $\bar{x} \in M$ be a maximum point of u. Then $A_{\hat{g}}(\bar{x}) \geq A_g(\bar{x})$. It follows that

$$\begin{aligned}\lambda(A_{\hat{g}}(\bar{x})) \equiv \lambda_{\hat{g}}(A_{\hat{g}}(\bar{x})) \geq \lambda_{\hat{g}}(A_g(\bar{x}))\\ = \lambda_g(A_g(\bar{x}))u(\bar{x})^{-\frac{4}{n-2}} \equiv \lambda(A_g(\bar{x}))u(\bar{x})^{-\frac{4}{n-2}}.\end{aligned}$$

By (1.32) and equation (1.43),

$$f\left(u(\bar{x})^{-\frac{4}{n-2}}\lambda(A_g(\bar{x}))\right) \leq f(\lambda(A_{\hat{g}}(\bar{x}))) = 1.$$

By (1.42), (1.43), and (1.40), $\{\lambda(A_g(x)) : x \in M\}$ is a compact subset of Γ. Therefore, estimate (4.4) for $\max_M u = u(\bar{x})$ follows from (1.41) and the above inequality.

By Theorem 1.20 and (4.1),

$$\|\nabla_g(\log u)\|_g + \|\nabla_g^2(\log u)\|_g \leq C \quad \text{on } M. \tag{4.5}$$

It follows that

$$\min_M u \geq \frac{1}{C}\max_M u \geq \frac{1}{C}. \tag{4.6}$$

Due to (4.1), (4.6), (4.5), and the hypothesis on (f, Γ), we can, as explained in Section 4, apply the interior estimates of Evans and Krylov to obtain, for some constants $0 < \alpha < 1$ and $C > 0$, depending only on (f, Γ) and (M, g), that

$$\|u\|_{C^{2,\alpha}(M,g)} \leq C.$$

Estimate (1.44) follows from the above and Schauder theory. □

PROOF OF THEOREM 1.25: We prove the existence part of Theorem 1.25 by using estimate (1.44) and a degree argument. We only need to consider the case where (M, g) is not conformally diffeomorphic to standard spheres.

For $0 \leq t \leq 1$, let f_t and Γ_t be defined as in (1.35) and (1.36). We consider

$$f_t(\lambda(A_{\hat{g}})) = 1, \quad \lambda(A_{\hat{g}}) \in \Gamma_t, \quad \text{on } M. \tag{4.7}$$

Here and below, $\hat{g} = u^{4/(n-2)}g$.

The proof of (1.44) yields a bound independent of t. In particular, there exist some constants $C > 0$ and $0 < \alpha < 1$ independent of $t \in [0, 1]$ such that for all solutions u of (4.7),

$$\|u\|_{C^{4,\alpha}} + \|u^{-1}\|_{C^{4,\alpha}} \leq C. \tag{4.8}$$

Indeed, by (i) in Proposition 1.19 and the following consequence of the concavity of f,

$$f_t(s\lambda) \geq tf(s\lambda) + (1-t)f(s\sigma_1(\lambda)), \quad 0 \leq t \leq 1,$$

the above proof of (1.44) works for f_t uniform in t.

By (1.40) and (4.8), there exists $\delta > 0$ independent of $t \in [0, 1]$ such that all solutions u of (4.7) satisfy

$$\operatorname{dist}(\lambda(A_{\hat{g}}), \partial\Gamma_t) \geq 2\delta .$$

Define, for $0 \leq t \leq 1$,

$$O_t^* = \Big\{u \in C^{4,\alpha}(M) : \lambda(A_{\hat{g}}) \in \Gamma_t,\ \operatorname{dist}(\lambda(A_{\hat{g}}), \partial\Gamma_t) > \delta, \\ u > 0,\ \|u\|_{C^{4,\alpha}} + \|u^{-1}\|_{C^{4,\alpha}} < 2C\Big\}$$

where C is the constant in (4.8). By [22],

$$d_t := \deg\big(F_t - 1, O_t^*, 0\big), \quad 0 \leq t \leq 1 ,$$

is well-defined where $F_t[u] := f_t(\lambda(A_{\hat{g}})) - 1$.

We first show that

$$d_t \equiv d_0 , \quad 0 \leq t \leq 1 . \tag{4.9}$$

For any $0 \leq \bar{t} \leq 1$, let

$$O^* = \Big\{u \in C^{4,\alpha}(M) : \lambda(A_{\hat{g}}) \in \Gamma_{\bar{t}},\ \operatorname{dist}(\lambda(A_{\hat{g}}), \partial\Gamma_{\bar{t}}) > \tfrac{1}{2}\delta, \\ u > 0, \|u\|_{C^{4,\alpha}} + \|u^{-1}\|_{C^{4,\alpha}} < 2C\Big\} .$$

Then for $\epsilon > 0$ small enough,

$$O_t^* \subset O^* \subset \Gamma_t , \quad (\overline{O^*} \setminus O_t^*) \cap F_t^{-1}(0) = \varnothing , \quad \forall |t - \bar{t}| < \epsilon .$$

By the excision property of the degree,

$$d_t = \deg\big(F_t, O^*, 0\big) \quad \forall |t - \bar{t}| < \epsilon .$$

By the homotopy invariance of the degree,

$$d_t = d_{\bar{t}} \quad \forall |t - \bar{t}| < \epsilon ,$$

from which (4.9) follows.

Now we consider another homotopy $H_t(\lambda) = tf_0 + (1 - t)\sigma_1(\lambda)$, $0 \leq t \leq 1$. Similarly, the total degree of solutions for H_1 $(= f_0)$ is equal to the total degree for H_0 $(= \sigma_1)$. By the result of Schoen [31] for the Yamabe problem, the total degree for σ_1 is equal to -1; thus the total degree for f is nonzero, and therefore equation (1.43) has a solution. □

5 Proof of Theorem 1.27

In this section we establish Theorem 1.27, the Harnack-type inequality. Our proof makes use of ideas in the proof of theorem 1.5 in [25]. By scaling, we only need to prove it for $R = 1$. Also by scaling, we may assume without loss of generality that $u \in C^2(B_6)$ is a positive solution of (1.46) and (1.47) in B_6, and we only need to prove $(\max_{\overline{B}_1} u)(\min_{\overline{B}_3} u) \leq C$.

We argue by contradiction. Suppose the contrary; then there exist positive solutions of (1.46) u_j, $j = 1, 2, \dots$, such that

$$u_j(\bar{x}_j) \min_{\overline{B}_3} u_j > j \quad \text{where } u_j(\bar{x}_j) = \max_{\overline{B}_1} u_j(y)\,. \tag{5.1}$$

As in the proof of theorem 1.5 in [25], there exist $x_j \in B_1(\bar{x}_j)$ such that

$$u_j(x_j) \geq 2^{\frac{2-n}{2}} \max_{B_{\sigma_j}(x_j)} u_j \tag{5.2}$$

and

$$\gamma_j := u_j(x_j)^{\frac{2}{n-2}} \sigma_j \geq \frac{1}{2} u_j(\bar{x}_j)^{\frac{2}{n-2}} \to \infty \tag{5.3}$$

where

$$\sigma_j = \frac{1}{2}(1 - |x_j - \bar{x}_j|) \leq \frac{1}{2}\,.$$

Set

$$w_j(y) = \frac{1}{u_j(x_j)} u_j\left(x_j + \frac{y}{u_j(x_j)^{\frac{2}{n-2}}}\right), \quad |y| < \Gamma_j$$

where

$$\Gamma_j := u_j(x_j)^{\frac{2}{n-2}}\,.$$

Let

$$U = \{M \in \mathcal{S}^{n \times n} : \lambda(M) \in \Gamma\}$$

and

$$F(M) = f(\lambda(M))\,, \quad M \in U\,.$$

Then U satisfies (1.4) and (1.5), and F satisfies (1.6) and (1.7). By the conformal invariance of the equation,

$$F(A^{w_j}) = 1\,, \quad w_j > 0\,, \quad \text{on } B_{\Gamma_j},$$

and by (5.2),

$$1 = w_j(0) \geq 2^{\frac{2-n}{2}} \max_{B_{\gamma_j}} w_j\,. \tag{5.4}$$

Since w_j is a superharmonic function, we have, by (5.1) and (5.3), that

$$\min_{\partial B_{\Gamma_j}} w_j = \min_{\overline{B}_{\Gamma_j}} w_j > j \Gamma_j^{2-n}\,.$$

For every $x \in \mathbb{R}^n$ satisfying $|x| < \frac{1}{2}\gamma_j$, we can find, as in [25], $0 < \lambda_{x,j} < 1$ such that, for all $0 < \lambda \leq \lambda_{x,j}$ and $y \in B_{\Gamma_j} \setminus B_\lambda(x)$, we have

$$w_{j,x,\lambda}(y) := \left(\frac{\lambda}{|y - x|}\right)^{n-2} w_j\left(x + \frac{\lambda^2(y - x)}{|y - x|^2}\right) \leq w_j(y)\,. \tag{5.5}$$

Because of (5.5), we can define

$$\bar{\lambda}_j(x) = \sup\left\{0 < \mu < \Gamma_j - |x| : w_{j,x,\lambda}(y) \leq w_j(y)\ \forall y \in B_{\Gamma_j} \setminus B_\lambda(x),\ 0 < \lambda < \mu\right\}.$$

LEMMA 5.1 *For every $m > 0$,*

$$\lim_{j\to\infty} \inf_{|x|\le m} \bar\lambda_j(x) = \infty\,.$$

PROOF: For simplicity, we only prove that $\bar\lambda_j := \bar\lambda_j(0) \to \infty$, since the general case is essentially the same. Suppose the contrary; then (along a subsequence),

$$\bar\lambda_j \le C < \gamma_j \tag{5.6}$$

for some constant C independent of j. Here we have used the fact $\gamma_j \to \infty$. By the definition of $\bar\lambda_j$,

$$w_j - w_{j,\bar\lambda_j} \ge 0 \quad \text{in } \Sigma_j := \{y;\, \bar\lambda_j < |y| < \Gamma_j\}\,.$$

By (5.4) and (5.6),

$$\max_{\partial B_{\Gamma_j}} w_{j,\bar\lambda_j} \le C(\Gamma_j)^{2-n} < j\Gamma_j^{2-n} < \min_{\partial B_{\Gamma_j}} w_j\,,$$

that is,

$$\min_{\partial B_{\Gamma_j}} (w_j - w_{j,\bar\lambda_j}) > 0\,. \tag{5.7}$$

Recall that

$$w_j - w_{j,\bar\lambda_j} = 0 \quad \text{on } \partial B_{\bar\lambda_j}\,.$$

Because of (5.7), we have, as in the proof of Theorem 1.4, that

$$(w_j - w_{j,\bar\lambda_j})(y) > 0\,, \quad \bar\lambda_j < |y| \le \Gamma_j\,,$$

and

$$\left.\frac{\partial(w_j - w_{j,\bar\lambda_j})}{\partial r}\right|_{\partial B_{\bar\lambda_j}} > 0\,,$$

and thus, for some $\epsilon_j > 0$, that

$$w_{j,\lambda}(y) \le w_j(y) \quad \forall\, \bar\lambda_j \le \lambda \le \bar\lambda_j + \epsilon_j,\ \lambda \le |y| \le \Gamma_j\,,$$

violating the definition of $\bar\lambda_j$. □

Since $\gamma_j \to \infty$, $w_j(0) = 1$, and $\{w_j\}$ is bounded on any compact subset of $\mathbb{R}^n$, we have (as explained in Section 4) by Theorem 1.20 and the $C^{2,\alpha}$ interior estimates of Evans and Krylov [9, 16] that (along a subsequence),

$$w_j \to w \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n)$$

for some positive solution w of

$$F(A^w) = 1 \quad \text{on } \mathbb{R}^n. \tag{5.8}$$

By Lemma 5.1 and the convergence of w_j to w, we have

$$w_{x,\lambda}(y) \le w(y) \quad \forall |y-x| \ge \lambda > 0\,.$$

It follows (see, e.g., [25, lemma 11.2]) that $w \equiv \text{const}$, which violates (5.8) in view of (1.45). Theorem 1.27 is established.

6 Proof of Theorem 1.28 and Theorem 1.10

In this section we establish Theorem 1.28 and Theorem 1.10.

6.1 Groundwork

The Harnack-type inequality (Theorem 1.27) yields the following consequence as established by Schoen in [30] for $(f, \Gamma) = (\sigma_1, \Gamma_1)$. See also an alternative proof of Chen and Lin [6], which we adapt for general (f, Γ).

LEMMA 6.1 *For $n \geq 3$, let (f, Γ) be as in Theorem 1.27, and let B_{3R}, $R > 0$, be a ball of radius R and $u \in C^2(B_{3R})$ be a positive solution of (1.46) satisfying (1.47). Then we have, for some positive constant C depending only on (f, Γ), that*

$$\int_{B_R} R_g u^{2n/(n-20} \leq C$$

where R_g is the scalar curvature of the metric $g = u^{4/(n-2)} g_{\text{flat}}$.

Remark 6.2. Since

$$-\Delta u = \frac{n-2}{4(n-1)} R_g u^{\frac{n+2}{n-2}}, \tag{6.1}$$

we have, by (1.49), that

$$R_g \geq \delta_2 > 0 \quad \text{on } \mathbb{R}^n.$$

A consequence of Lemma 6.1 and Remark 6.2 is the following:

COROLLARY 6.3 *For $n \geq 3$, let (f, Γ) be as in Theorem 1.28, and let u be a solution of (1.50) satisfying (1.51). Then for some positive constant C depending only on (f, Γ), we have*

$$\int_{\mathbb{R}^n} u^{\frac{2n}{n-2}} \leq C \int_{\mathbb{R}^n} R_g u^{\frac{2n}{n-2}} < \infty.$$

PROOF OF LEMMA 6.1: By scaling, we only need prove for the case $R = 1$. Let $u(y) = \min_{\overline{B}_2} u$ for some $y \in \overline{B}_2$, and let $G(x, y)$ be the Green's function of $-\Delta$ on $B_{5/2}$. Then, by (6.1),

$$\begin{aligned} u(y) &= \int_{B_{5/2}} G(y, z)(-\Delta u(z))dz - \int_{\partial B_{5/2}} \frac{\partial G(y, s)}{\partial \nu} u(s)ds \\ &\geq \int_{B_{5/2}} G(y, z) R_g(z) u(z)^{\frac{n+2}{n-2}} dz \geq \frac{1}{C} \int_{B_1} R_g u^{\frac{n+2}{n-2}}. \end{aligned}$$

By Theorem 1.27 and the above,

$$\int_{B_1} R_g u^{\frac{2n}{n-2}} \le (\max_{\bar{B}_1} u) \int_{B_1} R_g u^{\frac{n+2}{n-2}} \le C(\max_{\bar{B}_1} u)(\min_{\bar{B}_2} u) \le C\,.$$

□

A consequence of Theorem 1.20 is the following:

LEMMA 6.4 *Let* (f, Γ) *satisfy condition* (H_1)*. Then there exists some positive constant* δ_0*, depending only on* (f, Γ)*, such that for any positive* C^2 *solutions of*

$$f(\lambda(A^v)) = 1 \quad \textit{in } B_3 \quad \textit{satisfying} \quad \lambda(A^v) \in \Gamma \quad \textit{in } B_3 \quad \textit{and} \quad \int_{B_3} v^{\frac{2n}{n-2}} \le \delta_0\,,$$

we have

$$\sup_{B_2} v \le \frac{1}{\delta_0}\,.$$

PROOF: Our proof is an adaptation of the proof of proposition 2.1 in [24]. Suppose the contrary: there exist a sequence of solutions $\{v_j\}$ such that

$$\int_{B_3} v_j^{\frac{2n}{n-2}} \to 0$$

and

$$d(y_j)^{\frac{n-2}{2}} v_j(y_j) = \max_{y \in B_{2.9}} d(y)^{\frac{n-2}{2}} v_j(y) \to \infty$$

where $d(y) := \operatorname{dist}(y, \partial B_{2.9}) = (2.9 - |y|)$.

Let $\sigma_j = \frac{1}{2} d(y_j) > 0$, and set

$$w_j(z) = \frac{1}{v_j(y_j)} v_j\big(v_j(y_j)^{\frac{2}{2-n}} z + y_j\big)\,, \quad |z| \le r_j := v_j(y_j)^{\frac{2}{n-2}} \sigma_j \to \infty\,.$$

By the conformal invariance,

$$f(\lambda(A^{w_j})) = 1 \quad \text{in } B_{r_j}$$

and

$$\int_{B_{r_j}} w_j^{\frac{2n}{n-2}} \to 0\,. \tag{6.2}$$

It is easy to see that $w_j(0) = 1$ and $\sup_{B_{r_j}} w_j \le 2^{(n-2)/2}$. Thus, by Theorem 1.20, w_j converges uniformly in B_1 along a subsequence, violating (6.2). □

Now we are ready to present the proof of Theorem 1.28.

6.2 Proof of Theorem 1.28

By the maximum principle, using the positivity and the superharmonicity of u, we have

$$u(x) \geq \frac{\min_{\partial B_1} u}{|x|^{n-2}} \quad \forall |x| \geq 1\,. \tag{6.3}$$

By Theorem 1.27 and the above, for some positive universal constant C,

$$R^{2-n} \min_{\partial B_1} u \leq \min_{\partial B_R} u \leq Cu(0)^{-1} R^{2-n}\,. \tag{6.4}$$

For $R > 1$, let

$$u_R(x) := R^{\frac{n-2}{2}} u(Rx)\,, \quad 1 \leq |x| \leq 9\,.$$

By the conformal invariance,

$$f\big(\lambda(A^{u_R}(x))\big) = 1\,, \quad 1 \leq |x| \leq 9\,.$$

By Corollary 6.3,

$$\max_{|x|=4} \int_{B_3(x)} (u_R)^{\frac{2n}{n-2}} \leq \int_{|y|\geq R} u^{\frac{2n}{n-2}} \to 0 \quad \text{as } R \to \infty\,.$$

Fix some positive number $R_0 \geq 1$ such that

$$\max_{|x|=4} \int_{B_3(x)} (u_R)^{\frac{2n}{n-2}} \leq \int_{|y|\geq R_0} u^{\frac{2n}{n-2}} \leq \delta_0 \quad \forall R \geq R_0\,.$$

Thus, by Lemma 6.4,

$$\max_{\overline{B}_2(x)} u_R \leq \frac{1}{\delta_0} \quad \forall |x| = 4\,.$$

By Theorem 1.20,

$$\max_{\overline{B}_1(x)} u_R \leq C \min_{\overline{B}_1(x)} u_R \quad \forall |x| = 4\,.$$

It follows from (6.4) and the above that

$$C^{-1}(\min_{\partial B_1} u) R^{\frac{2-n}{2}} \leq \min_{\overline{B}_{1/2}(x)} u_R \leq \max_{\overline{B}_{1/2}(x)} u_R \leq Cu(0)^{-1} R^{\frac{2-n}{2}} \quad \forall |x| = 4\,.$$

For $R \geq \max\{R_0, u(0)^{-2/(n-2)}\}$,

$$f^{(R^2)}\big(\lambda(A^{R^{(n-2)/2}u_R})\big) = R^{-2} f\big(R^2 \lambda(A^{R^{(n-2)/2}u_R})\big) = R^{-2} \quad \text{on } B_{\frac{1}{2}}(x)$$

where $f^{(s)}(\lambda) := s^{-1} f(s\lambda)$.

We already know that

$$C^{-1} \leq R^{\frac{n-2}{2}} u_R \leq C \quad \text{on } B_{\frac{1}{2}}(x)\,.$$

Applying Theorem 1.20 to $R^{(n-2)/2}u_R$, and in view of Proposition 1.19(ii), we have

$$\sum_{i=0}^{2} \left|\nabla^i (R^{\frac{n-2}{2}} u_R)(x)\right| \leq C \quad \forall |x| \geq R_0$$

where C is some positive constant independent of R. Estimate (1.53) follows from (6.3) and the above.

Integrating equation (6.1) on B_r ($r > 1$) leads to

$$\int_{B_r} R_g(x)u^{\frac{n+2}{n-2}}\,dx = \frac{4(n-1)}{2-n}\int_{\partial B_r}\frac{\partial u}{\partial \nu} \le \frac{4(n-1)}{n-2}\big(\max_{|x|=r}|\nabla u(x)|\big)(|\partial B_r|)\,.$$

Estimate (1.54) follows from the above and (1.53).

6.3 Proof of Theorem 1.10

In the remainder of this section we establish Theorem 1.10 in two cases: $\frac{n}{2} < k \le n$ and $1 \le k \le \frac{n}{2}$.

Case 1. $\frac{n}{2} < k \le n$.

We first make a simple but useful observation.

LEMMA 6.5 *For $n \ge 3$ and $1 \le k \le n$, assume $w \in C^2(B_1 \setminus \{0\})$, $w > 0$ in $B_1 \setminus \{0\}$, and $\lambda(A^w(x)) \in \Gamma_k$ $\forall x \in B_1 \setminus \{0\}$. Let*

$$\xi(x) = \frac{n-2}{2}w(x)^{-\frac{2}{n-2}}\,, \quad x \in B_1 \setminus \{0\}\,.$$

Then

$$\lambda(D^2\xi(x)) \in \Gamma_k \quad \forall x \in B_1 \setminus \{0\}\,.$$

PROOF: A calculation shows

$$D^2\xi = \frac{n-2}{2}w^{\frac{n+2}{n-2}}A^w + \frac{1}{n-2}w^{\frac{2-2n}{n-2}}|\nabla w|^2 I \ge \frac{n-2}{2}w^{\frac{n+2}{n-2}}A^w\,.$$

The lemma follows from the above and the fact that $\lambda(A^w) \in \Gamma_k$. □

LEMMA 6.6 *Let B_1 be the unit ball in $\mathbb{R}^n$ centered at the origin. Assume, for some constants $0 < \alpha < 1$ and $C_1 > 0$, that $\xi \in C^2(B_1 \setminus \{0\}) \bigcap C^\alpha(B_1)$ satisfies*

$$|\xi(x) - \xi(y)| \le C_1|x-y|^\alpha\,, \quad |x|^2|\nabla^2\xi(x)| \le C_1\,, \quad \forall x, y \in B_1 \setminus \{0\}\,. \tag{6.5}$$

Then there exists some constant $C > 0$, depending only on n, α, and C_1, such that

$$|\nabla\xi(x)| \le C|x|^{\frac{\alpha}{2}-1} \quad \forall x \in B_{\frac{1}{4}} \setminus \{0\}\,.$$

PROOF: For $0 < |x| \le \frac{1}{4}$, let $\eta(y) = \xi(|x|y) - \xi(0)$, $\frac{1}{2} \le |y| \le 2$. Then by inequality (6.5),

$$|\eta(y)| \le C|x|^\alpha\,, \quad |\nabla^2\eta(y)|C\,, \quad \forall\,\frac{1}{2} \le |y| \le 2\,.$$

It follows that

$$\|\nabla\eta\|_{L^\infty(\frac{1}{2}\le|y|\le 2)} \le C\|\eta\|^{\frac{1}{2}}_{L^\infty(\frac{1}{2}\le|y|\le 2)}\left(\|\nabla^2\eta\|_{L^\infty(\frac{1}{2}\le|y|\le 2)} + \|\eta\|_{L^\infty(\frac{1}{2}\le|y|\le 2)}\right)^{\frac{1}{2}}$$
$$\le C|x|^{\frac{\alpha}{2}}\,.$$

In particular, $|\nabla\xi(x)| \le C|x|^{\frac{\alpha}{2}-1}$. □

Let u be the solution in Theorem 1.10, and let

$$w(x) = \frac{1}{|x|^{n-2}}u(\frac{x}{|x|^2})\,, \quad x \in \mathbb{R}^n \setminus \{0\}\,.$$

By Theorem 1.28, for some positive constant C_1, we have

$$\frac{1}{C_1} \le w(x) \le C_1 \quad \text{and} \quad |x||\nabla w(x)| + |x|^2|\nabla^2 w(x)| \le C_1 \quad \forall x \in B_2 \setminus \{0\}\,.$$

Let

$$\xi(x) = \frac{n-2}{2}w(x)^{-\frac{2}{n-2}}\,, \quad x \in B_2 \setminus \{0\}\,.$$

To fix the ideas, we consider the following two subcases:

Case 1.1: $k = n$

Case 1.2: $\frac{n}{2} < k < n$.

In case 1.1, since w is bounded above and below by positive constants near the origin, we have, for some positive constant C_2, that

$$\frac{1}{C_2} \le \xi(x) \le C_2 \quad \text{and} \quad |x|\,|\nabla\xi(x)| + |x|^2|\nabla^2\xi(x)| \le C_2 \quad \forall x \in B_2 \setminus \{0\}\,.$$

By Lemma 6.5, ξ is convex, and therefore

$$\sup_{B_1\setminus\{0\}} |\nabla\xi| < \infty\,,$$

from which we deduce

$$\sup_{B_1\setminus\{0\}} |\nabla w| < \infty\,.$$

Theorem 1.10 in case 1 follows from Theorem 1.4.

In case 1.2, we consider, for $0 < \epsilon < \frac{1}{9}$, the mollification of ξ:

$$\xi_\epsilon(x) = \int \xi(x-y)\rho_\epsilon(y)dy\,,$$

where $\rho \in C_c^\infty(B_1)$ with $\rho \geq 0$ and $\int \rho = 1$. Due to the above behavior of ξ and $|\nabla \xi|$ near the origin, we have

$$\begin{aligned} D^2\xi_\epsilon(x) &= \int \xi(y) D^2\rho_\epsilon(x-y)dy \\ &= \lim_{\delta\to 0} \int_{|y|>\delta} \xi(y) D^2\rho_\epsilon(x-y)dy \\ &= \lim_{\delta\to 0} \int_{|y|>\delta} D^2\xi(y)\rho_\epsilon(x-y)dy \\ &= \int D^2\xi(x-y)\rho_\epsilon(y)dy\,. \end{aligned}$$

It follows from the above and the properties of Γ_k that $D^2\xi_\epsilon(x) \in \overline{\Gamma}_k$ for any $x \in B_{1/2}$.

For a nonnegative and nonzero smooth function h that is supported in $B_{1/8} \setminus B_{1/9}$, there exists some positive number δ such that $\zeta h(x) + D^2\xi(x) \in \Gamma_k$ away from the origin for any vector $\zeta \in \mathbb{R}^n$ satisfying $|\zeta| \leq \delta$. Here we have used the property that $D^2\xi$ is continuous and is in Γ_k away from the origin. It follows that

$$\zeta\left(\int h(x-y)\rho_\epsilon(y)dy\right) + D^2\xi_\epsilon(x) = \int (\zeta h + D^2\xi)(x-y)\rho_\epsilon(y)dy \quad \forall |\zeta| \leq \delta\,.$$

Thus

$$D^2\xi_\epsilon(x) \in \Gamma_k \quad \forall x \in B_{\frac{1}{2}}\,.$$

By [34, theorem 2.7] (for this application, knowing that $D^2\rho_\epsilon(x) \in \overline{\Gamma}_k$ for all $x \in B_{1/2}$ is enough), there exist some constants $0 < \alpha < 1$ and C_3, independent of ϵ, such that

$$\|\xi_\epsilon\|_{C^\alpha(\overline{B}_{1/4})} \leq C_3\,.$$

It follows that ξ can be extended to a function in $C^\alpha(\overline{B}_{1/4})$. Applying Lemma 6.6 to ξ, we have, for some constant C,

$$|\nabla\xi(x)| \leq C|x|^{\frac{\alpha}{2}-1} \quad \forall x \in B_1 \setminus \{0\}\,.$$

It follows that

$$|\nabla w(x)| \leq C|x|^{\frac{\alpha}{2}-1} \quad \forall x \in B_1 \setminus \{0\}\,.$$

Again, Theorem 1.10 in case 2 follows from Theorem 1.4.

Case 2. $1 \le k \le \frac{n}{2}$.

Let V be an n-dimensional inner product space, and let $A : V \to V$ be a symmetric linear transformation. For $1 \le k \le n$, the k^{th} Newton transformation associated with A is

$$T_k(A) = \sigma_k(A)I - \sigma_{k-1}(A)A + \cdots + (-1)^k A^k\,,$$

where I is the identity, $\sigma_i(A)$ denotes $\sigma_i(\lambda(A))$, and $\lambda(A)$ denotes eigenvalues of A with respect to an orthonormal basis. We also let

$$L_k(A) = \frac{n-k}{n}\sigma_k(A)I - T_k(A)\,.$$

Recall that

$$A_g = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{R_g}{2(n-1)}g\right)$$

and let

$$E = \mathrm{Ric}_g - \frac{R_g}{n}g\,.$$

Using the metric g we view A_g as a (1, 1)–tensor; then $\sigma_k(A_g)$ and $L_k(A_g)$ are well-defined.

In the rest of this section, we assume that g is locally conformally flat. We list some properties of $\sigma_k(A_g)$ and $L_k(A_g)$ that can be found in [28, 35].

PROPOSITION 6.7 *We have the following*:

(i) *For* $1 \le k \le n$,

$$\sigma_1(L_k(A_g)) = (n-k)\sigma_k(A_g) - \sigma_1(T_k(A_g)) = 0\,,$$
$$\delta_g(L_k(A_g)) = \frac{n-k}{n}d(\sigma_k(A_g))\,,$$

and, for $1 \le k \le n-1$,

$$\sigma_1(T_k(A_g)\cdot A_g) = (k+1)\sigma_{k+1}(A_g)\,.$$

(ii) *For* $1 \le k \le n-1$ *and for* $A_g \in \Gamma_k$,

$$\sigma_{k+1}(A_g) \le \frac{n-k}{(k+1)n}\sigma_k(A_g)\sigma_1(A_g)\,,$$

and equality holds if and only if $A_g = \lambda g(x)$, *i.e.,* $E = 0$.

LEMMA 6.8 *For* $1 \le k \le n-1$, $g(L_k(A_g), E) = \sigma_1(L_k(A_g)\cdot E) \ge 0$, *and equality holds if and only if* $E = 0$.

PROOF: Write

$$A_g = \frac{1}{n-2}(E + \alpha R_g g) \quad \text{where } \alpha = \frac{(n-2)}{2n(n-1)}\,.$$

By Proposition 6.7,

$$\begin{aligned} g(L_k(A_g), E) &= g(L_k(A_g), (n-2)A_g - \alpha R_g g) \\ &= (n-2) g(L_k(A_g), A_g) \\ &= (n-2) g\left(\frac{n-k}{n}\sigma_k(A_g) g, A_g\right) - (n-2) g(T_k(A_g), A_g) \\ &= (n-2)\left\{\frac{n-k}{n}\sigma_k(A_g)\sigma_1(A_g) - (k+1)\sigma_{k+1}(A_g)\right\} \geq 0\,, \end{aligned}$$

where equality holds if and only if A_g is a multiple of g, i.e., $E = 0$. □

Now let $u \in C^2(\mathbb{R}^n)$ be a positive solution of (1.22) satisfying (1.23), and let $g = u^{4/(n-2)} g_{\text{flat}}$.

LEMMA 6.9 *There exists some positive constant C such that*

$$\|A_g\|_g(x) \leq C|x|^2 \quad \forall |x| \geq 1\,.$$

Consequently, for $1 \leq k \leq n$,

$$R_g(x) \leq C|x|^2\,, \quad \|L_k(A_g)\|_g(x) \leq C|x|^{2k}\,, \quad \forall |x| \geq 1\,,$$

where the norm is taken with respect to g.

PROOF: By (1.53) and (6.3),

$$\|A_g\|_g(x) = \|A^u(x)\| \leq C|x|^2 \quad \forall |x| \geq 1\,.$$

The rest follows easily. □

LEMMA 6.10 *For $n \geq 2$,*

$$|\lambda| \leq \sigma_1(\lambda) \quad \forall \lambda \in \overline{\Gamma}_2\,.$$

COROLLARY 6.11 *For $n \geq 2$ and $2 \leq k \leq n$, if $\lambda(A_g) \in \overline{\Gamma}_2$, then*

$$\|L_k(A_g)\|_g \leq C(n)\|A_g\|_g^k \leq C(n) R_g^k\,.$$

PROOF OF LEMMA 6.10: On $\overline{\Gamma}_2$, both σ_1 and σ_2 are nonnegative. So, for $\lambda \in \overline{\Gamma}_2$, $\sigma_1(\lambda)^2 = 2\sigma_2(\lambda) + |\lambda|^2 \geq |\lambda|^2$. □

We resume our proof of Theorem 1.10 when $1 \leq k \leq \frac{n}{2}$.

Writing $g = u^{4/(n-2)} g_{\text{flat}} = v^2 g_{\text{flat}}$, i.e., $v = u^{2/(n-2)}$. By Theorem 1.28, we have, for some positive constant C, that

$$\frac{1}{C|x|^2} \leq v(x) \leq \frac{C}{|x|^2}\,, \quad |\nabla v(x)| \leq \frac{C}{|x|^3}\,, \quad |\nabla^2 v(x)| \leq \frac{C}{|x|^4}\,, \quad \forall |x| \geq 1\,. \tag{6.6}$$

It is elementary to deduce, by using the formula of A_g under a conformal change of metrics, that

$$A_g = -\frac{1}{v}\nabla_g^2 v + \frac{|\nabla_g v|_g^2}{2v^2} g\,, \quad R_g = -\frac{2(n-1)}{v}\Delta_g v + \frac{n(n-1)}{v^2}|\nabla_g v|_g^2\,,$$

and

$$E = \frac{n-2}{n} v^{-1}\Delta_g v g - (n-2)v^{-1}\nabla_g^2 v\,.$$

For any $r \geq 1$, let η be a cutoff function that is equal to 1 in B_r, equal to 0 outside B_{2r}, and satisfies $|\nabla\eta| \leq 2r^{-1}$.

By Proposition 6.7(i),

$$\begin{aligned}\int g(L_k(A_g), E)v\eta^2\, dv_g &= \int g\Big(L_k(A_g), \frac{n-2}{n}v^{-1}\Delta_g v g\Big)v\eta^2\, dv_g \\ &\quad - \int g\big(L_k(A_g), (n-2)v^{-1}\nabla_g^2 v\big)v\eta^2\, dv_g \\ &= \frac{n-2}{n}\int (\Delta_g v)\eta^2\sigma_1(L_k(A_g))dv_g \\ &\quad - (n-2)\int g\big(L_k(A_g), \nabla_g^2 v\big)\eta^2\, dv_g \\ &= -(n-2)\int g\big(L_k(A_g), \nabla_g^2 v\big)\eta^2\, dv_g\,.\end{aligned}$$

Integrating by parts and using Proposition 6.7(i), we have, in view of equation (1.22) ($\sigma_k(A_g) \equiv 1$ on $\mathbb{R}^n$), that

$$\begin{aligned}\int g(L_k(A_g), E)v\eta^2\, dv_g &\leq (n-2)\int g(\delta_g(L_k(A_g)), \nabla_g v)\eta^2\, dv_g \\ &\quad + 2(n-2)\int L_k(A_g)(\nabla_g v, \nabla_g\eta)\eta\, dv_g \\ &= 2(n-2)\int L_k(A_g)(\nabla_g v, \nabla_g\eta)\eta\, dv_g\,.\end{aligned}$$

Using Lemma 6.9 and (6.6),

$$\begin{aligned}\int g(L_k(A_g), E)v\eta^2\, dv_g &\leq 2(n-2)\int \|L_k(A_g)\|_g\|\nabla_g v\|_g\|\nabla_g\eta\|_g\eta\, dv_g \\ &\leq C\int_{r\leq|x|\leq 2r} |x|^{2k-2n}\, dx\,.\end{aligned}$$

For $1 \leq k < \frac{n}{2}$, we deduce from the above, by sending r to infinity, that

$$g(L_k(A_g), E) \equiv 0 \quad \text{on } \mathbb{R}^n\,. \tag{6.7}$$

For $2 \le k \le \frac{n}{2}$, we have, by Lemma 6.9 and Corollary 6.11,

$$\int g(L_k(A_g), E)v\eta^2\, dv_g \le 2(n-2)\int \|L_k(A_g)\|_g \|\nabla_g v\|_g \|\nabla_g \eta\|_g \eta\, dv_g$$
$$\le C \int_{r\le|x|\le 2r} R_g^k v^n\, dx$$
$$\le C \int_{r\le|x|\le 2r} |x|^{2k-n} R_g u^{\frac{n+2}{n-2}}\, dx\,.$$

Sending r to ∞, we deduce (6.7) by using (1.54). By Lemma 6.8, $E \equiv 0$ on $\mathbb{R}^n$. Since

$$E = \frac{n-2}{w}\left(\nabla^2 w - \frac{\Delta w}{n} g_{\text{flat}}\right),$$

where $w := u^{-2/(n-2)}$, $\nabla^2 = \nabla^2_{g_{\text{flat}}}$, and $\Delta = \Delta_{g_{\text{flat}}}$. We have

$$w_{ij} = \frac{\Delta w}{n}\delta_{ij} \quad \forall\, 1 \le i, j \le n\,.$$

It follows that $w_{ijk} = 0$ on $\mathbb{R}^n$ for all $1 \le i, j, k \le n$. Consequently, w is a quadratic polynomial, and therefore, by elementary arguments, u is of the form (1.24).

Another way to conclude the proof is to deduce from $E = \text{Ric}_g - \frac{R_g}{n} g = 0$, by using the second Bianchi identity, that $R_g \equiv$ const. Then, by (1.53) and equation (6.1), u is regular at infinity and therefore u must be of the form (1.24) due to the result of Obata and of Gidas, Ni, and Nirenberg. Theorem 1.10 in the case $1 \le k \le \frac{n}{2}$ is established.

7 Proof of Theorem 1.2

In this section we prove Theorem 1.2.

Let $x \in \mathbb{R}^n$, $s > 0$, $p \in \mathbb{R}^n$, $M \in \mathcal{S}^{n\times n}$, and $O \in O(n)$. We first define a positive smooth function u on $\mathbb{R}^n$ satisfying $u(x) = s$, $\nabla u(x) = p$, and $\nabla^2 u(x) = M$, and consider $\psi(z) = z + x$. Evaluating (1.2) at 0 leads to

$$H(x, s, p, M) = H(0, s, p, M)\,,$$

so H is independent of x. In the following we denote $H(s, p, M) = H(0, s, p, M)$.

Next we define another function u that satisfies $u(0) = s$, $\nabla u(0) = 0$, and $\nabla^2 u(0) = M$, and we consider $\psi(z) = Oz$. Evaluating (1.2) at 0 leads to

$$H(s, 0, O^{-1}MO) = H(s, 0, M)\,.$$

Now pick a function u satisfying $u(0) = s$, $\nabla u(0) = 0$, and $\nabla^2 u(0) = M$, and consider $\psi(z) = s^{2/(2-n)}z$. Evaluating (1.2) at 0 leads to

$$H(s, 0, M) = H\big(1, 0, s^{\frac{n+2}{2-n}} M\big)\,.$$

Finally, restricting $p \neq 0$, setting

$$y = \frac{(2-n)s}{|p|^2} p\,, \quad \lambda = |y|\,,$$

and picking a function u satisfying $u(y) = s$, $u(y) = p$, and $\nabla^2 u(y) = M$. Consider $\psi(z) = \lambda^2 z/|z|^2$. Evaluating (1.2) at $\psi^{-1}(y)$ leads to

$$H(s, p, M) = H\big(u_\psi, \nabla u_\psi, \nabla^2 u_\psi\big) \circ \psi^{-1}(y)\,.$$

A calculation gives

$$J_\psi \circ \psi^{-1}(y) = -1\,, \quad u_\psi \circ \psi^{-1}(y) = u(y)\,, \quad \text{and} \quad \nabla u_\psi \circ \psi^{-1}(y) = 0\,.$$

Recall that

$$A^{u_\psi} \circ \psi^{-1}(y) \sim A^u(y)\,.$$

Thus

$$-\frac{2}{n-2} u(y)^{-\frac{n+2}{n-2}} \nabla^2 u_\psi \circ \psi^{-1}(y) \sim A^u(y)\,.$$

Here and in the following $M \sim N$ means $M = O^{-1} N O$ for some $O \in O(n)$. It follows that

$$\begin{aligned} H(s, p, M) &= H\left(s, 0, -\frac{n-2}{2} s^{\frac{n+2}{n-2}} A^u(y)\right) \\ &= H\left(1, 0, -\frac{n-2}{2} A^u(y)\right) = H\left(1, 0, -\frac{n-2}{2} A(s, p, M)\right) \end{aligned}$$

where

$$A(s, p, M) := \frac{2n}{(n-2)^2} s^{\frac{-2n}{n-2}} p \otimes p - \frac{2}{n-2} s^{\frac{-2-n}{n-2}} M - \frac{2}{(n-2)^2} s^{\frac{-2n}{n-2}} |p|^2 I_{n\times n}\,.$$

Theorem 1.2 follows from the above.

8 Proof of Theorem 1.31

We first claim that for some positive constant C,

$$\frac{1}{C} u(0) \le \inf_{\mathbb{R}^n} u \le \sup_{\mathbb{R}^n} u \le C u(0)\,. \tag{8.1}$$

Indeed for any $R > 1$, let $\{u_i\}$ be a sequence of functions in $C^2(B_{3R})$ satisfying, for some positive constants $\epsilon_i \to 0$,

$$f(\lambda(A^{u_i})) = \epsilon_i \quad \text{on } B_{3R}$$

and $u_i \to u$ in $C^0(\bar{B}_{2R})$. Let

$$\tilde{u}_i(x) = \frac{u_i(Rx)}{M_R}\,, \quad x \in B_2\,,$$

where $M_R = \max_{\bar{B}_{2R}} u + R^{(n-2)/2}$. Then $\lambda(A^{\tilde{u}_i}) \in \Gamma$ on B_2 and

$$f(\lambda(A^{\tilde{u}_i})) = R^{2\beta} M_R^{\frac{4\beta}{n-2}} \epsilon_i \quad \text{in } B_2\,.$$

For large i, $0 < \tilde{u}_i < 1$ on B_2. By Theorem 1.20,

$$\sup_{B_1} \tilde{u}_i \le C \inf_{B_1} \tilde{u}_i \,.$$

Sending i to infinity, we have

$$u(0) \le \sup_{B_R} u \le C \inf_{B_R} u \le C u(0)\,,$$

from which we deduce (8.1).

Next we prove that for any $x \in \mathbb{R}^n$ and for any $\lambda > 0$, we have

$$u_{x,\lambda}(y) \le u(y) \quad \forall |y-x| \ge \lambda\,. \tag{8.2}$$

To prove the above, we only need

$$\limsup_{r\to\infty} \left(r^{n-2} \min_{\partial B_r} u \right) = \infty\,, \tag{8.3}$$

which follows from (8.1).

Without loss of generality, we will establish (8.2) for $x = 0$. We denote $u_\lambda = u_{0,\lambda}$. For any $R > 1$, there exists, by (8.3), $R_j \to \infty$ such that for any $0 < \lambda \le R$,

$$\max_{|y|=R_j} u_\lambda(y) \le \frac{R^{n-2}}{R_j^{n-2}} \max_{\bar{B}_R} u \le \frac{1}{2} \min_{\partial B_{R_j}} u. \tag{8.4}$$

For every fixed $R_j > 2R$, there exist, by the hypotheses, a sequence of positive numbers $\epsilon_i \to 0$ and a sequence of positive functions $\{u_i\}$ in $C^2(B_{2R_j})$ satisfying $\lambda(A^{u_i}) \in \Gamma$ on B_{2R_j}, $u_i \to u$ in $C^0(\overline{B}_{R_j})$, and $f(\lambda(A^{u_i})) = \epsilon_i$ in B_{2R_j}. Arguing as usual by using (8.4), we have, for large i, that

$$(u_i)_\lambda(y) \le u_i(y) \quad \forall\, 0 < \lambda \le R,\ \forall\, \lambda \le |y| \le R_j\,.$$

Sending i to infinity first, then sending j to infinity, and finally sending R to infinity, we obtain (8.2) for $x = 0$. Theorem 1.31 follows from (8.2) together with a calculus lemma (see, e.g., [25, lemma 11.2]).

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AOBING LI
Rutgers–The State University of New Jersey
Department of Mathematics
Hill Center, Busch Campus
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
E-mail: abli@math.rutgers.edu

YANYAN LI
Rutgers–The State University of New Jersey
Department of Mathematics
Hill Center, Busch Campus
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
E-mail: yyli@math.rutgers.edu

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