

A SHARP SOBOLEV INEQUALITY ON RIEMANNIAN MANIFOLDS*

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ABSTRACT. Let (M, g) be a smooth compact Riemannian manifold without boundary of dimension $n \geq 6$. We prove that

$$\|u\|_{L^{2^*}(M, g)}^2 \leq K^2 \int_M \{|\nabla_g u|^2 + c(n)R_g u^2\} dv_g + A\|u\|_{L^{2n/(n+2)}(M, g)}^2,$$

for all $u \in H^1(M)$, where $2^* = 2n/(n-2)$, $c(n) = (n-2)/[4(n-1)]$, R_g is the scalar curvature, $K^{-1} = \inf \|\nabla u\|_{L^2(\mathbb{R}^n)} \|u\|_{L^{2n/(n-2)}(\mathbb{R}^n)}^{-1}$ and $A > 0$ is a constant depending on (M, g) only. The inequality is *sharp* in the sense that on any (M, g) , K can not be replaced by any smaller number and R_g can not be replaced by any continuous function which is smaller than R_g at some point. If (M, g) is not locally conformally flat, the exponent $2n/(n+2)$ can not be replaced by any smaller number. If (M, g) is locally conformally flat, a stronger inequality, with $2n/(n+2)$ replaced by 1, holds in all dimensions $n \geq 3$.

1. Introduction. Considerable work has been devoted to the analysis of sharp Sobolev-type inequalities, very often in connection with concrete problems from geometry and physics. See, e.g., Trudinger [44], Moser [34], Aubin [3, 4], Talenti [43], Brezis and Nirenberg [10], Lieb [32], Lieb and Thirring [33], Carlen and Loss [14], Carleson and Chang [15], Struwe [42], Escobar [22], and Beckner [8].

In order to fix notation, we recall a classical result. For $n \geq 3$ and $2^* = 2n/(n-2)$, it was shown by Aubin [3] and Talenti [43] that

$$K^{-1} = \inf \left\{ \frac{\|\nabla u\|_{L^2(\mathbb{R}^n)}}{\|u\|_{L^{2^*}(\mathbb{R}^n)}} : u \in L^{2^*}(\mathbb{R}^n) \setminus \{0\}, |\nabla u| \in L^2(\mathbb{R}^n) \right\}, \quad (1.1)$$

where $K^2 = 4/[n(n-2)\sigma_n^{2/n}]$ and where σ_n is the volume of the standard n -sphere. They also showed that the infimum is attained and, modulo non-zero constant

1991 *Mathematics Subject Classification.* 35J60, 58E35.

Key words and phrases. sharp Sobolev inequality, critical exponent, Yamabe problem.

*Y.Y.L. was partially supported by National Science Foundation Grant DMS-0100819 and a Rutgers University Research Council Grant. T.R. was partially supported by CNR Fellowship 203.01.69 (19/01/98) and by PRIN 2000 "Variational Methods and Nonlinear Differential Equations".

multiples, the set of minimizers is given by

$$\{U_{y,\lambda} ; y \in \mathbb{R}^n, \lambda > 0\},$$

where

$$U_{y,\lambda}(x) = \lambda^{(n-2)/2} U(\lambda(x-y)),$$

$$U(x) = U_{0,1}(x) = \left(\frac{1}{1 + \lambda^2 |x|^2} \right)^{\frac{n-2}{2}}$$

and $\bar{\lambda}^2 = [n(n-2)]^{-1} K^{-2}$. The function U is characterized as the unique solution of the equation

$$-\Delta U = K^{-2} U^{2^*-1} \quad \text{in } \mathbb{R}^n \quad (1.2)$$

satisfying

$$U \in D^{1,2}(\mathbb{R}^n), \quad 0 < U \leq 1,$$

$$U(0) = 1, \quad \int_{\mathbb{R}^n} U^{2^*} dx = 1.$$

A conjecture was made by Aubin [3]: On any smooth compact Riemannian manifold (M, g) of dimension $n \geq 3$, there exists a constant $A > 0$ depending only on (M, g) , such that

$$\|u\|_{L^{2^*}(M,g)}^2 \leq K^2 \|\nabla_g u\|_{L^2(M,g)}^2 + A \|u\|_{L^2(M,g)}^2, \quad \forall u \in H^1(M). \quad (1.3)$$

The conjecture was proved in [3] for manifolds of constant sectional curvature. He also proved a weaker version of (1.3), where for any $\varepsilon > 0$, K is replaced by $K + \varepsilon$ and where A is allowed to depend on ε .

Various related questions in bounded domains Ω of \mathbb{R}^n have been extensively studied. In particular, the following result was proved by Brezis and Nirenberg [10]: For $n = 3$, there exists a constant $\lambda^* > 0$ such that

$$\|\nabla u\|_{L^2(\Omega)}^2 \geq K^{-2} \|u\|_{L^6(\Omega)}^2 + \lambda^* \|u\|_{L^2(\Omega)}^2, \quad \forall u \in H_0^1(\Omega),$$

where λ^* depends on Ω ; when Ω is a ball, λ^* can be taken as $\frac{1}{4}\pi^2(3|\Omega|/(4\pi))^{-2/3}$ which is sharp. They also showed that for $n \geq 4$ and for all $q < n/(n-2)$,

$$\|\nabla u\|_{L^2(\Omega)}^2 \geq K^{-2} \|u\|_{L^{2^*}(\Omega)}^2 + \lambda_q \|u\|_{L^q(\Omega)}^2, \quad \forall u \in H_0^1(\Omega),$$

where $\lambda_q > 0$ depends on Ω and q . On the other hand they pointed out that, on any Ω , such an inequality can not hold with $q = n/(n-2)$. Further results were obtained by Brezis and Lieb [9], and closely related ones by Adimurthi and Yadava [1]. Results of similar nature, concerning the Hardy-Littlewood inequality for functions with support in a ball of \mathbb{R}^n , were obtained and used by Daubechies and Lieb [16]. We refer to Brezis and Marcus [11], Brezis, Marcus and Shafrir [12], and Shafrir [38] for more recent related works on the sharp Hardy-Littlewood inequality.

The conjecture (1.3) was proved by Hebey and Vaugon [27]. Results of similar nature for manifolds with boundary were established by Li and Zhu [30, 31], with improvements given by Zhu [45, 46]. A $W^{1,p}$ version of (1.3) with $p \neq 2$, also conjectured by Aubin [3], was proved through the work of Aubin and Li [6], and Druet [18, 19]. It should be mentioned that in two dimensions, the corresponding inequality discovered by Trudinger [44] has also been widely investigated and applied in its sharp form, due to Moser [34]. A sharp Moser-Trudinger inequality on Riemannian 2-manifolds was established by Fontana [23], and used by Gillet and Soulé [25]. Alternative proofs of Fontana's result and connections to the analysis of

vortices in the Chern-Simon-Higgs gauge theory were given by Ding, Jost, Li and Wang [17] and by Nolasco and Tarantello [35, 36].

Statement of the main results. Our main result in this paper is the following sharp Sobolev inequality on Riemannian manifolds of dimension $n \geq 6$:

THEOREM 1.1 (Main Result). *Let (M, g) be a smooth compact Riemannian manifold without boundary of dimension $n \geq 6$. There exists a constant $A > 0$, depending on (M, g) only, such that for all $u \in H^1(M)$ there holds:*

$$\|u\|_{L^{2^*}(M, g)}^2 \leq K^2 \int_M \{|\nabla_g u|^2 + c(n)R_g u^2\} dv_g + A\|u\|_{L^{\bar{r}}(M, g)}^2, \quad (1.4)$$

where 2^* and K are defined above, $c(n) = (n-2)/[4(n-1)]$, $\bar{r} = 2n/(n+2) = 2^{*'}$, R_g is the scalar curvature of g .

We point out that our proof of Theorem 1.1 does not make any use of inequality (1.3), which on the other hand is an easy consequence.

REMARK 1.2 (Sharpness). *Theorem 1.1 is sharp, in the sense that one can neither replace K by any smaller number, nor replace R_g by any $R_g + f$ with $f \in C^0$ negative somewhere. Moreover, if (M, g) is not locally conformally flat, one cannot replace \bar{r} by any smaller number.*

The case of locally conformally flat manifolds is completely described by our next result:

THEOREM 1.3. *Let (M, g) be a smooth compact locally conformally flat Riemannian manifold without boundary of dimension $n \geq 3$. There exists a constant $A > 0$, depending on (M, g) only, such that for all $u \in H^1(M)$ there holds:*

$$\|u\|_{L^{2^*}(M, g)}^2 \leq K^2 \int_M \{|\nabla_g u|^2 + c(n)R_g u^2\} dv_g + A\|u\|_{L^1(M, g)}^2. \quad (1.5)$$

In view of the work of Schoen [39], we expect a positive answer to

QUESTION 1.4. *For locally conformally flat manifolds and for manifolds of dimension $3 \leq n \leq 5$, are there some Sobolev type inequalities involving global geometric quantities?*

For manifolds with positive total scalar curvature $\int_M R_g dv_g$, a natural global geometric quantity is the ‘‘mass’’, which corresponds to the leading term of the regular part of the Green’s function for the conformal Laplacian, see [39, 28].

The *sharpness* of Theorem 1.1 as stated in Remark 1.2 can be deduced from the following expansions due to Aubin [4] (see also [5, 28]). Let $P \in M$, $\lambda > 0$ and let h be a Riemannian metric on M ; denote by $\xi_{P, \lambda}^h$ the ‘‘ h -bubble’’ defined for $x \in M$ by

$$\xi_{P, \lambda}^h(x) = \left(\frac{\lambda}{1 + (\lambda \bar{\lambda})^2 \text{dist}_h^2(x, P)} \right)^{\frac{n-2}{2}}.$$

Let η be a smooth cutoff function supported near P , and set $\tilde{\xi}_{P, \lambda}^h = \eta \xi_{P, \lambda}^h$. Then, as $\lambda \rightarrow \infty$,

$$Y_h(\tilde{\xi}_{P, \lambda}^h) = \begin{cases} K^{-2} - \gamma_n |W_h(P)|^2 \lambda^{-4} + o(\lambda^{-4}), & \text{if } n \geq 7, \\ K^{-2} - \gamma_n |W_h(P)|^2 \lambda^{-4} \log \lambda + o(\lambda^{-4} \log \lambda), & \text{if } n = 6, \end{cases} \quad (1.6)$$

where $\gamma_n > 0$ is a dimensional constant, $W_h(P)$ is the Weyl tensor of h at P and Y_h denotes the Yamabe functional:

$$Y_h(u) = \frac{\int_M \{|\nabla_h u|^2 + c(n)R_h u^2\} dv_h}{\left(\int_M |u|^{2^*} dv_h\right)^{2/2^*}}, \quad u \in H^1(M) \setminus \{0\}. \quad (1.7)$$

To see the sharpness of Theorem 1.1, we note that if K is replaced by any smaller number, then (1.4) is violated by $u = \tilde{\xi}_{P,\lambda}^g$ for large λ (fixing any $P \in M$); if R_g is replaced by $R_g + f$ with $f(\bar{P}) < 0$ for some $\bar{P} \in M$, then (1.4) is violated by $u = \tilde{\xi}_{\bar{P},\lambda}^g$ for large λ ; if \bar{r} is replaced by some $1 \leq s < \bar{r}$, then we have $\|\tilde{\xi}_{P,\lambda}^g\|_{L^s(M)} = o(\|\tilde{\xi}_{P,\lambda}\|_{L^{\bar{r}}(M)})$, and thus inequality (1.4) and (1.6)–(1.7) imply that $|W_g(P)| = 0$ for all $P \in M$, i.e., (M, g) is locally conformally flat.

In view of (1.6) and our results we expect a positive answer to

QUESTION 1.5. *Are there some refined versions of (1.4) involving the Weyl tensor?*

Outline of the proofs. We first sketch the proof of Theorem 1.3, which is simple, and relies on a “local to global” argument, given in the Appendix. By a local to global argument, we mean that we first establish the inequality for all functions $u \in H^1(M)$ supported in a ball of fixed diameter $\varepsilon > 0$, and then we extend the inequality to arbitrary $u \in H^1(M)$. Another ingredient is the following well-known transformation property of the conformal Laplacian, see, e.g., [40]:

$$-\Delta_{\hat{h}} u + c(n)R_{\hat{h}} u = \varphi^{1-2^*} \{-\Delta_h(u\varphi) + c(n)R_h(u\varphi)\}, \quad (1.8)$$

for all $u \in H^1(M)$, where $\hat{h} = \varphi^{4/(n-2)}h$, $\varphi \in C^\infty(M)$, $\varphi > 0$.

Proof of Theorem 1.3. Since (M, g) is locally conformally flat, for some $\varepsilon > 0$ independent of $P \in M$, we have $(B_\varepsilon(P), g) \cong (B, \varphi^{4/(n-2)}\mathcal{E})$, for some $\varphi > 0$ (under control), with \mathcal{E} the Euclidean metric. Since

$$\left(\int_B |u|^{2^*} dx\right)^{2/2^*} \leq K^2 \int_B |\nabla u|^2 dx, \quad \forall u \in H_0^1(B),$$

we have by (1.8),

$$\left(\int_{B_\varepsilon(P)} |u|^{2^*} dv_g\right)^{2/2^*} \leq K^2 \int_{B_\varepsilon(P)} \{|\nabla_g u|^2 + c(n)R_g u^2\} dv_g, \quad \forall u \in H_0^1(B_\varepsilon(P)).$$

Now Theorem 1.3 follows from the above and from Lemma 7.1 in the Appendix. \square

The “local to global” approach has been systematically used by Aubin [3], Hebey and Vaugon [27], Aubin and Li [6], Druet, Hebey and Vaugon [20], and others. In [30, 31], Li and Zhu introduced a global approach by attacking the problem directly on the whole manifold. Such an approach should be useful in obtaining a positive answer to Question 1.4, since the inequality would involve global quantities and therefore could not be obtained by a local to global approach.

We shall now provide a brief sketch of the proof of Theorem 1.1, which will occupy the main part of this paper. For simplicity of exposition, we shall restrict ourselves in the present sketch to the case $n \geq 7$. We argue by contradiction, and we take a global approach. Namely, for all $\alpha > 0$ we define:

$$I_\alpha(u) = \frac{\int_M \{|\nabla_g u|^2 + c(n)R_g u^2\} dv_g + \alpha \|u\|_{L^{\bar{r}}(M,g)}^2}{\|u\|_{L^{2^*}(M,g)}^2}, \quad u \in H^1(M) \setminus \{0\}.$$

Negating (1.4), we assume that

$$\inf_{H^1(M) \setminus \{0\}} I_\alpha < K^{-2}, \quad \forall \alpha > 0. \quad (1.9)$$

It is straightforward to check that inequality (1.4) holds for the family $\{t\tilde{\xi}_{P,\lambda}^g\}$ defined above, uniformly in $t > 0, P \in M, \lambda > 0$. The underlying idea of the proof is that if (1.9) holds for all $\alpha > 0$, then for all $\alpha > 0$ there exist minimizers u_α of I_α , which approach $\{t\tilde{\xi}_{P,\lambda}^g\}$ as $\alpha \rightarrow +\infty$, and the convergence rate is sufficiently rapid to ensure that for some suitable $A > 0$, u_α also satisfies (1.4), uniformly in α . But then $\alpha \leq C$, a contradiction.

In *Section 2*, for the reader's convenience, we establish some preliminary results by suitably adapting to our needs some well-known techniques from [44, 4, 10, 27, 6]. We show that (1.9) implies the existence of a minimizer $u_\alpha \in H^1(M)$ for I_α satisfying $u_\alpha \in H^1(M)$, $u_\alpha \geq 0$, $\int_M u_\alpha^{2^*} dv_g = 1$ and such that

$$\mu_\alpha^{(n-2)/2} := \max_M u_\alpha^{-1} =: u_\alpha(x_\alpha)^{-1} \rightarrow 0.$$

We fix some small $\delta_0 > 0$ which depends only on (M, g) . We show:

$$\begin{aligned} \|\nabla_g(u_\alpha - \xi_{x_\alpha, \mu_\alpha^{-1}}^g)\|_{L^2(B_{\delta_0})} + \|u_\alpha - \xi_{x_\alpha, \mu_\alpha^{-1}}^g\|_{L^{2^*}(B_{\delta_0})} &\rightarrow 0 \\ \mu_\alpha^{(n-2)/2} u_\alpha(\exp_{x_\alpha}^g(\mu_\alpha \cdot)) &\rightarrow U \quad \text{in } C_{\text{loc}}^2(\mathbb{R}^n). \end{aligned}$$

The $C_{\text{loc}}^2(\mathbb{R}^n)$ -convergence and a change of variables imply the lower bound:

$$\|u_\alpha\|_{L^{\bar{r}}(M,g)} \geq C^{-1} \mu_\alpha^2. \quad (1.10)$$

In *Section 3* we prove the *uniform estimate*:

$$u_\alpha(x) \leq C \mu_\alpha^{(n-2)/2} \text{dist}_g(x, x_\alpha)^{2-n}, \quad \forall x \in M.$$

This estimate ensures a suitable decay of u_α away from x_α ; it is a key step. We note that pointwise estimates for minimizers to critical exponent equations have been established and used by Brezis and Peletier [13], Atkinson and Peletier [2], Rey [37], Han [26], Hebey and Vaugon [27], Li and Zhu [30, 31], Aubin and Li [6], and others. We derive our pointwise estimate along the line of [30, 31], by working directly on u_α ; new ingredients are needed in deriving our estimate.

In *Section 4*, in order to simplify calculations, we introduce a conformal metric $\hat{g} = \psi^{4/(n-2)}g$, with $\psi \in C^\infty(M)$, $\psi(x_\alpha) = 1$, $\frac{1}{2} \leq \psi \leq 2$, $\|\psi\|_{C^2} \leq C$, such that $R_{\hat{g}} \equiv 0$ in $B_{\delta_0}(x_\alpha)$. Our pointwise estimates in Section 2 allow us to adapt ideas of Bahri and Coron [7] to make an energy estimate of the difference: $u_\alpha/\psi - t_\alpha \xi_{\tilde{x}_\alpha, \lambda_\alpha}^{\hat{g}}$ in a small ball $B_{\delta_\alpha}(x_\alpha)$, where $\delta_\alpha \in [\delta_0/2, \delta_0]$, $t_\alpha > 0$, $\mu_\alpha^{-1}|\tilde{x}_\alpha - x_\alpha| \rightarrow 0$, $\lambda_\alpha > 0$ are ‘‘optimal’’ in a suitable sense. The main result of Section 4 is the estimate for the projection $u_\alpha/\psi - t_\alpha \xi_{\tilde{x}_\alpha, \lambda_\alpha}^{\hat{g}}$ on $H_0^1(B_{\delta_\alpha}(x_\alpha))$, denoted w_α , as in Proposition 4.1.

In *Section 5* we show that by choosing a ‘‘good radius’’ $\delta_\alpha \in [\delta_0/2, \delta_0]$, the ‘‘boundary part’’ of $u_\alpha/\psi - t_\alpha \xi_{\tilde{x}_\alpha, \lambda_\alpha}^{\hat{g}}$ may be controlled in $H^1(\partial B_\alpha)$, see Lemma 5.3. For $n \geq 7$, the estimate resulting from our pointwise estimates, Proposition 4.1, Lemma 5.3 and taking into account (2.10) is given by:

$$\|\nabla_{\hat{g}}\left(\frac{u_\alpha}{\psi} - t_\alpha \xi_{\tilde{x}_\alpha, \lambda_\alpha}^{\hat{g}}\right)\|_{L^2(B_{\delta_\alpha}(x_\alpha))} \leq C\left(\mu_\alpha^2 + (1 + \mu_\alpha^{-2+\beta})\alpha\|u_\alpha\|_{L^{\bar{r}}(M,g)}^2\right), \quad (1.11)$$

where $\beta = (n-6)(n-2)/[2(n+2)] > 0$ is *strictly positive*, since $n \geq 7$. By carefully exploiting orthogonality, we prove the following lower bound:

$$Y_g(u_\alpha) \geq Y_{\widehat{g}}(\widetilde{\xi}_{\widehat{x}_\alpha, \lambda_\alpha}^g) + O(\mu_\alpha^2 \|\nabla_{\widehat{g}}(\frac{u_\alpha}{\psi} - t_\alpha \xi_{\widehat{x}_\alpha, \lambda_\alpha}^g)\|_{L^2(B_{\delta_\alpha}(x_\alpha))} + \mu_\alpha^{n-2}), \quad (1.12)$$

see Proposition 5.1.

At this point we have all the necessary ingredients to conclude the proof in the case $n \geq 7$. We note that the contradiction assumption (1.9) implies:

$$K^{-2} > I_\alpha(u_\alpha) = Y_g(u_\alpha) + \alpha \|u_\alpha\|_{L^{\bar{r}}(M,g)}^2.$$

By the above inequality and (1.12), we obtain

$$\begin{aligned} \alpha \|u_\alpha\|_{L^{\bar{r}}(M,g)}^2 &\leq K^{-2} - Y_{\widehat{g}}(\widetilde{\xi}_{\widehat{x}_\alpha, \lambda_\alpha}^g) \\ &\quad + C(\mu_\alpha^2 \|\nabla_{\widehat{g}}(\frac{u_\alpha}{\psi} - t_\alpha \xi_{\widehat{x}_\alpha, \lambda_\alpha}^g)\|_{L^2(B_{\delta_\alpha}(x_\alpha))} + \mu_\alpha^{n-2}). \end{aligned} \quad (1.13)$$

By (1.6) (or an easy calculation since we do not need the explicit coefficient of λ^{-4}),

$$|K^{-2} - Y_{\widehat{g}}(\widetilde{\xi}_{\widehat{x}_\alpha, \lambda_\alpha}^g)| \leq C\mu_\alpha^4. \quad (1.14)$$

Inserting (1.11) and (1.14) into (1.13), and recalling that $\beta > 0$, we derive

$$(1 + o(1))\alpha \|u_\alpha\|_{L^{\bar{r}}(M,g)}^2 \leq C\mu_\alpha^4.$$

In view of (1.10), the desired contradiction $\alpha \leq C$ follows, and Theorem 1.1 is established.

Finally, *Section 6* is devoted to the proof of Theorem 1.1 in the remaining case $n = 6$. This is more delicate than the case $n \geq 7$. Nevertheless, we can still obtain the inequality (1.4) with the aid of a uniform *lower* bound, reminiscent of an argument in [30].

Notation. Henceforth, $C > 0$ always denotes a general constant independent of α , and subsequences of $\alpha \rightarrow +\infty$ are taken without further notice. Denoting by (Ω, h) a Riemannian manifold (possibly with boundary), we set

$$\begin{aligned} \langle \varphi, \psi \rangle_h &= \int_\Omega \nabla_h \varphi \cdot \nabla_h \psi \, dv_h = \int_\Omega h^{ij} \frac{\partial \varphi}{\partial x^i} \frac{\partial \psi}{\partial x^j} \, dv_h, \quad \forall \varphi, \psi \in H_0^1(\Omega), \\ \|\varphi\|_h &= \sqrt{\langle \varphi, \varphi \rangle_h}, \quad \forall \varphi \in H_0^1(\Omega). \end{aligned}$$

We note that the metrics g and \widehat{g} defined above are both equivalent to the Euclidean metric \mathcal{E} . When the specific metric is clear from the context, or irrelevant up to equivalence to g , we do not indicate it explicitly. Furthermore, for $q \geq 1$ we denote:

$$\begin{aligned} \|\varphi\|_q &= \|\varphi\|_{L^q(B_\alpha)}, \\ \|U^q\|_{2^{*'}, \mu_\alpha^{-1}} &= \left(\int_{B_{\mu_\alpha^{-1}}(0)} U^{2^{*'}q} \, dy \right)^{1/2^{*'}}, \quad 2^{*'} = \frac{2n}{n+2}, \end{aligned}$$

where U is the standard minimizer on \mathbb{R}^n defined above.

For ease of future reference, we prove our estimates for $n \geq 3$. Moreover, we obtain our estimates for a general exponent $r \in (1, 2)$, which could even depend on α (this will also be convenient for the local to global argument sketched in the Appendix). The actual value $r = \bar{r} = 2n/(n+2)$ and the condition $n \geq 6$ are used only in the final part of the proof of Theorem 1.1, in Section 5 and in Section 6.

Theorem 1.1 in the case $n \geq 7$ has been presented at the 966th AMS Meeting at Hoboken, NJ, April 28–29, 2001. The results in the present paper were announced in [29].

2. Preliminaries. The preliminary results in this section are obtained by adapting standard methods to our situation, see, e.g., [44, 4, 10, 27, 6]. For the reader's convenience, we sketch their proofs. Throughout this section, we assume $n \geq 3$.

For every $\alpha > 0$ and for $r \in (1, 2)$ (possibly depending on α) we consider the functional:

$$I_\alpha(u) = \frac{\int_M \{|\nabla_g u|^2 + c(n)R_g u^2\} dv_g + \alpha \|u\|_{L^r(M)}^2}{\|u\|_{L^{2^*}(M)}^2},$$

defined for all $u \in H^1(M) \setminus \{0\}$. If (1.4) is false, then for all $\alpha > 0$ we have

$$\inf_{H^1(M) \setminus \{0\}} I_\alpha < K^{-2}. \quad (2.1)$$

PROPOSITION 2.1 (Existence of a minimizer). *For all $\alpha > 0$ there exists a non-negative minimizer $u_\alpha \in H^1(M)$ such that*

$$I_\alpha(u_\alpha) = \ell_\alpha = \inf_{H^1(M) \setminus \{0\}} I_\alpha < K^{-2},$$

$$\int_M u_\alpha^{2^*} dv_g = 1.$$

Moreover, $u_\alpha \in C^{2,r-1}(M)$ is a classical solution of the Euler-Lagrange equation:

$$-\Delta_g u_\alpha + c(n)R_g u_\alpha + \alpha \|u_\alpha\|_{L^r(M)}^{2-r} u_\alpha^{r-1} = \ell_\alpha u_\alpha^{2^*-1} \quad \text{on } M. \quad (2.2)$$

Proof. By homogeneity, it is equivalent to minimize I_α on the set

$$\mathcal{A} = \{u \in H^1(M) : \int_M |u|^{2^*} dv_g = 1\}.$$

However, \mathcal{A} is not sequentially weakly closed in $H^1(M)$. Therefore, as usual, for fixed α and for all $1 \leq q < 2^*$ we define:

$$\mathcal{A}_q = \{u \in H^1(M) : \int_M |u|^q dv_g = 1\}$$

and we consider the functional

$$I_q(u) = \frac{\int_M \{|\nabla_g u|^2 + c(n)R_g u^2\} dv_g + \alpha \left(\int_M |u|^r dv_g\right)^{2/r}}{\left(\int_M |u|^q dv_g\right)^{2/q}}$$

on \mathcal{A}_q . By standard arguments $\inf_{\mathcal{A}_q} I_q$ is attained, i.e., for every $1 \leq q < 2^*$ there exists $u_q \in \mathcal{A}_q$ such that

$$I_q(u_q) = \inf_{\mathcal{A}_q} I_q =: \ell_q.$$

The minimizer u_q satisfies the Euler-Lagrange equation:

$$-\Delta_g u_q + c(n)R_g u_q + \alpha \|u_q\|_{L^r(M)}^{2-r} u_q^{r-1} = \ell_q u_q^{q-1} \quad \text{on } M. \quad (2.3)$$

The sequence u_q is bounded in $H^1(M)$; therefore passing to a subsequence we can assume that there exists $u_\alpha \in H^1(M)$ such that $u_q \rightharpoonup u_\alpha$ weakly in $H^1(M)$, strongly in $L^2(M)$ and a.e. Since for every fixed u we have $I_q(u) \rightarrow I_\alpha(u)$ as $q \rightarrow 2^*$, it is clear that

$$\limsup_{q \rightarrow \infty} \ell_q \leq \ell_\alpha < K^{-2}.$$

Consequently, for every $0 < 2^* - q \ll 1$, we can apply the Moser iteration technique to (2.3) to derive a uniform bound $\sup_M u_q \leq C(\alpha)$, where $C(\alpha) > 0$ is a constant independent of q (see, e.g., [6]). Then by dominated convergence, $u_\alpha \in \mathcal{A}$ and by weak semicontinuity $I_\alpha(u_\alpha) \leq \liminf \ell_q \leq \ell_\alpha$. The u_α is a desired minimizer.

The proof of the existence of the minimizer shows that u_α is in $L^\infty(M)$ for every fixed α . Then standard elliptic theory implies that $u_\alpha \in C^{1,\beta}(M)$ for some $0 < \beta < 1$. Therefore $u_\alpha^{r-1} \in C^{0,r-1}(M)$, and by Schauder estimates $u_\alpha \in C^{2,r-1}(M)$. \square

REMARK 2.2. *Since $0 < r - 1 < 1$, the nonlinearity u^{r-1} is sublinear and therefore we can not use the maximum principle to conclude $u_\alpha > 0$ on M .*

PROPOSITION 2.3 (Standard blowup). *As $\alpha \rightarrow +\infty$, we have:*

$$u_\alpha \rightarrow 0, \quad \text{weakly in } H^1(M), \text{ strongly in } L^p(M) \forall 1 \leq p < 2^* \text{ and a.e.}, \quad (\text{i})$$

$$\int_M |\nabla_g u_\alpha|^2 dv_g \rightarrow K^{-2}, \quad (\text{ii})$$

$$\alpha \|u_\alpha\|_{L^r(M)}^2 \rightarrow 0, \quad (\text{iii})$$

$$\ell_\alpha \rightarrow K^{-2}, \quad (\text{iv})$$

$$\max_M u_\alpha \rightarrow +\infty. \quad (\text{v})$$

Proof. By compactness, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that:

$$\|u\|_{L^2(M)}^2 \leq \varepsilon \int_M |\nabla_g u|^2 dv_g + C_\varepsilon \|u\|_{L^r(M)}^2.$$

So,

$$\begin{aligned} I_\alpha(u_\alpha) = \ell_\alpha &= \int_M \{|\nabla_g u_\alpha|^2 + c(n)R_g u_\alpha^2\} dv_g + \alpha \|u_\alpha\|_{L^r(M)}^2 \\ &\geq (1 - \varepsilon c(n) \max_M |R_g|) \int_M |\nabla_g u_\alpha|^2 dv_g + (\alpha - C_\varepsilon) \|u_\alpha\|_{L^r(M)}^2. \end{aligned}$$

Fixing a small ε we obtain:

$$\frac{1}{2} \int_M |\nabla_g u_\alpha|^2 dv_g + (\alpha - C_\varepsilon) \|u_\alpha\|_{L^r(M)}^2 \leq I_\alpha(u_\alpha) < K^{-2}.$$

Consequently,

$$\int_M |\nabla_g u_\alpha|^2 dv_g \leq C, \quad \alpha \|u_\alpha\|_{L^r(M)}^2 \leq C,$$

and therefore,

$$\int_M u_\alpha^r dv_g \rightarrow 0 \quad \text{as } \alpha \rightarrow +\infty.$$

Passing to a subsequence, we have (i). Furthermore, we can assume that for some $\theta, \eta \in [0, +\infty)$ there holds (along a subsequence):

$$\int_M |\nabla_g u_\alpha|^2 dv_g \rightarrow \theta \quad \text{and} \quad \alpha \|u_\alpha\|_{L^r(M)}^2 \rightarrow \eta, \quad \text{as } \alpha \rightarrow +\infty.$$

Proof of (ii)–(iii). We have to show that $\theta = K^{-2}$ and $\eta = 0$. By the Sobolev inequality as in [5], for every $\varepsilon > 0$ there exists $A_\varepsilon > 0$ such that:

$$\|u_\alpha\|_{L^{2^*}(M)}^2 \leq K^2(1 + \varepsilon) \int_M |\nabla_g u_\alpha|^2 dv_g + A_\varepsilon \|u_\alpha\|_{L^r(M)}^2. \quad (2.4)$$

Letting $\alpha \rightarrow +\infty$ in (2.4) we obtain:

$$1 = \left(\int_M u_\alpha^{2^*} dv_g \right)^{2/2^*} \leq K^2(1 + \varepsilon)\theta.$$

Sending $\varepsilon \rightarrow 0$, we conclude $1 \leq K^2\theta$. On the other hand, we have by definition of u_α :

$$\int_M \{|\nabla_g u_\alpha|^2 + c(n)R_g u_\alpha^2\} dv_g + \alpha \|u_\alpha\|_{L^r(M)}^2 = \ell_\alpha < K^{-2}.$$

Sending $\alpha \rightarrow +\infty$ we find $\theta + \eta \leq K^{-2}$. It follows that $\theta = K^{-2}$ and $\eta = 0$, as asserted.

Proof of (iv). This is an immediate consequence of (i)–(ii)–(iii) and the definition of ℓ_α .

Proof of (v). We have:

$$1 = \int_M u_\alpha^{2^*} dv_g \leq (\max_M u_\alpha)^{2^*-r} \int_M u_\alpha^r dv_g = o(1)(\max_M u_\alpha)^{2^*-r}.$$

□

Our next aim is to show that, after rescaling, the limit profile of u_α is the standard minimizer U , and that u_α approaches this limit “in energy”, as in Proposition 2.4. Let $x_\alpha \in M$ be a maximum point of u_α , namely $u_\alpha(x_\alpha) = \max_M u_\alpha$, then by Proposition 2.3–(v) we have

$$\mu_\alpha := u_\alpha(x_\alpha)^{-2/(n-2)} \rightarrow 0 \quad \text{as } \alpha \rightarrow +\infty. \quad (2.5)$$

Let $\delta_0 > 0$ be a small constant to be fixed below (e.g., less than injectivity radius). Let $\delta_0/2 \leq \delta_\alpha \leq \delta_0$.

PROPOSITION 2.4 (Convergence in energy).

$$\lim_{\alpha \rightarrow +\infty} \int_{B_{\delta_\alpha}(x_\alpha)} \{|\nabla_g(u_\alpha - \xi_{x_\alpha, \mu_\alpha^{-1}}^g)|^2 + |u_\alpha - \xi_{x_\alpha, \mu_\alpha^{-1}}^g|^{2^*}\} dv_g = 0. \quad (2.6)$$

Proof. We consider the following rescaling of u_α on the geodesic ball $B_{\delta_\alpha}(x_\alpha)$:

$$v_\alpha(y) = \mu_\alpha^{(n-2)/2} u_\alpha(\exp_{x_\alpha}(\mu_\alpha y)), \quad y \in \Omega_\alpha, \quad (2.7)$$

where

$$\Omega_\alpha = \mu_\alpha^{-1} \exp_{x_\alpha}^{-1}(B_{\delta_\alpha}(x_\alpha)) = \mu_\alpha^{-1} B_{\delta_\alpha}(0). \quad (2.8)$$

v_α satisfies

$$-\Delta_{g_\alpha} v_\alpha + c(n)R_{g_\alpha} v_\alpha + \varepsilon_\alpha v_\alpha^{r-1} = \ell_\alpha v_\alpha^{2^*-1} \quad \text{in } \Omega_\alpha, \quad (2.9)$$

where

$$g_\alpha(y) = g(\exp_{x_\alpha}(\mu_\alpha y)), \quad |R_{g_\alpha}| \leq C\mu_\alpha^2,$$

and

$$\varepsilon_\alpha := \alpha \mu_\alpha^{n - \frac{n-2}{2}r} \|u_\alpha\|_{L^r(M)}^{2-r}.$$

We observe that the rescaled metric g_α converges to the Euclidean metric (δ_{ij}) on \mathbb{R}^n uniformly on compact subsets, and it is equivalent to (δ_{ij}) , uniformly in α , i.e., there exists $C > 0$ independent of α such that $C^{-1}\delta_{ij} \leq g_{\alpha,ij}(y) \leq C\delta_{ij}$. We claim that

$$\varepsilon_\alpha \leq \alpha \|u_\alpha\|_{L^r(M)}^2 \rightarrow 0 \quad \text{as } \alpha \rightarrow +\infty. \quad (2.10)$$

Indeed, by the definition of μ_α and ε_α ,

$$\varepsilon_\alpha = \frac{\alpha \|u_\alpha\|_{L^r(M)}^2}{(\max_M u_\alpha)^{2^*-r} \int_M u_\alpha^r dv_g}$$

and

$$1 = \int_M u_\alpha^{2^*} dv_g \leq (\max_M u_\alpha)^{2^*-r} \int_M u_\alpha^r dv_g.$$

Property (2.10) now follows by Proposition 2.3–(iii). By a change of variables,

$$\int_{\Omega_\alpha} v_\alpha^{2^*} dv_{g_\alpha} = \int_{B_\alpha} u_\alpha^{2^*} dv_g$$

and

$$\int_{\Omega_\alpha} |\nabla_{g_\alpha} v_\alpha|^2 dv_{g_\alpha} = \int_{B_\alpha} \{|\nabla_g u_\alpha|^2 + c(n)R_g u_\alpha^2\} dv_g.$$

Consequently, by the definition of u_α

$$\limsup_{\alpha \rightarrow +\infty} \int_{\Omega_\alpha} v_\alpha^{2^*} dv_{g_\alpha} \leq 1 \quad (2.11)$$

and by Proposition 2.3–(i)–(ii),

$$\limsup_{\alpha \rightarrow +\infty} \int_{\Omega_\alpha} |\nabla_{g_\alpha} v_\alpha|^2 dv_{g_\alpha} \leq K^{-2}. \quad (2.12)$$

By the definition of μ_α , $v_\alpha(y) \leq v_\alpha(0) = 1$, thus, by standard elliptic estimates, there exists $v \in C_{\text{loc}}^1(\mathbb{R}^n)$ such that, along a subsequence, $v_\alpha \rightarrow v$ in $C_{\text{loc}}^1(\mathbb{R}^n)$, and $v(0) = 1$. Furthermore, v satisfies:

$$\int_{\mathbb{R}^n} |\nabla v|^2 dy = \lim_{R \rightarrow +\infty} \int_{B_R} |\nabla v|^2 dy = \lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{B_R} |\nabla_{g_\alpha} v_\alpha|^2 dv_{g_\alpha} \leq K^{-2}, \quad (2.13)$$

and

$$\int_{\mathbb{R}^n} v^{2^*} dy = \lim_{R \rightarrow +\infty} \int_{B_R} v^{2^*} dy = \lim_{R \rightarrow +\infty} \lim_{\alpha \rightarrow +\infty} \int_{B_R} v_\alpha^{2^*} dv_{g_\alpha} \leq 1. \quad (2.14)$$

In particular, $v \in D^{1,2}(\mathbb{R}^n)$, and taking pointwise limits in (2.9) we find that v satisfies:

$$\begin{aligned} -\Delta v &= K^{-2} v^{2^*-1}, \quad \text{in } \mathbb{R}^n \\ 0 &\leq v \leq 1, \quad v(0) = 1. \end{aligned}$$

Multiplying the above equation by v and integrating by parts, and recalling the definition of K we have:

$$K^{-2} \int_{\mathbb{R}^n} v^{2^*} dy = \int_{\mathbb{R}^n} |\nabla v|^2 dy \geq K^{-2} \left(\int_{\mathbb{R}^n} v^{2^*} dy \right)^{2/2^*}.$$

Therefore, $\left(\int_{\mathbb{R}^n} v^{2^*} dy \right)^{1-2/2^*} \geq 1$, which together with (2.13) and (2.14) implies $\int_{\mathbb{R}^n} v^{2^*} dy = 1$ and $\int_{\mathbb{R}^n} |\nabla v|^2 dy = K^{-2}$ and thus necessarily $v = U$. Since the limit v is independent of subsequences, the convergence is for all $\alpha \rightarrow +\infty$ with $x_\alpha \rightarrow P$. At this point, it is intuitively clear that Proposition 2.3–(ii) should imply the “strong convergence” (2.6); however we face some minor technicality due to the

fact that v_α does not necessarily vanish on $\partial\Omega_\alpha$. Using the elementary calculus inequality:

$$||a + b|^p - |a|^p - |b|^p| \leq C(p)(|a|^{p-1}|b| + |a||b|^{p-1}), \quad \forall a, b \in \mathbb{R}^n, p \geq 1$$

with $p = 2^*$, $a = U$, and $b = v_\alpha - U$, we have:

$$\begin{aligned} \int_{\Omega_\alpha} |v_\alpha - U|^{2^*} dv_{g_\alpha} &\leq \int_{\Omega_\alpha} v_\alpha^{2^*} dv_{g_\alpha} - \int_{\Omega_\alpha} U^{2^*} dv_{g_\alpha} \\ &\quad + C \left(\int_{\Omega_\alpha} U^{2^*-1} |v_\alpha - U| dv_{g_\alpha} + \int_{\Omega_\alpha} U |v_\alpha - U|^{2^*-1} dv_{g_\alpha} \right) \\ &\leq o(1) + C \left(\int_{\Omega_\alpha} U^{2^*-1} |v_\alpha - U| dv_{g_\alpha} + \int_{\Omega_\alpha} U |v_\alpha - U|^{2^*-1} dv_{g_\alpha} \right). \end{aligned}$$

The right hand side is easily seen to vanish as $\alpha \rightarrow +\infty$:

$$\begin{aligned} &\int_{\Omega_\alpha} U |v_\alpha - U|^{2^*-1} dv_{g_\alpha} \\ &= \int_{B_R} U |v_\alpha - U|^{2^*-1} dv_{g_\alpha} + \int_{\Omega_\alpha \setminus B_R} U |v_\alpha - U|^{2^*-1} dv_{g_\alpha} \\ &\leq \int_{B_R} U |v_\alpha - U|^{2^*-1} dv_{g_\alpha} \\ &\quad + \left(\int_{\Omega_\alpha \setminus B_R} U^{2^*} dv_{g_\alpha} \right)^{1/2^*} \left(\int_{\Omega_\alpha \setminus B_R} |v_\alpha - U|^{2^*} dv_{g_\alpha} \right)^{1/2^{*'}} \\ &\leq \int_{B_R} U |v_\alpha - U|^{2^*-1} dv_{g_\alpha} + C \int_{\mathbb{R}^n \setminus B_R} U^{2^*} dy. \end{aligned}$$

By taking R large, the second integral can be made arbitrarily small; then, by C_{loc}^1 -convergence, the first integral is small for large α . Hence,

$$\lim_{\alpha \rightarrow +\infty} \int_{\Omega_\alpha} U |v_\alpha - U|^{2^*-1} dv_{g_\alpha} = 0.$$

Similarly, one easily checks that

$$\lim_{\alpha \rightarrow +\infty} \int_{\Omega_\alpha} U^{2^*-1} |v_\alpha - U| dv_{g_\alpha} = 0.$$

The strong convergence of the gradients is straightforward:

$$\begin{aligned} \left| \int_{\Omega_\alpha} \nabla_{g_\alpha}(v_\alpha - U) \cdot \nabla_{g_\alpha} U dv_{g_\alpha} \right| &\leq \int_{B_R} |\nabla_{g_\alpha}(v_\alpha - U)| |\nabla_{g_\alpha} U| dv_{g_\alpha} \\ &\quad + \left(\int_{\Omega_\alpha \setminus B_R} |\nabla_{g_\alpha}(v_\alpha - U)|^2 dv_{g_\alpha} \right)^{1/2} \left(\int_{\Omega_\alpha \setminus B_R} |\nabla_{g_\alpha} U|^2 dv_{g_\alpha} \right)^{1/2} \\ &\leq \int_{B_R} |\nabla_{g_\alpha}(v_\alpha - U)| |\nabla_{g_\alpha} U| dv_{g_\alpha} + C \left(\int_{\Omega_\alpha \setminus B_R} |\nabla U|^2 dy \right)^{1/2} \end{aligned}$$

and therefore

$$\lim_{\alpha \rightarrow +\infty} \int_{\Omega_\alpha} \nabla_{g_\alpha}(v_\alpha - U) \cdot \nabla_{g_\alpha} U dv_{g_\alpha} = 0.$$

Consequently, by (2.12) and since $\int_{\Omega_\alpha} |\nabla_{g_\alpha} U|^2 dv_{g_\alpha} \rightarrow K^{-2}$, we conclude:

$$\begin{aligned} \int_{\Omega_\alpha} |\nabla_{g_\alpha} (v_\alpha - U)|^2 dv_{g_\alpha} &= \int_{\Omega_\alpha} |\nabla_{g_\alpha} v_\alpha|^2 dv_{g_\alpha} - \int_{\Omega_\alpha} |\nabla_{g_\alpha} U|^2 dv_{g_\alpha} \\ &\quad - 2 \int_{\Omega_\alpha} \nabla_{g_\alpha} (v_\alpha - U) \cdot \nabla_{g_\alpha} U dv_{g_\alpha} \leq o(1), \end{aligned}$$

and (2.6) follows after a change of variables. \square

COROLLARY 2.5 (One point concentration for u_α). *For any $\varepsilon > 0$ there exist $\delta_\varepsilon > 0$ and $\alpha_\varepsilon > 0$ such that*

$$\int_{M \setminus B_{\mu_\alpha/\delta_\varepsilon}(x_\alpha)} \{|\nabla_g u_\alpha|^2 + u_\alpha^{2^*}\} dv_g \leq \varepsilon$$

for all $\alpha \geq \alpha_\varepsilon$. In particular, for any fixed $\rho > 0$,

$$\lim_{\alpha \rightarrow +\infty} \int_{M \setminus B_\rho(x_\alpha)} \{|\nabla_g u_\alpha|^2 + u_\alpha^{2^*}\} dv_g = 0.$$

Proof. For any $\varepsilon > 0$, by (2.6) and a change of variable, there exists $\delta_\varepsilon > 0$ and α'_ε such that for all $\alpha \geq \alpha'_\varepsilon$,

$$\int_{B_{\mu_\alpha/\delta_\varepsilon}(x_\alpha)} |\nabla_g u_\alpha|^2 dv_g \geq \int_{\mathbb{R}^n} |\nabla U|^2 - \frac{\varepsilon}{4} = K^{-2} - \frac{\varepsilon}{4},$$

and

$$\int_{B_{\mu_\alpha/\delta_\varepsilon}(x_\alpha)} u_\alpha^{2^*} dv_g \geq \int_{\mathbb{R}^n} U^{2^*} - \frac{\varepsilon}{4} = 1 - \frac{\varepsilon}{4}.$$

Recall that $\int_M |\nabla_g u_\alpha|^2 dv_g \rightarrow K^{-2}$ and $\int_M u_\alpha^{2^*} dv_g = 1$, we can take some $\alpha_\varepsilon \geq \alpha'_\varepsilon$ such that for all $\alpha \geq \alpha_\varepsilon$,

$$\int_{M \setminus B_{\mu_\alpha/\delta_\varepsilon}(x_\alpha)} \{|\nabla_g u_\alpha|^2 + u_\alpha^{2^*}\} dv_g \leq \varepsilon.$$

\square

COROLLARY 2.6. *For any fixed $\rho > 0$,*

$$\lim_{\alpha \rightarrow +\infty} \|u_\alpha\|_{L^\infty(M \setminus B_\rho(x_\alpha))} = 0.$$

Proof. Equation (2.2) implies the differential inequality

$$-\Delta_g u_\alpha + [c(n)R_g - \ell_\alpha u_\alpha^{2^*-2}] u_\alpha \leq 0 \quad \text{on } M.$$

By Corollary 2.5 we have, for all $x \in M \setminus B_\rho(x_\alpha)$, that

$\|u_\alpha^{2^*-2}\|_{L^{n/2}(B_{\rho/2}(x))} = \|u_\alpha\|_{L^{2^*}^{2^*-2}(B_{\rho/2}(x))} \leq \|u_\alpha\|_{L^{2^*}(M \setminus B_{\rho/2}(x_\alpha))} = o(1)$. By Moser iterations, we derive

$$\|u_\alpha\|_{L^\infty(B_{\rho/4}(x))} \leq C \|u_\alpha\|_{L^1(B_{\rho/2}(x))} \leq C \|u_\alpha\|_{L^1(M)},$$

and the claim follows by Proposition 2.3-(i). \square

3. Uniform Estimate. The $C_{\text{loc}}^2(\mathbb{R}^n)$ -convergence of the rescaled minimizer v_α to U readily provides a complete description of u_α in a ball of *shrinking* radius $B_{\rho\mu_\alpha}(x_\alpha)$, for any $\rho > 0$. In particular, it implies the estimate:

$$u_\alpha(x) = \mu_\alpha^{-(n-2)/2} v_\alpha(\mu_\alpha^{-1}(\exp_{x_\alpha} g)^{-1}(x)) \leq C(\rho)\mu_\alpha^{-(n-2)/2}, \quad \forall x \in B_{\rho\mu_\alpha}(x_\alpha),$$

and consequently

$$u_\alpha(x) \leq C(\rho)\mu_\alpha^{(n-2)/2} \text{dist}_g(x, x_\alpha)^{2-n}, \quad \forall x \in B_{\rho\mu_\alpha}(x_\alpha). \quad (3.1)$$

Our aim in this section is to show that (3.1) holds *uniformly on M* . This type of estimate for minimizers has been obtained by Brezis and Peletier [13] and by Atkinson and Peletier [2] in the radially symmetric case on Euclidean balls, by Rey [37] and Han [26] on general domains in \mathbb{R}^n , and by Hebey and Vaugon [27], Li and Zhu [30] and Aubin and Li [6] on Riemannian manifolds. Our approach, similar in spirit to [30], requires new ingredients. Throughout this section, we assume $n \geq 3$.

PROPOSITION 3.1. *For every α sufficiently large, u_α satisfies*

$$u_\alpha(x) \leq C\mu_\alpha^{(n-2)/2} \text{dist}_g(x, x_\alpha)^{2-n}, \quad \forall x \in M. \quad (3.2)$$

Here $C > 0$ is a constant depending on (M, g) only. Consequently, we have the following uniform estimate for v_α :

$$v_\alpha(y) \leq \frac{C}{1 + |y|^{n-2}}, \quad \forall y \in \Omega_\alpha. \quad (3.3)$$

We shall prove Proposition 3.1 by showing that

$$u_\alpha(x) \leq C\varphi_\alpha(x), \quad \forall x \in M \quad (3.4)$$

for some $\varphi_\alpha > 0$ satisfying:

$$C^{-1}\mu_\alpha^{(n-2)/2} \text{dist}_g(x, x_\alpha)^{2-n} \leq \varphi_\alpha(x) \leq C\mu_\alpha^{(n-2)/2} \text{dist}_g(x, x_\alpha)^{2-n}, \quad \forall x \in M \quad (3.5)$$

for every α sufficiently large. In fact, our main effort will be to construct a suitable such φ_α . We set

$$\zeta_\alpha = \frac{u_\alpha}{\varphi_\alpha}.$$

We have to show $\zeta_\alpha \leq C$ pointwise on M . By the conformal invariance, ζ_α satisfies

$$\begin{aligned} -\Delta_{\widehat{g}_\alpha} \zeta_\alpha &= \ell_\alpha \zeta_\alpha^{2^*-1} - \varphi_\alpha^{1-2^*} (-\zeta_\alpha \Delta_g \varphi_\alpha + c(n)R_g u_\alpha) \\ &\quad + \alpha \|u_\alpha\|_{L^r(M)}^{2-r} u_\alpha^{r-1} \quad \text{in } M \setminus \{x_\alpha\}, \end{aligned} \quad (3.6)$$

where \widehat{g}_α is the metric conformal to g defined in terms of φ_α by $\widehat{g}_\alpha = \varphi_\alpha^{4/(n-2)} g$.

Indeed, we have

$$-\Delta_{\widehat{g}_\alpha} \frac{u}{\varphi_\alpha} + c(n)R_{\widehat{g}_\alpha} \frac{u}{\varphi_\alpha} = \varphi_\alpha^{1-2^*} (-\Delta_g u + c(n)R_g u), \quad \forall u \in C^2(M \setminus \{x_\alpha\}).$$

Taking $u = \varphi_\alpha$, we obtain

$$c(n)R_{\widehat{g}_\alpha} = \varphi_\alpha^{1-2^*} (-\Delta_g \varphi_\alpha + c(n)R_g \varphi_\alpha).$$

Taking $u = u_\alpha$, we find

$$-\Delta_{\widehat{g}_\alpha} \zeta_\alpha = -\varphi_\alpha^{1-2^*} \Delta_g u_\alpha - c(n)(R_{\widehat{g}_\alpha} - \varphi_\alpha^{-4/(n-2)} R_g) \zeta_\alpha.$$

It follows that

$$\begin{aligned} -\Delta_{\widehat{g}_\alpha} \zeta_\alpha &= \varphi_\alpha^{-(n+2)/(n-2)} (-c(n)R_g u_\alpha - \alpha \|u_\alpha\|_{L^r(M)}^{2-r} u_\alpha^{r-1} + \ell_\alpha u_\alpha^{2^*-1}) \\ &\quad + \varphi_\alpha^{1-2^*} \zeta_\alpha \Delta_g \varphi_\alpha, \end{aligned}$$

which implies (3.6).

By the uniform estimate (3.5), the metrics \widehat{g}_α satisfy a Sobolev inequality with a constant *independent* of α :

LEMMA 3.2. *There exists a constant $C > 0$ independent of α such that for all $u \in H^1(M)$, $u \equiv 0$ in a neighborhood of x_α :*

$$\left(\int_M |u|^{2^*} dv_{\widehat{g}_\alpha} \right)^{2/2^*} \leq C \int_M |\nabla_{\widehat{g}_\alpha} u|^2 dv_{\widehat{g}_\alpha}. \quad (3.7)$$

Proof. It is well-known (see, e.g., Appendix A in [30]) that there exists a constant $C = C(M, g)$ such that for all $x_0 \in M$, $u \in H^1(M)$, $u \equiv 0$ in a neighborhood of x_0 , there holds:

$$\left(\int_M \frac{|u|^{2^*}}{\text{dist}_g(x, x_0)^{2n}} dv_g \right)^{2/2^*} \leq C \int_M \frac{|\nabla_g u|^2}{\text{dist}_g(x, x_0)^{2n-4}} dv_g. \quad (3.8)$$

Now it suffices to observe that by conformality of \widehat{g}_α we have:

$$dv_{\widehat{g}_\alpha} = \varphi_\alpha^{2^*} dv_g \quad \text{and} \quad |\nabla_{\widehat{g}_\alpha} u|^2 = \varphi_\alpha^{-4/(n-2)} |\nabla_g u|^2,$$

and to recall (3.5). \square

At this point it is clear from (3.6) that if we can find a function $\varphi_\alpha > 0$ satisfying (3.5) and such that:

$$-\frac{u_\alpha}{\varphi_\alpha} \Delta_g \varphi_\alpha + c(n)R_g u_\alpha + \alpha \|u_\alpha\|_{L^r(M)}^{2-r} u_\alpha^{r-1} \geq 0 \quad \text{in } M \setminus \{x_\alpha\}, \quad (3.9)$$

then the corresponding ζ_α will satisfy:

$$-\Delta_{\widehat{g}_\alpha} \zeta_\alpha \leq \ell_\alpha \zeta_\alpha^{2^*-1} \quad \text{in } M \setminus \{x_\alpha\} \quad (3.10)$$

$$\int_{M \setminus B_{\mu_\alpha/\delta_1}(x_\alpha)} \zeta_\alpha^{2^*} dv_{\widehat{g}_\alpha} = \int_{M \setminus B_{\mu_\alpha/\delta_1}(x_\alpha)} u_\alpha^{2^*} dv_g \leq \varepsilon,$$

(recall Corollary 2.5 in Section 2). For any $\rho > 0$, let

$$R_i := \frac{[2 - 2^{-(i-1)}]\mu_\alpha}{\rho}, \quad i = 1, 2, 3, \dots$$

By (3.5) we may choose cutoff functions η_i (depending on α) satisfying:

$$\begin{aligned} \eta_i &\equiv 1 \quad \text{in } M \setminus B_{R_{i+1}} \\ \eta_i &\equiv 0 \quad \text{in } M \setminus B_{R_i} \\ |\nabla_{\widehat{g}_\alpha} \eta_i| &\leq C(\rho)2^i, \quad |\nabla_{\widehat{g}_\alpha}^2 \eta_i| \leq C(\rho)4^i. \end{aligned}$$

Then we shall have all necessary ingredients to apply the Moser iteration technique to (3.10) and to derive:

LEMMA 3.3. *The following pointwise upper bound holds:*

$$\zeta_\alpha \leq C \quad \text{in } M \setminus B_{\mu_\alpha/\delta_0}(x_\alpha). \quad (3.11)$$

Proof. By applying Moser iterations to (3.10), see [30] for the detailed proof. \square

Estimates (3.1) and (3.11) will then imply (3.4) and thus Proposition 3.1 will be established.

We note that (3.9) is trivially satisfied if $u_\alpha = 0$. In $(M \setminus \{x_\alpha\}) \cap \{u_\alpha > 0\}$, (3.9) is equivalent to:

$$-\Delta_g \varphi_\alpha + \left[c(n)R_g + \alpha \left(\frac{\|u_\alpha\|_{L^r(M)}}{u_\alpha} \right)^{2-r} \right] \varphi_\alpha \geq 0, \quad (3.12)$$

and the operator on the left hand side above is linear in φ_α . Furthermore, the blowup rate as in (3.5) is satisfied if $\mu_\alpha^{(2-n)/2} \varphi_\alpha$ has the blowup rate of the Green's function with pole at x_α . In fact, we shall obtain a φ_α of the form $\varphi_\alpha = \mu_\alpha^{(n-2)/2} G_\alpha$, with G_α the Green's function for the operator $-\Delta_g + \tilde{V}_\alpha$ with pole at x_α , and where \tilde{V}_α is a truncation of the "potential" $c(n)R_g + \alpha(\|u_\alpha\|_{L^r(M)}/u_\alpha)^{2-r}$ appearing in (3.12). The detailed proof follows.

We define a function \tilde{V}_α in the following way:

$$\tilde{V}_\alpha := \begin{cases} \min \left\{ c(n)R_g + \alpha \left(\frac{\|u_\alpha\|_{L^r(M)}}{u_\alpha} \right)^{2-r}, 1 \right\}, & \text{if } u_\alpha \neq 0, \\ 1, & \text{if } u_\alpha = 0. \end{cases}$$

Note that \tilde{V}_α is Lipschitz on M (with Lipschitz constant depending on α) and it is uniformly bounded:

$$-c(n)\|R_g\|_\infty \leq \tilde{V}_\alpha \leq 1. \quad (3.13)$$

We shall prove (3.4) with $\varphi_\alpha = \mu_\alpha^{(n-2)/2} G_\alpha$ and G_α defined in the following

PROPOSITION 3.4. *The operators $-\Delta_g + \tilde{V}_\alpha$ are coercive on $H^1(M)$ for sufficiently large α , with coercivity constant uniform in α . Consequently, for every α sufficiently large there exists a unique (distributional) solution G_α to the equation:*

$$-\Delta_g G_\alpha + \tilde{V}_\alpha G_\alpha = \delta_{x_\alpha}, \quad \text{on } M. \quad (3.14)$$

Furthermore, the first nonzero eigenvalue of $-\Delta_g + \tilde{V}_\alpha$ is bounded away from zero and therefore G_α satisfies, for some constant $C > 0$ independent of α ,

- (i) $G_\alpha \in C_{loc}^2(M \setminus \{x_\alpha\})$;
- (ii) $C^{-1} \text{dist}_g(x, x_\alpha)^{2-n} \leq G_\alpha(x) \leq C \text{dist}_g(x, x_\alpha)^{2-n} \quad \forall x \in M$;
- (iii) $u_\alpha \Delta_g G_\alpha \leq [c(n)R_g u_\alpha + \alpha \|u_\alpha\|_r^{2-r} u_\alpha^{r-1}] G_\alpha$ in $M \setminus \{x_\alpha\}$.

In order to prove Proposition 3.4 we need the following

LEMMA 3.5. *The functions \tilde{V}_α satisfy:*

$$\lim_{\alpha \rightarrow +\infty} \text{vol}_g \left\{ \tilde{V}_\alpha < \frac{1}{2} \right\} = 0.$$

Proof. Note that for every measurable set E such that $\bar{E} \subset M \cap \{u_\alpha > 0\}$ we have the lower bound:

$$\|u_\alpha\|_{L^r(E)} \|u_\alpha^{-1}\|_{L^r(E)} \geq (\text{vol}_g E)^{2/r}.$$

Indeed, using the Hölder inequality we find:

$$\text{vol}_g E = \int_E dv_g = \int_E u_\alpha^{r/2} u_\alpha^{-r/2} dv_g \leq \|u_\alpha\|_{L^r(E)}^{r/2} \|u_\alpha^{-1}\|_{L^r(E)}^{r/2}.$$

It follows that

$$\begin{aligned} \|(\|u_\alpha\|_{L^r(M)}u_\alpha^{-1})^{2-r}\|_{L^{r/(2-r)}(E)} &= \|u_\alpha\|_{L^r(M)}^{2-r}\|u_\alpha^{-(2-r)}\|_{L^{r/(2-r)}(E)} \\ &\geq \|u_\alpha\|_{L^r(E)}^{2-r}\|u_\alpha^{-1}\|_{L^r(E)}^{2-r} \geq |E|^{(2-r)2/r}. \end{aligned} \quad (3.15)$$

Let $E_\alpha := \{\tilde{V}_\alpha < 1/2\}$. Then $\bar{E}_\alpha \subset M \cap \{u_\alpha > 0\}$ and therefore, by (3.15),

$$(\text{vol}_g E_\alpha)^{(2-r)2/r} \leq \|(\|u_\alpha\|_{L^r(M)}u_\alpha^{-1})^{2-r}\|_{L^{r/(2-r)}(E_\alpha)}.$$

On the other hand, since

$$\alpha(\|u_\alpha\|_{L^r(M)}u_\alpha^{-1})^{2-r} < \frac{1}{2} + c(n)|R_g|, \quad \text{on } E_\alpha,$$

we have

$$\alpha\|(\|u_\alpha\|_{L^r(M)}u_\alpha^{-1})^{2-r}\|_{L^{r/(2-r)}(E_\alpha)} \leq \left(\frac{1}{2} + c(n)\|R_g\|_{L^\infty(M)}\right)(\text{vol}_g M)^{(2-r)/r},$$

and consequently,

$$\alpha(\text{vol}_g E_\alpha)^{(2-r)/r} \leq C,$$

for some $C > 0$ independent of α . Now Lemma 3.5 follows immediately. \square

Proof of Proposition 3.4. Proof of the coercivity. For $\tilde{\gamma} = 1/2$ and $u \in H^1(M)$, by the Sobolev inequality and a straightforward computation we have:

$$\begin{aligned} \int_M \{|\nabla_g u|^2 + \tilde{V}_\alpha u^2\} dv_g &= \int_M \{|\nabla_g u|^2 + \tilde{\gamma} u^2 + (\tilde{V}_\alpha - \tilde{\gamma})u^2\} dv_g \\ &\geq \int_M \{|\nabla_g u|^2 + \tilde{\gamma} u^2 - (\tilde{V}_\alpha - \tilde{\gamma})_+ u^2\} dv_g \\ &\geq \int_M \{|\nabla_g u|^2 + \tilde{\gamma} u^2\} dv_g - \|(\tilde{V}_\alpha - \tilde{\gamma})_-\|_{L^{n/2}(M)} \|u\|_{L^{2^*}(M)}^2 \\ &\geq \int_M \{|\nabla_g u|^2 + \tilde{\gamma} u^2\} dv_g - C \text{vol}_g \{\tilde{V}_\alpha < 1/2\} \int_M \{|\nabla_g u|^2 + u^2\} dv_g, \end{aligned}$$

where $(\tilde{V}_\alpha - \tilde{\gamma})_+ \geq 0$ denotes the negative part of $\tilde{V}_\alpha - \tilde{\gamma}$. The coercivity and its uniformity in α follow from the above and Lemma 3.5.

Proof of (i) and (ii). Because of the coercivity of $-\Delta_g + \tilde{V}_\alpha$, the Lipschitz regularity and the uniform L^∞ bound of \tilde{V}_α , it follows from standard elliptic theories (see e.g., [24], [41] and [21]) that G_α is uniquely defined by (3.14) and it satisfies (i) and (ii).

Proof of (iii). Since $G_\alpha \in C_{\text{loc}}^2(M \setminus \{x_\alpha\})$ we only need to check the inequality pointwise. If $u_\alpha = 0$ it is trivial. So assume $u_\alpha > 0$. By (3.14) we have

$$-\Delta_g G_\alpha + \tilde{V}_\alpha G_\alpha = 0 \quad \text{pointwise in } M \setminus \{x_\alpha\}.$$

Since $G_\alpha > 0$, using the definition of \tilde{V}_α , we have

$$\Delta_g G_\alpha = \tilde{V}_\alpha G_\alpha \leq [c(n)R_g + \alpha\left(\frac{\|u_\alpha\|_{L^r(M)}}{u_\alpha}\right)^{2-r}]G_\alpha,$$

pointwise in $(M \setminus \{x_\alpha\}) \cap \{u_\alpha > 0\}$. Multiplying the inequality above by u_α , we again obtain (iii). Proposition 3.4 is established. \square

Proof of Proposition 3.1. The estimate for u_α follows by (3.1) and Lemma 3.3. Since v_α is uniformly bounded in $|y| < 1$, (3.3) follows from the estimate of u_α by a change of variables. \square

4. Energy Estimate. We shall need estimates for the convergence rates of the limits “in energy” obtained in Section 2. The pointwise estimates obtained in Section 2 allow us to adapt the energy estimates of Bahri-Coron [7].

In order to simplify calculations, we introduce a conformal metric $\widehat{g} = \psi^{4/(n-2)}g$, with $\psi \in C^\infty(M)$, $\psi(x_\alpha) = 1$, $\frac{1}{2} \leq \psi \leq 2$, $\|\psi\|_{C^2} \leq C$, such that $R_{\widehat{g}} \equiv 0$ in $B_{\delta_0}(P)$, and where δ_0 is a suitably chosen small constant and both δ_0 and C depend only on (M, g) . Such a metric may be obtained by locally solving

$$-\Delta_g \psi + c(n)R_g \psi = 0 \quad \text{in } B_{\delta_0}$$

and then extending ψ smoothly to M . We denote, for $\delta_0/2 \leq \delta_\alpha \leq \delta_0$,

$$B_\alpha = B_{\delta_\alpha}^{\widehat{g}}(x_\alpha).$$

For $\tilde{x} \in B_\alpha$ and $\lambda > 0$, we consider

$$\xi_{\tilde{x}, \lambda}^{\widehat{g}}(x) = \left(\frac{\lambda}{1 + (\lambda\bar{\lambda})^2 \text{dist}_{\widehat{g}}^2(x, \tilde{x})^2} \right)^{\frac{n-2}{2}}, \quad \forall x \in B_\alpha.$$

It follows from Proposition 2.4 that

$$\lim_{\alpha \rightarrow +\infty} \int_{B_\alpha} \{ |\nabla_{\widehat{g}}(\frac{u_\alpha}{\psi} - \xi_{x_\alpha, \mu_\alpha^{-1}}^{\widehat{g}})|^2 + |\frac{u_\alpha}{\psi} - \xi_{x_\alpha, \mu_\alpha^{-1}}^{\widehat{g}}|^2 \} dv_{\widehat{g}} = 0. \quad (4.1)$$

We follow the idea in [7] of selecting for every α an optimal multiple of a \widehat{g} -bubble, denoted $t_\alpha \xi_\alpha = t_\alpha \xi_{\tilde{x}_\alpha, \lambda_\alpha}^{\widehat{g}}$, and of estimating the difference $u_\alpha/\psi - t_\alpha \xi_\alpha$ by exploiting orthogonality. For future convenience, we prove our estimates for $n \geq 3$. For $\tilde{x} \in B_{\mu_\alpha \delta_\alpha / 2}^{\widehat{g}}(x_\alpha)$ and $\lambda > 0$, let $h_{\tilde{x}, \lambda}$ be defined by:

$$\begin{cases} \Delta_{\widehat{g}} h_{\tilde{x}, \lambda} = 0 & \text{in } B_\alpha, \\ h_{\tilde{x}, \lambda} = \xi_{\tilde{x}, \lambda}^{\widehat{g}} & \text{on } \partial B_\alpha, \end{cases} \quad (4.2)$$

and let χ_α be defined by

$$\begin{cases} \Delta_{\widehat{g}} \chi_\alpha = 0 & \text{in } B_\alpha, \\ \chi_\alpha = \frac{u_\alpha}{\psi} & \text{on } \partial B_\alpha. \end{cases} \quad (4.3)$$

Then $u_\alpha/\psi - \chi_\alpha \in H_0^1(B_\alpha)$, $\xi_{\tilde{x}, \lambda}^{\widehat{g}} - h_{\tilde{x}, \lambda} \in H_0^1(B_\alpha)$ are the projections of u_α and $\xi_{\tilde{x}, \lambda}^{\widehat{g}}$, respectively, on $H_0^1(B_\alpha)$. We set

$$\sigma_{\tilde{x}, \lambda} = \xi_{\tilde{x}, \lambda}^{\widehat{g}} - h_{\tilde{x}, \lambda}.$$

Then $\sigma_{\tilde{x}, \lambda} \leq \xi_{\tilde{x}, \lambda}^{\widehat{g}}$ satisfies:

$$\begin{cases} \Delta_{\widehat{g}} \sigma_{\tilde{x}, \lambda} = \Delta_{\widehat{g}} \xi_{\tilde{x}, \lambda}^{\widehat{g}} & \text{in } B_\alpha, \\ \sigma_{\tilde{x}, \lambda} = 0 & \text{on } \partial B_\alpha. \end{cases}$$

Let $(t_\alpha, \tilde{x}_\alpha, \lambda_\alpha) \in [\frac{1}{2}, \frac{3}{2}] \times \overline{B_{\mu_\alpha \delta_\alpha / 2}^{\widehat{g}}(x_\alpha)} \times [\frac{1}{2\mu_\alpha}, \frac{3}{2\mu_\alpha}]$ be such that

$$\begin{aligned} & \left\| \frac{u_\alpha}{\psi} - \chi_\alpha - t_\alpha \sigma_{\tilde{x}_\alpha, \lambda_\alpha} \right\|_{\widehat{g}} \\ &= \min \left\{ \left\| \frac{u_\alpha}{\psi} - \chi_\alpha - t \sigma_{\tilde{x}, \lambda} \right\|_{\widehat{g}} : \begin{array}{l} |t-1| \leq 1/2, \tilde{x} \in \overline{B_{\mu_\alpha \delta_\alpha / 2}^{\widehat{g}}(x_\alpha)} \\ |\mu_\alpha \lambda - 1| \leq 1/2 \end{array} \right\}. \end{aligned}$$

To simplify notation, henceforth we denote:

$$\sigma_\alpha = \sigma_{\tilde{x}_\alpha, \lambda_\alpha}, \quad \xi_\alpha = \xi_{\tilde{x}_\alpha, \lambda_\alpha}^{\hat{g}}, \quad h_\alpha = h_{\tilde{x}_\alpha, \lambda_\alpha},$$

and we set:

$$w_\alpha = \frac{u_\alpha}{\psi} - \chi_\alpha - t_\alpha \sigma_\alpha.$$

The main result in this section is the following estimate for w_α :

PROPOSITION 4.1 (Energy estimate). *For $n \geq 3$, we have:*

$$\begin{aligned} & \|w_\alpha\| + |t_\alpha^{2^*-2} \ell_\alpha - K^{-2}| \\ & \leq C(\mu_\alpha^2 \|U\|_{2^{*'}, \mu_\alpha^{-1}} + \varepsilon_\alpha \|U^{r-1}\|_{2^{*'}, \mu_\alpha^{-1}} + \mu_\alpha^{n-2} \|U^{2^*-2}\|_{2^{*'}, \mu_\alpha^{-1}}). \end{aligned}$$

Recall from Section 2 that $\varepsilon_\alpha = \mu_\alpha^{n-\frac{n-2}{2}r} \alpha \|u_\alpha\|_{L^r(M)}^{2-r} = o_\alpha(1)$. We define

$$W_\alpha = \left\{ w \in H_0^1(B_\alpha) : \begin{array}{l} \langle \sigma_\alpha, w \rangle_{\hat{g}} = 0 \\ \langle f, w \rangle_{\hat{g}} = 0 \quad \forall f \in E \end{array} \right\},$$

where $E \subset H_0^1(B_\alpha)$ is the tangent space at $\sigma_{\tilde{x}_\alpha, \lambda_\alpha}$ of the finite dimensional surface $\{\sigma_{\tilde{x}, \lambda} : \tilde{x} \in B_{\mu_\alpha \delta_\alpha}(x_\alpha), \lambda > 0\} \subset H_0^1(B_\alpha)$, with respect to the metric induced by the inner product $\langle u, v \rangle_{\hat{g}} = \int_{B_\alpha} \nabla_{\hat{g}} u \cdot \nabla_{\hat{g}} v \, dv_{\hat{g}}$. We work with coordinates given by the exponential map $\exp_{\tilde{x}_\alpha}(y)$, $y = (y^i)$, $i = 1, \dots, n$, we can write

$$E = \text{span} \left\{ \frac{\partial \sigma_\alpha}{\partial y^i}, i = 1, \dots, n, \frac{\partial \sigma_\alpha}{\partial \lambda} \right\},$$

where

$$\frac{\partial \sigma_\alpha}{\partial y^i} = \frac{\partial \sigma_{\exp_{\tilde{x}_\alpha}(y), \lambda_\alpha}}{\partial y^i} \Big|_{y=0}, \quad \frac{\partial \sigma_\alpha}{\partial \lambda} = \frac{\partial \sigma_{\tilde{x}_\alpha, \lambda}}{\partial \lambda} \Big|_{\lambda=\lambda_\alpha}.$$

LEMMA 4.2. *For some constant C independent of α ,*

$$\int_{B_\alpha} |\nabla_{\hat{g}} h_\alpha|^2 \, dv_{\hat{g}} \leq C \mu_\alpha^{n-2}.$$

Proof. By standard elliptic estimates and properties of ξ_α , we have that

$$\int_{B_\alpha} |\nabla_{\hat{g}} h_\alpha|^2 \, dv_{\hat{g}} \leq C \left(\int_{\partial B_\alpha} |\nabla_{\hat{g}} \xi_\alpha|^2 \, ds_{\hat{g}} + \int_{\partial B_\alpha} \xi_\alpha^2 \, ds_{\hat{g}} \right) \leq C \mu_\alpha^{n-2}.$$

□

We observe that by the uniform estimate (3.2) and by the maximum principle,

$$\|h_\alpha\|_{L^\infty(B_\alpha)} + \|\chi_\alpha\|_{L^\infty(B_\alpha)} \leq C \mu_\alpha^{(n-2)/2}. \quad (4.4)$$

It follows that $|\sigma_\alpha| \leq C \xi_\alpha$ on B_α . Using Proposition 2.4, it is not difficult to see that:

LEMMA 4.3. *As $\alpha \rightarrow +\infty$, we have $\|w_\alpha\| \rightarrow 0$, $t_\alpha \rightarrow 1$, $\mu_\alpha^{-1} \text{dist}_{\hat{g}}(x_\alpha, \tilde{x}_\alpha) \rightarrow 0$, $\mu_\alpha \lambda_\alpha \rightarrow 1$. Furthermore, $w_\alpha \in W_\alpha$.*

Proof. By definition of t_α and σ_α ,

$$\begin{aligned} \|t_\alpha \sigma_\alpha - \sigma_{x_\alpha, \mu_\alpha^{-1}}\| & \leq \left\| \frac{u_\alpha}{\psi} - \chi_\alpha - t_\alpha \sigma_\alpha \right\| + \left\| \frac{u_\alpha}{\psi} - \chi_\alpha - \sigma_{x_\alpha, \mu_\alpha^{-1}} \right\| \\ & \leq 2 \left\| \frac{u_\alpha}{\psi} - \chi_\alpha - \sigma_{x_\alpha, \mu_\alpha^{-1}} \right\| \leq 2 \left\| \frac{u_\alpha}{\psi} - \xi_{x_\alpha, \mu_\alpha^{-1}}^{\hat{g}} \right\|. \end{aligned}$$

In the last step we have used $\Delta_{\hat{g}}(\frac{u_\alpha}{\psi} - \xi_{x_\alpha, \mu_\alpha^{-1}}^{\hat{g}}) = \Delta_{\hat{g}}(\frac{u_\alpha}{\psi} - \chi_\alpha - \sigma_{x_\alpha, \mu_\alpha^{-1}})$ in B_α and $\frac{u_\alpha}{\psi} - \chi_\alpha - \sigma_{x_\alpha, \mu_\alpha^{-1}} = 0$ on ∂B_α . Hence, in view of (4.1), $\|t_\alpha \sigma_\alpha - \sigma_{x_\alpha, \mu_\alpha^{-1}}\| \rightarrow 0$ and $\|w_\alpha\| \rightarrow 0$. By the arguments in Lemma 4.2, we have $\|h_{\tilde{x}, \lambda}\| \leq C\lambda^{-(n-2)/2}$ if $\text{dist}_{\hat{g}}(\tilde{x}, x_\alpha) \leq \delta_\alpha/2$. Consequently, we derive

$$\|t_\alpha \xi_\alpha - \xi_{x_\alpha, \mu_\alpha^{-1}}^{\hat{g}}\| \leq \|t_\alpha \sigma_\alpha - \sigma_{x_\alpha, \mu_\alpha^{-1}}\| + \|t_\alpha h_\alpha\| + \|h_{x_\alpha, \mu_\alpha^{-1}}\| \rightarrow 0,$$

as $\alpha \rightarrow +\infty$. It follows that $t_\alpha \rightarrow 1$, $\mu_\alpha^{-1} \text{dist}_{\hat{g}}(x_\alpha, \tilde{x}_\alpha) \rightarrow 0$, and $\mu_\alpha \lambda_\alpha \rightarrow 1$. Therefore the minimum of the norm is attained in the interior of $[\frac{1}{2}, \frac{3}{2}] \times \overline{B_{\mu_\alpha \delta_\alpha/2}(x_\alpha)} \times [\frac{1}{2\mu_\alpha}, \frac{3}{2\mu_\alpha}]$. Now a straightforward variational argument yields $w_\alpha \in W_\alpha$. \square

In order to estimate w_α , we begin by writing an equation for w_α :

LEMMA 4.4. w_α satisfies:

$$-\Delta_{\hat{g}} w_\alpha - k_\alpha |\Theta_\alpha|^{2^*-3} \Theta_\alpha w_\alpha + b' |\Theta_\alpha|^{2^*-3} w_\alpha^2 + b'' |w_\alpha|^{2^*-1} = f_\alpha \quad \text{in } B_\alpha, \quad (4.5)$$

where

$$k_\alpha = (2^* - 1)\ell_\alpha,$$

$$\Theta_\alpha = t_\alpha \sigma_\alpha + \chi_\alpha,$$

$$f_\alpha = \ell_\alpha (t_\alpha \xi_\alpha)^{2^*-1} + t_\alpha \Delta_{\hat{g}} \xi_\alpha - \alpha \|u_\alpha\|_{L^r(M)}^{2-r} \psi^{1-2^*} u_\alpha^{r-1} + O(\mu_\alpha^{(n-2)/2} \xi_\alpha^{2^*-2}),$$

and where b', b'' are bounded functions with $b' \equiv 0$ if $n \geq 6$.

Proof. From (2.2), using the conformal invariance (1.8) and recalling that $R_{\hat{g}} \equiv 0$ in B_α , we have that u_α/ψ satisfies:

$$-\Delta_{\hat{g}} \frac{u_\alpha}{\psi} + \alpha \|u_\alpha\|_{L^r(M)}^{2-r} \psi^{1-2^*} u_\alpha^{r-1} = \ell_\alpha \left(\frac{u_\alpha}{\psi}\right)^{2^*-1} \quad \text{in } B_\alpha.$$

Consequently, w_α satisfies:

$$-\Delta_{\hat{g}} w_\alpha = \ell_\alpha (\Theta_\alpha + w_\alpha)^{2^*-1} + t_\alpha \Delta_{\hat{g}} \sigma_\alpha - \alpha \|u_\alpha\|_{L^r(M)}^{2-r} \psi^{1-2^*} u_\alpha^{r-1} \quad \text{in } B_\alpha. \quad (4.6)$$

In order to simplify the right hand side in (4.6), we use the elementary expansion:

$$\begin{aligned} (x+y)^{2^*-1} &= |x|^{2^*-2}x + (2^*-1)|x|^{2^*-3}xy \\ &\quad + b'(x,y)|x|^{2^*-3}y^2 + b''(x,y)|y|^{2^*-1}, \end{aligned}$$

for all $x, y \in \mathbb{R}$ such that $x+y \geq 0$, where b', b'' are bounded functions and $b' \equiv 0$ if $n \geq 6$. For $x = \Theta_\alpha$ and $y = w_\alpha$, we obtain:

$$\begin{aligned} (\Theta_\alpha + w_\alpha)^{2^*-1} &= |\Theta_\alpha|^{2^*-2} \Theta_\alpha + (2^*-1)|\Theta_\alpha|^{2^*-3} \Theta_\alpha w_\alpha \\ &\quad + b' |\Theta_\alpha|^{2^*-3} w_\alpha^2 + b'' |w_\alpha|^{2^*-1}. \end{aligned}$$

Note that $\Theta_\alpha = t_\alpha \xi_\alpha - t_\alpha h_\alpha + \chi_\alpha$. By (4.4) and properties of ξ_α , we have $|\chi_\alpha - t_\alpha h_\alpha| \leq C\mu_\alpha^{(n-2)/2} \leq Ct_\alpha \xi_\alpha$, and thus by simple calculus:

$$\begin{aligned} |\Theta_\alpha|^{2^*-2} \Theta_\alpha &= |t_\alpha \xi_\alpha - t_\alpha h_\alpha + \chi_\alpha|^{2^*-2} (t_\alpha \xi_\alpha - t_\alpha h_\alpha + \chi_\alpha) \\ &= (t_\alpha \xi_\alpha)^{2^*-1} + O(\mu_\alpha^{(n-2)/2} \xi_\alpha^{2^*-2}). \end{aligned}$$

Inserting the above expansions into (4.6), we obtain (4.5). \square

The proof of Proposition 4.1 relies on the coercivity property as in Lemma 4.5 below. Recall that $\delta_0 > 0$ was introduced in Section 2 as an upper bound for the radii δ_α of the balls $B_\alpha = B_{\delta_\alpha}(x_\alpha)$. Here is where we fix δ_0 . We denote by Q_α the continuous bilinear form defined for $\varphi, \psi \in H_0^1(B_\alpha)$ by:

$$Q_\alpha(\varphi, \psi) = \int_{B_\alpha} \{\nabla_{\hat{g}}\varphi \cdot \nabla_{\hat{g}}\psi - k_\alpha |\Theta_\alpha|^{2^*-3} \Theta_\alpha \varphi \psi\} dv_{\hat{g}},$$

where k_α and Θ_α are defined in Lemma 4.4.

LEMMA 4.5. *There exist $0 < \delta_0 \ll 1$, $\alpha_0 \gg 1$ and $c_0 > 0$ independent of $\alpha \geq \alpha_0$ such that*

$$Q_\alpha(w, w) \geq c_0 \int_{B_\alpha} |\nabla_{\hat{g}} w|^2 dv_{\hat{g}}, \quad \forall w \in W_\alpha, \quad \forall \alpha \geq \alpha_0.$$

Lemma 4.5 is a consequence of the following general perturbation result:

LEMMA 4.6. *Let $\Omega \subset \mathbb{R}^n$, let h be a metric on Ω , $k > 0$ and $\Theta \in L^{2^*}(\Omega)$. Denote by Q the continuous bilinear form defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ by*

$$Q(\varphi, \psi) = \int_{\Omega} \{\nabla_h \varphi \cdot \nabla_h \psi - k |\Theta|^{2^*-3} \Theta \varphi \psi\} dv_h.$$

There exist $\varepsilon_0 > 0$ and $c_1 > 0$, depending only on n , such that if

$$\|\Theta - U\|_{L^{2^*}(\Omega)} + |k - (2^* - 1)K^{-2}| + \|h - \mathcal{E}\|_{L^\infty(\Omega)} \leq \varepsilon_0,$$

where \mathcal{E} denotes the Euclidean metric, then

$$Q(\varphi, \varphi) \geq \frac{c_1}{2} \int_{\Omega} |\nabla_h \varphi|^2 dv_h, \quad \forall \varphi \in H_0^1(\Omega) : \langle \varphi, e_i \rangle_h \leq \varepsilon_0 \|\varphi\|_h, \quad 0 \leq i \leq n+1,$$

where $e_0 = U$, $e_i = \partial U_{y,1}/\partial y^i|_{y=0}$, $i = 1, \dots, n$, $e_{n+1} = \partial U_{0,\lambda}/\partial \lambda|_{\lambda=1}$.

Proof. We denote by \tilde{Q} the continuous bilinear form on $D^{1,2}(\mathbb{R}^n)$ defined by

$$\tilde{Q}(\tilde{\varphi}, \tilde{\psi}) = \int_{\mathbb{R}^n} \{\nabla \tilde{\varphi} \cdot \nabla \tilde{\psi} - \frac{2^* - 1}{K^{2^*}} U^{2^*-2} \tilde{\varphi} \tilde{\psi}\} dy.$$

It is well-known (and it may be verified by pull-back to the standard n -sphere, in stereographic projection coordinates) that there exists $c_1 > 0$ such that

$$\tilde{Q}(\tilde{\varphi}, \tilde{\varphi}) \geq c_1 \int_{\mathbb{R}^n} |\nabla \tilde{\varphi}|^2 dy, \quad \forall \tilde{\varphi} \in D^{1,2}(\mathbb{R}^n) : \langle \tilde{\varphi}, e_i \rangle_{\mathcal{E}} = 0, \quad 0 \leq i \leq n+1. \quad (4.7)$$

Now the claim follows by elementary considerations. Indeed, there exist unique μ^j , $|\mu^j| = O(\varepsilon_0 \|\varphi\|_h)$, such that $\tilde{\varphi} := \varphi - \mu^j e_j$ satisfies $\langle \tilde{\varphi}, e_i \rangle_{\mathcal{E}} = 0$, $\forall 0 \leq i \leq n+1$. (4.7) holds for $\tilde{\varphi}$, and the claim follows easily. \square

We introduce some notations. We set

$$\tilde{\Omega}_\alpha = \mu_\alpha^{-1}(\exp_{\hat{x}_\alpha}^{\hat{g}})^{-1}(B_\alpha) \subset \mathbb{R}^n.$$

We denote by T_{μ_α} the transformation which maps $f : B_\alpha \rightarrow \mathbb{R}$ into $T_{\mu_\alpha} f : \tilde{\Omega}_\alpha \rightarrow \mathbb{R}$ defined by

$$(T_{\mu_\alpha} f)(y) = \mu_\alpha^{(n-2)/2} f(\exp_{\hat{x}_\alpha}^{\hat{g}}(\mu_\alpha y)), \quad \forall y \in \tilde{\Omega}_\alpha.$$

We denote by g_α the metric on $\tilde{\Omega}_\alpha$ defined by $g_\alpha(y) = \widehat{g}(\exp_{\tilde{x}_\alpha}^{\widehat{g}}(\mu_\alpha y))$. The following transformation properties hold:

$$\int_{B_\alpha} \nabla_{\widehat{g}} \varphi \cdot \nabla_{\widehat{g}} \psi \, dv_{\widehat{g}} = \int_{\tilde{\Omega}_\alpha} \nabla_{g_\alpha} T_{\mu_\alpha} \varphi \cdot \nabla_{g_\alpha} T_{\mu_\alpha} \psi \, dv_{g_\alpha}, \quad \forall \varphi, \psi \in H_0^1(B_\alpha). \quad (4.8)$$

If $p_1 + \dots + p_k = 2^*$, then

$$\int_{B_\alpha} |\varphi_1|^{p_1} \dots |\varphi_k|^{p_k} \, dv_{\widehat{g}} = \int_{\tilde{\Omega}_\alpha} |T_{\mu_\alpha} \varphi_1|^{p_1} \dots |T_{\mu_\alpha} \varphi_k|^{p_k} \, dv_{g_\alpha}, \quad (4.9)$$

$$\forall \varphi_1, \dots, \varphi_k \in H_0^1(B_\alpha).$$

Proof of Lemma 4.5. Observe that by the transformation properties (4.8)–(4.9) we have

$$Q_\alpha(\varphi, \varphi) = \int_{\tilde{\Omega}_\alpha} \{|\nabla_{g_\alpha} T_{\mu_\alpha} \varphi|^2 - k_\alpha |T_{\mu_\alpha} \Theta|^{2^*-3} T_{\mu_\alpha} \Theta (T_{\mu_\alpha} \varphi)^2\} \, dv_{g_\alpha}.$$

By taking δ_0 small, we achieve $|g_\alpha - \mathcal{E}| \leq \varepsilon_0$. By taking $\alpha_0 \gg 1$, we achieve $\|T_{\mu_\alpha} \Theta_\alpha - U\|_{L^{2^*}(\tilde{\Omega}_\alpha)} \leq \varepsilon_0$ and $|k_\alpha - (2^* - 1)K^{-2}| \leq \varepsilon_0$. It remains to check that by taking a possibly smaller δ_0 and a possibly larger α_0 , we have for all $\varphi \in W_\alpha$:

$$|\langle T_{\mu_\alpha} \varphi, e_i \rangle_{\mathcal{E}}| \leq \varepsilon_0 \|T_{\mu_\alpha} \varphi\|_{g_\alpha}.$$

We check the above for $i = 0$. Since $\varphi \in W_\alpha$,

$$\begin{aligned} 0 &= \int_{B_\alpha} \nabla_{\widehat{g}} \varphi \cdot \nabla_{\widehat{g}} \sigma_\alpha \, dv_{\widehat{g}} = \int_{B_\alpha} \nabla_{\widehat{g}} \varphi \cdot \nabla_{\widehat{g}} \xi_\alpha \, dv_{\widehat{g}} \\ &= \int_{\tilde{\Omega}_\alpha} \nabla_{g_\alpha} T_{\mu_\alpha} \varphi \cdot \nabla_{g_\alpha} T_{\mu_\alpha} \xi_\alpha \, dv_{g_\alpha} = \int_{\tilde{\Omega}_\alpha} g_\alpha^{ij} \frac{\partial T_{\mu_\alpha} \varphi}{\partial y^i} \frac{\partial \xi_\alpha}{\partial y^j} \sqrt{\det g_\alpha} \, dy. \end{aligned}$$

Therefore:

$$\begin{aligned} & \left| \int_{\tilde{\Omega}_\alpha} \nabla T_{\mu_\alpha} \varphi \cdot \nabla U \, dy \right| \\ & \leq \left| \int_{\tilde{\Omega}_\alpha} \nabla T_{\mu_\alpha} \varphi \cdot \nabla (U - T_{\mu_\alpha} \xi_\alpha) \, dy \right| + \left| \int_{\tilde{\Omega}_\alpha} (\delta^{ij} - g_\alpha^{ij} \sqrt{\det g_\alpha}) \frac{\partial T_{\mu_\alpha} \varphi}{\partial y^i} \frac{\partial \xi_\alpha}{\partial y^j} \, dy \right| \\ & \leq (\|U - T_{\mu_\alpha} \xi_\alpha\| + \sup_{\tilde{\Omega}_\alpha} |\delta^{ij} - g_\alpha^{ij} \sqrt{\det g_\alpha}| \|\nabla T_{\mu_\alpha} \xi_\alpha\|) \times \|T_{\mu_\alpha} \varphi\| \\ & = (\circ_\alpha(1) + O_{\delta_0}(\delta_0^2)) \|T_{\mu_\alpha} \varphi\|_{g_\alpha}. \end{aligned}$$

The remaining conditions are verified similarly. Taking into account (4.8), we conclude by Lemma 4.6 that for all $\varphi \in W_\alpha$,

$$Q_\alpha(\varphi, \varphi) \geq \frac{c_1}{2} \int_{\tilde{\Omega}_\alpha} |\nabla_{g_\alpha} T_{\mu_\alpha} \varphi|^2 \, dv_{g_\alpha} = \frac{c_1}{2} \int_{B_\alpha} |\nabla_{\widehat{g}} \varphi|^2 \, dv_{\widehat{g}},$$

as asserted. \square

Proof of Proposition 4.1. Multiplying (4.5) by w_α and integrating over B_α we obtain:

$$Q_\alpha(w_\alpha, w_\alpha) + \circ_\alpha(\|w_\alpha\|^2) = \int_{B_\alpha} f_\alpha w_\alpha \, dv_{\widehat{g}}.$$

By Lemma 4.5, in view of the form of f_α and recalling the orthogonality property $\int_{B_\alpha} w_\alpha \Delta_{\tilde{g}} \xi_\alpha dv_{\tilde{g}} = 0$, we derive from the above:

$$\begin{aligned} \|w_\alpha\| &\leq C(\|\xi_\alpha^{2^*-1} + K^2 \Delta_{\tilde{g}} \xi_\alpha\|_{2^{*'}} \\ &\quad + \alpha \|u_\alpha\|_{L^r(M)}^{2-r} \|u_\alpha^{r-1}\|_{2^{*'}} + \mu_\alpha^{(n-2)/2} \|\xi_\alpha^{2^*-2}\|_{2^{*'}}). \end{aligned}$$

ξ_α satisfies

$$-\Delta_{\tilde{g}} \xi_\alpha = K^{-2} \xi_\alpha^{2^*-1} + O(\xi_\alpha). \quad (4.10)$$

It follows that

$$\|\xi_\alpha^{2^*-1} + K^2 \Delta_{\tilde{g}} \xi_\alpha\|_{2^{*'}} \leq C \|\xi_\alpha\|_{2^{*'}} \leq C \mu_\alpha^2 \|U\|_{2^{*'}, \mu_\alpha^{-1}},$$

where we have used $\lambda_\alpha \mu_\alpha \leq C$ in the last inequality. In order to estimate the second term, we note that the uniform estimate (3.2) implies: $u_\alpha \leq C \xi_\alpha$ in B_α . Consequently,

$$\|u_\alpha^{r-1}\|_{2^{*'}} \leq C \left(\int_{B_\alpha} \xi_{\tilde{x}_\alpha, \mu_\alpha^{-1}}^{(r-1)2^{*'}} dv_{\tilde{g}} \right)^{1/2^{*'}} \leq C \mu_\alpha^{n - \frac{n-2}{2}r} \|U^{r-1}\|_{2^{*'}, \mu_\alpha^{-1}}.$$

Similarly, we compute:

$$\|\xi_\alpha^{2^*-2}\|_{2^{*'}} \leq C \mu_\alpha^{(n-2)/2} \|U^{2^*-2}\|_{2^{*'}, \mu_\alpha^{-1}}.$$

The asserted decay estimate for $\|w_\alpha\|$ follows.

In order to estimate $|\ell_\alpha t_\alpha^{2^*-1} - t_\alpha K^{-2}|$, in view of (4.10), we write f_α in the form:

$$\begin{aligned} f_\alpha &= (\ell_\alpha t_\alpha^{2^*-1} - t_\alpha K^{-2}) \xi_\alpha^{2^*-1} - \alpha \|u_\alpha\|_{L^r(M)}^{2-r} \psi^{1-2^*} u_\alpha^{r-1} \\ &\quad + O(\xi_\alpha) + O(\mu_\alpha^{(n-2)/2} \xi_\alpha^{2^*-2}). \end{aligned}$$

Multiplying (4.5) by σ_α , integrating over B_α and taking into account that $\int_{B_\alpha} \sigma_\alpha \Delta_{\tilde{g}} w_\alpha dv_{\tilde{g}} = 0$, we have:

$$\begin{aligned} &-k_\alpha \int_{B_\alpha} |\Theta_\alpha|^{2^*-3} \Theta_\alpha w_\alpha \sigma_\alpha dv_{\tilde{g}} + \int_{B_\alpha} b' |\Theta|^{2^*-3} w_\alpha^2 \sigma_\alpha dv_{\tilde{g}} + \int_{B_\alpha} b'' |w_\alpha|^{2^*-1} \sigma_\alpha dv_{\tilde{g}} \\ &= (\ell_\alpha t_\alpha^{2^*-1} - t_\alpha K^{-2}) \int_{B_\alpha} \xi_\alpha^{2^*-1} \sigma_\alpha dv_{\tilde{g}} - \alpha \|u_\alpha\|_{L^r(M)}^{2-r} \int_{B_\alpha} \psi^{1-2^*} u_\alpha^{r-1} \sigma_\alpha dv_{\tilde{g}} \\ &\quad + O\left(\int_{B_\alpha} \xi_\alpha \sigma_\alpha dv_{\tilde{g}}\right) + O(\mu_\alpha^{(n-2)/2}) \int_{B_\alpha} \xi_\alpha^{2^*-2} \sigma_\alpha dv_{\tilde{g}} \end{aligned}$$

and thus, using $|w_\alpha| + |\Theta_\alpha| + |\sigma_\alpha| + u_\alpha \leq C \xi_\alpha$, we derive:

$$\begin{aligned} |\ell_\alpha t_\alpha^{2^*-1} - t_\alpha K^{-2}| \int_{B_\alpha} \xi_\alpha^{2^*-1} \sigma_\alpha dv_{\tilde{g}} &\leq C \left(\int_{B_\alpha} \xi_\alpha^{2^*-1} |w_\alpha| dv_{\tilde{g}} + \int_{B_\alpha} \xi_\alpha^2 dv_{\tilde{g}} \right. \\ &\quad \left. + \alpha \|u_\alpha\|_{L^r(M)}^{2-r} \int_{B_\alpha} \xi_\alpha^r dv_{\tilde{g}} + \mu_\alpha^{(n-2)/2} \int_{B_\alpha} \xi_\alpha^{2^*-1} dv_{\tilde{g}} \right). \end{aligned}$$

In order to compare with the decay rate of $\|w_\alpha\|$, it is convenient to estimate as follows:

$$\begin{aligned} \int_{B_\alpha} \xi_\alpha^{2^*-1} |w_\alpha| dv_{\tilde{g}} &\leq C \|w_\alpha\|; \\ \int_{B_\alpha} \xi_\alpha^r dv_{\tilde{g}} &\leq C \|\xi_\alpha^{r-1}\|_{2^{*'}} \|\xi_\alpha\|_{2^*} \leq C \mu_\alpha^{n - \frac{n-2}{2}r} \|U^{r-1}\|_{2^{*'}, \mu_\alpha^{-1}}; \end{aligned}$$

$$\begin{aligned} \mu_\alpha^{(n-2)/2} \int_{B_\alpha} \xi_\alpha^{2^*-1} dv_{\tilde{g}} &= \mu_\alpha^{(n-2)/2} \|\xi_\alpha^{2^*-2}\|_{2^{*'}} \|\xi_\alpha\|_{2^*} \leq C \mu_\alpha^{n-2} \|U^{2^*-2}\|_{2^{*'}, \mu_\alpha^{-1}}; \\ \int_{B_\alpha} \xi_\alpha^2 dv_{\tilde{g}} &\leq C \|\xi_\alpha\|_{2^*} \|\xi_\alpha\|_{2^{*'}} \leq C \mu_\alpha^2 \|U\|_{2^{*'}, \mu_\alpha^{-1}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} |\ell_\alpha t_\alpha^{2^*-1} - t_\alpha K^{-2}| \int_{B_\alpha} \xi_\alpha^{2^*-1} \sigma_\alpha dv_{\tilde{g}} &\geq |\ell_\alpha t_\alpha^{2^*-1} - t_\alpha K^{-2}| \left(\int_{B_\alpha} \xi_\alpha^{2^*} dv_{\tilde{g}} + O(\mu_\alpha^{n-2}) \right) \\ &\geq C^{-1} |\ell_\alpha t_\alpha^{2^*-1} - t_\alpha K^{-2}| + O(\mu_\alpha^{n-2}). \end{aligned}$$

The estimate for $|\ell_\alpha t_\alpha^{2^*-1} - t_\alpha K^{-2}|$ is established. \square

5. Lower Bound for Y_g and Proof of Theorem 1.1 for $n \geq 7$. In this section we shall carefully exploit orthogonality in order to derive a lower bound for $Y_g(u_\alpha)$, as in Proposition 5.1 below. Together with the estimates from the previous sections, it will readily imply the proof of Theorem 1.1 in the case $n \geq 7$. We shall need an L^2 -estimate of $|\nabla_g u_\alpha|$ on ∂B_α . This can be achieved by selecting a suitable ‘‘good radius’’ $\delta_\alpha \in [\delta_0/2, \delta_0]$, see Lemma 5.3 below. Here is where we fix δ_α . Unless otherwise stated, we assume $n \geq 3$.

The main step towards obtaining a contradiction is the following lower bound for $Y_g(u_\alpha)$:

PROPOSITION 5.1 (Lower bound for Y_g). *Let δ_α be a ‘‘good radius’’. Then, for all α sufficiently large,*

$$Y_g(u_\alpha) \geq Y_{\tilde{g}}(\tilde{\xi}_{\tilde{x}_\alpha, \lambda_\alpha}^{\tilde{g}}) + O(\mu_\alpha^2 \|U\|_{2^{*'}, \mu_\alpha^{-1}} \|w_\alpha\| + \mu_\alpha^{n-2}).$$

Proposition 5.1 readily implies:

COROLLARY 5.2. *The following estimates hold:*

$$\alpha \|u_\alpha\|_{L^r(M)}^2 \leq C(\mu_\alpha^2 \|U\|_{2^{*'}, \mu_\alpha^{-1}} + \mu_\alpha^{n-2}) + |K^{-2} - Y_{\tilde{g}}(\tilde{\xi}_{\tilde{x}_\alpha, \lambda_\alpha}^{\tilde{g}})| \quad (\text{i})$$

$$\begin{aligned} \alpha \|u_\alpha\|_{L^r(M)}^2 &\leq C(\mu_\alpha^4 \|U\|_{2^{*'}, \mu_\alpha^{-1}}^2 + \varepsilon_\alpha \mu_\alpha^2 \|U\|_{2^{*'}, \mu_\alpha^{-1}} \|U^{r-1}\|_{2^{*'}, \mu_\alpha^{-1}}) \\ &\quad + \mu_\alpha^n \|U\|_{2^{*'}, \mu_\alpha^{-1}} \|U^{2^*-2}\|_{2^{*'}, \mu_\alpha^{-1}} + \mu_\alpha^{n-2} \\ &\quad + |K^{-2} - Y_{\tilde{g}}(\tilde{\xi}_{\tilde{x}_\alpha, \lambda_\alpha}^{\tilde{g}})|. \end{aligned} \quad (\text{ii})$$

Proof. By the initial assumption on I_α , we have:

$$K^{-2} > I_\alpha(u_\alpha) = Y_g(u_\alpha) + \alpha \|u_\alpha\|_{L^r(M)}^2.$$

Therefore, the lower bound as in Proposition 5.1 implies:

$$K^{-2} > \alpha \|u_\alpha\|_{L^r(M)}^2 + Y_{\tilde{g}}(\tilde{\xi}_{\tilde{x}_\alpha, \lambda_\alpha}^{\tilde{g}}) + O(\mu_\alpha^2 \|U\|_{2^{*'}, \mu_\alpha^{-1}} \|w_\alpha\| + \mu_\alpha^{n-2}),$$

which in turn yields:

$$\alpha \|u_\alpha\|_{L^r(M)}^2 \leq |Y_{\tilde{g}}(\tilde{\xi}_{\tilde{x}_\alpha, \lambda_\alpha}^{\tilde{g}}) - K^{-2}| + C(\mu_\alpha^2 \|U\|_{2^{*'}, \mu_\alpha^{-1}} \|w_\alpha\| + \mu_\alpha^{n-2}).$$

Using $\|w_\alpha\| \leq C$, we obtain (i). Using the energy estimate as in Proposition 4.1, we obtain (ii). \square

The proof of Proposition 5.1 relies on some boundary estimates and on consequences of orthogonality, which we proceed to derive.

LEMMA 5.3 (Choice of “good radius”). *There exists $C > 0$ independent of α such that:*

$$\int_{M \setminus B_{\delta_0/2}(x_\alpha)} |\nabla_g u_\alpha|^2 dv_g \leq C \mu_\alpha^{n-2}.$$

Consequently, for every α we can select $\delta_\alpha \in [\delta_0/2, \delta_0]$ such that on $B_\alpha = B_{\delta_\alpha}(x_\alpha)$ we have:

$$\int_{\partial B_\alpha} |\nabla_g u_\alpha|^2 ds_g \leq C \mu_\alpha^{n-2}. \quad (5.1)$$

Furthermore, for such a δ_α we have:

$$\int_{B_\alpha} |\nabla_{\widehat{g}} \chi_\alpha|^2 dv_{\widehat{g}} \leq C \mu_\alpha^{n-2}. \quad (5.2)$$

Proof. Denote by η a smooth cutoff function to be fixed below, satisfying $0 \leq \eta \leq 1$. Multiplying (2.2) by $\eta^2 u_\alpha$ and integrating by parts on M we have:

$$\int_M \nabla_g u_\alpha \cdot \nabla_g (\eta^2 u_\alpha) dv_g \leq -c(n) \int_M R_g \eta^2 u_\alpha^2 dv_g + \ell_\alpha \int_M \eta^2 u_\alpha^{2^*} dv_g.$$

It follows that:

$$\int_M \eta^2 |\nabla_g u_\alpha|^2 dv_g \leq C \left(\int_M u_\alpha^2 (|\nabla_g \eta|^2 + \eta^2) dv_g + \int_M \eta^2 u_\alpha^{2^*} dv_g \right).$$

Choosing η such that $\eta \equiv 1$ in $M \setminus B_{\delta_0}(x_\alpha)$, $\text{supp } \eta \subset M \setminus B_{\delta_0/2}(x_\alpha)$, we obtain:

$$\int_{M \setminus B_{\delta_0}(x_\alpha)} |\nabla_g u_\alpha|^2 dv_g \leq C \left(\int_{M \setminus B_{\delta_0/2}(x_\alpha)} u_\alpha^2 dv_g + \int_{M \setminus B_{\delta_0/2}(x_\alpha)} u_\alpha^{2^*} dv_g \right).$$

Now the statement follows by the uniform estimate as in Proposition 3.1.

Since $u_\alpha \in C^1(M)$, we can choose δ_α such that:

$$\int_{\partial B_\alpha} |\nabla_g u_\alpha|^2 dv_g = \min_{\delta \in [\delta_0/2, \delta_0]} \int_{\partial B_\delta(x_\alpha)} |\nabla_g u_\alpha|^2 ds_g \leq C \mu_\alpha^{n-2}.$$

Recalling the definition of χ_α , we have by standard elliptic estimates and equivalence of g and \widehat{g} :

$$\int_{\partial B_\alpha} |\nabla_{\widehat{g}} \chi_\alpha|^2 dv_{\widehat{g}} \leq C \int_{\partial B_\alpha} \{ |\nabla_g u_\alpha|^2 + u_\alpha^2 \} ds_g \leq C \mu_\alpha^{n-2}.$$

□

Proof of Proposition 5.1. By the uniform estimate (3.2) and by Lemma 5.3, we have

$$Y_g(u_\alpha) = \frac{\int_{B_\alpha} \{ |\nabla_g u_\alpha|^2 + c(n) R_g u_\alpha^2 \} dv_g}{\left(\int_{B_\alpha} u_\alpha^{2^*} dv_g \right)^{2/2^*}} + O(\mu_\alpha^{n-2}). \quad (5.3)$$

By conformal invariance (1.8), together with (5.1) and (3.2),

$$\int_{B_\alpha} \{ |\nabla_g u_\alpha|^2 + c(n) R_g u_\alpha^2 \} dv_g = \int_{B_\alpha} |\nabla_{\widehat{g}} \frac{u_\alpha}{\psi}|^2 dv_{\widehat{g}} + O(\mu_\alpha^{n-2}).$$

Recall from Section 4 that $u_\alpha/\psi = t_\alpha \xi_\alpha - t_\alpha h_\alpha + \chi_\alpha + w_\alpha$. By Lemma 4.2, (4.4), (5.2), and the fact

$$\int_{B_\alpha} \nabla_{\widehat{g}} h_\alpha \cdot \nabla_{\widehat{g}} w_\alpha dv_{\widehat{g}} = 0 = \int_{B_\alpha} \nabla_{\widehat{g}} \chi_\alpha \cdot \nabla_{\widehat{g}} w_\alpha dv_{\widehat{g}},$$

we have

$$Y_g(u_\alpha) = F(w_\alpha) + O(\mu_\alpha^{n-2}), \quad (5.4)$$

where

$$F(w) := \frac{\int_{B_\alpha} |\nabla_{\hat{g}}(t_\alpha \xi_\alpha + w)|^2 dv_{\hat{g}}}{\left(\int_{B_\alpha} |t_\alpha \xi_\alpha + w|^{2^*} dv_{\hat{g}}\right)^{2/2^*}}, \quad w \in H_0^1(B_\alpha).$$

A Taylor expansion yields:

$$F(w_\alpha) = F(0) + F'(0)w_\alpha + \frac{1}{2}\langle F''(0)w_\alpha, w_\alpha \rangle + o(\|w_\alpha\|^2), \quad (5.5)$$

where F' , F'' denote Fréchet derivatives. We compute:

$$\begin{aligned} F'(0)w_\alpha &= \frac{2}{\left(\int_{B_\alpha} (t_\alpha \xi_\alpha dv_{\hat{g}})^{2^*}\right)^{2/2^*}} \times \\ &\times \left\{ \int_{B_\alpha} \nabla_{\hat{g}}(t_\alpha \xi_\alpha) \cdot \nabla_{\hat{g}} w_\alpha dv_{\hat{g}} - \frac{\int_{B_\alpha} |\nabla_{\hat{g}} t_\alpha \xi_\alpha|^2 dv_{\hat{g}}}{\int_{B_\alpha} (t_\alpha \xi_\alpha)^{2^*} dv_{\hat{g}}} \int_{B_\alpha} (t_\alpha \xi_\alpha)^{2^*-1} w_\alpha dv_{\hat{g}} \right\}. \end{aligned}$$

By orthogonality, $\int_{B_\alpha} \nabla_{\hat{g}} \xi_\alpha \cdot \nabla_{\hat{g}} w_\alpha dv_{\hat{g}} = 0$ and by (4.10)

$$K^2 \int_{B_\alpha} \xi_\alpha^{2^*-1} w_\alpha dv_{\hat{g}} = \int_{B_\alpha} (-\Delta_{\hat{g}} \xi_\alpha + O(\xi_\alpha)) w_\alpha dv_{\hat{g}} = O\left(\int_{B_\alpha} \xi_\alpha w_\alpha dv_{\hat{g}}\right).$$

Hence,

$$\left| \int_{B_\alpha} \xi_\alpha^{2^*-1} w_\alpha dv_{\hat{g}} \right| \leq C \|\xi_\alpha\|_{2^{*'}} \|w_\alpha\| \leq C \mu_\alpha^2 \|U\|_{2^{*'}, \mu_\alpha^{-1}} \|w_\alpha\|$$

and consequently

$$|F'(0)w_\alpha| \leq C \mu_\alpha^2 \|U\|_{2^{*'}, \mu_\alpha^{-1}} \|w_\alpha\|.$$

Similarly, we compute:

$$\begin{aligned} \langle F''(0)w_\alpha, w_\alpha \rangle &= \frac{2}{\left(\int_{B_\alpha} (t_\alpha \xi_\alpha)^{2^*}\right)^{2/2^*}} \times \\ &\times \left\{ \int_{B_\alpha} |\nabla_{\hat{g}} w_\alpha|^2 dv_{\hat{g}} - (2^* - 1) \frac{\int_{B_\alpha} |\nabla_{\hat{g}} \xi_\alpha|^2 dv_{\hat{g}}}{\int_{B_\alpha} \xi_\alpha^{2^*} dv_{\hat{g}}} \int_{B_\alpha} \xi_\alpha^{2^*-2} w_\alpha^2 dv_{\hat{g}} \right\} \\ &+ O\left(\int_{B_\alpha} \xi_\alpha^{2^*-1} w_\alpha dv_{\hat{g}}\right)^2. \end{aligned}$$

By the transformations (4.8)–(4.9) and by Lemma 4.6 with

$$\begin{aligned} \Omega &= \tilde{\Omega}_\alpha, \quad \Theta = T_{\mu_\alpha} \xi_\alpha, \quad h = g_\alpha, \\ k &= (2^* - 1) \frac{\int_{B_\alpha} |\nabla_{\hat{g}} \xi_\alpha|^2 dv_{\hat{g}}}{\int_{B_\alpha} \xi_\alpha^{2^*} dv_{\hat{g}}}, \end{aligned}$$

we obtain, for large α , that

$$\int_{B_\alpha} |\nabla_{\hat{g}} w_\alpha|^2 dv_{\hat{g}} - (2^* - 1) \frac{\int_{B_\alpha} |\nabla_{\hat{g}} \xi_\alpha|^2 dv_{\hat{g}}}{\int_{B_\alpha} \xi_\alpha^{2^*} dv_{\hat{g}}} \int_{B_\alpha} \xi_\alpha^{2^*-2} w_\alpha^2 dv_{\hat{g}} \geq \frac{c_1}{2} \|w_\alpha\|^2.$$

Consequently,

$$\langle F''(0)w_\alpha, w_\alpha \rangle \geq \frac{c_1}{2} \|w_\alpha\|^2 + O(\mu_\alpha^4) \|U\|_{2^{*'}, \mu_\alpha^{-1}}^2 \|w_\alpha\|^2.$$

Inserting into (5.5) and observing that $\mu_\alpha^2 \|U\|_{2^{*'}, \mu_\alpha^{-1}} = o_\alpha(1)$, we derive:

$$F(w_\alpha) \geq F(0) + O(\mu_\alpha^2 \|U\|_{2^{*'}, \mu_\alpha^{-1}} \|w_\alpha\| + \mu_\alpha^{n-2}). \quad (5.6)$$

Returning to (5.3) and taking into account that

$$F(0) = Y_{\tilde{g}}(\tilde{\xi}_{x_\alpha, \lambda_\alpha}^{\tilde{g}}) + O(\mu_\alpha^{n-2}),$$

we obtain the asserted lower bound. \square

Proof of Theorem 1.1 for $n \geq 7$. By straightforward computations,

$$\|U^q\|_{2^{*'}, \mu_\alpha^{-1}} \leq \begin{cases} C, & \text{if } q > (n+2)/[2(n+2)], \\ (\log \mu_\alpha^{-1})^{1/2^{*'}}, & \text{if } q = (n+2)/[2(n+2)], \\ \mu_\alpha^{-\frac{n+2}{2} + q(n-2)}, & \text{if } q < (n+2)/[2(n+2)]. \end{cases}$$

We take $r = \bar{r} = 2n/(n+2)$. Then, since $n \geq 7$, we have:

$$\begin{aligned} \|U\|_{2^{*'}, \mu_\alpha^{-1}} &\leq C, \\ \|U^{\bar{r}-1}\|_{2^{*'}, \mu_\alpha^{-1}} &\leq C(1 + \mu_\alpha^{-2+\beta}), \\ \|U^{2^*-2}\|_{2^{*'}, \mu_\alpha^{-1}} &\leq C\mu_\alpha^{-(n-6)/2}, \end{aligned}$$

where $\beta = (n-6)(n-2)/[2(n+2)]$ is *strictly positive*. Hence, (ii) in Corollary 5.2 yields:

$$\alpha \|u_\alpha\|_{L^{\bar{r}}(M)}^2 \leq |Y_{\tilde{g}}(\xi_\alpha, B_\alpha) - K^{-2}| + C[\mu_\alpha^4 + \varepsilon_\alpha(\mu_\alpha^2 + \mu_\alpha^\beta)]. \quad (5.7)$$

By (1.6),

$$|Y_{\tilde{g}}(\tilde{\xi}_{x_\alpha, \lambda_\alpha}^{\tilde{g}}) - K^{-2}| \leq C\mu_\alpha^4.$$

In view of (2.10), we derive:

$$\alpha \|u_\alpha\|_{L^{\bar{r}}(M)}^2 \leq C\mu_\alpha^4. \quad (5.8)$$

On the other hand, rescaling, we have:

$$\|u_\alpha\|_{L^{\bar{r}}(M)} \geq \|u_\alpha\|_{L^{\bar{r}}(B_\alpha)} \geq C^{-1}\mu_\alpha^2 \|U\|_{L^{\bar{r}}(B_1(0))} \geq C^{-1}\mu_\alpha^2 \quad (5.9)$$

and inserting into (5.8) we obtain $\alpha \leq C$, a contradiction. Hence, Theorem 1.1 is established for all $n \geq 7$. \square

6. Proof of Theorem 1.1 for $n = 6$. In order to prove Theorem 1.1 in the remaining case $n = 6$ we need a uniform *lower* bound for u_α . Indeed we shall prove:

PROPOSITION 6.1 (Uniform lower bound). *For $n = 6$, $r = \bar{r} = 3/2$, and any $1/2 < \gamma < 1$, there exists some constant $C > 0$, which is independent of α , such that u_α satisfies:*

$$u_\alpha(x) \geq C^{-1}\mu_\alpha^2 \text{dist}_g(x, \tilde{x}_\alpha)^{-4}, \quad \forall x \in B_{\delta_\alpha \mu_\alpha^\gamma}(x_\alpha) \setminus B_{\mu_\alpha}(x_\alpha),$$

for all $\alpha \gg 1$.

Proof. We equivalently show that

$$v_\alpha(y) \geq C^{-1}|y|^{-4}, \quad \forall y \in B_{\delta_\alpha \mu_\alpha^{\gamma-1}}(0) \setminus B_1.$$

Here v_α is defined on Ω_α as in (2.7) and (2.8). Recall the $\delta_0/2 \leq \delta_\alpha \leq \delta_0$. $0 < \delta_0 < 1$ will be small and fixed below. We define a comparison function

$$H_\alpha(y) = \tau \left(\frac{\delta_\alpha^4}{|y|^4} - \mu_\alpha^4 \right) + L \mu_\alpha^2 \left(\log \frac{1}{\mu_\alpha} \right)^{2/3} \log \left(\frac{\mu_\alpha |y|}{\delta_\alpha} \right), \quad y \in \Omega_\alpha \setminus B_1$$

where $\tau > 0$, $L > 0$ will be chosen below.

Since $v_\alpha \rightarrow U$ uniformly on ∂B_1 , we first fix some $0 < \tau = \tau(\delta_0) < 1$ such that $v_\alpha \geq H_\alpha$ on ∂B_1 for large α . Since $H_\alpha = 0$ on $\partial\Omega_\alpha$, we also have $v_\alpha \geq H_\alpha$ on $\partial\Omega_\alpha$. We know that

$$C_1^{-1}\delta_0^{-2}\mu_\alpha^2 \leq |y|^{-2} \leq 1, \quad \text{on } \Omega_\alpha \setminus B_1.$$

Here and in the following, $C_1 > 1$ denotes some constant depending only on (M, g) . Setting $g_\alpha(y) = g(\exp_{x_\alpha}^g(\mu_\alpha y))$ we have,

$$\begin{aligned} |\Delta_{g_\alpha}|y|^{-4}| &\leq C_1\mu_\alpha^2|y|^{-4}, \\ |\Delta_{g_\alpha} \log |y| - 4|y|^{-2}| &\leq C_1\mu_\alpha^2. \end{aligned}$$

Hence,

$$\Delta_{g_\alpha} H_\alpha(y) \geq 4L\mu_\alpha^2 \left(\log \frac{1}{\mu_\alpha}\right)^{2/3} |y|^{-2} - C_1\mu_\alpha^2 |y|^{-4} - C_1L\mu_\alpha^4 \left(\log \frac{1}{\mu_\alpha}\right)^{2/3}.$$

Recall from Section 1 that

$$\varepsilon_\alpha = \alpha\mu_\alpha^{n-\frac{n-2}{2}r} \|u_\alpha\|_{L^r(M)}^{2-r} \leq \alpha \|u_\alpha\|_{L^r(M)}^2.$$

By (i) in Corollary 5.2,

$$\alpha \|u_\alpha\|_{L^r(M)}^2 \leq C_2(\mu_\alpha^2 \|U\|_{2^{*'}, \mu_\alpha^{-1}} + \mu_\alpha^4) + |Y_{\tilde{g}}(\tilde{\xi}_{\tilde{x}_\alpha, \lambda_\alpha}^g) - K^{-2}|.$$

Here and in the following, $C_2 > 1$ denotes some constant independent of α and L . By the expansion (1.6),

$$|Y_{\tilde{g}}(\tilde{\xi}_{\tilde{x}_\alpha, \lambda_\alpha}^g) - K^{-2}| \leq C_2\mu_\alpha^4 \log \frac{1}{\mu_\alpha}, \quad (6.1)$$

and, clearly,

$$\|U\|_{2^{*'}, \mu_\alpha^{-1}} \leq C_2 \left(\log \frac{1}{\mu_\alpha}\right)^{2/3}.$$

It follows that

$$\alpha \|u_\alpha\|_{L^r(M)}^2 \leq C_2\mu_\alpha^2 \left(\log \frac{1}{\mu_\alpha}\right)^{2/3}.$$

Together with the uniform estimate from Section 2: $v_\alpha(y) \leq C_2|y|^{-4}$, we obtain

$$\varepsilon_\alpha v_\alpha^{1/2} \leq C_2\mu_\alpha^2 \left(\log \frac{1}{\mu_\alpha}\right)^{2/3} |y|^{-2}.$$

Hence, by the equation of v_α , we have, on $\Omega_\alpha \setminus B_1$, that

$$\begin{aligned} & -\Delta_{g_\alpha}(v_\alpha - H_\alpha)(y) \\ & \geq (4L - C_2)\mu_\alpha^2 \left(\log \frac{1}{\mu_\alpha}\right)^{2/3} |y|^{-2} - C_2\mu_\alpha^2 |y|^{-4} - C_1L\mu_\alpha^4 \left(\log \frac{1}{\mu_\alpha}\right)^{2/3} \\ & \geq (4L - C_2 - C_1\delta_0^2 L)\mu_\alpha^2 \left(\log \frac{1}{\mu_\alpha}\right)^{2/3} |y|^{-2}. \end{aligned}$$

We first fix $\delta_0 > 0$ small ($C_1\delta_0 < 1$), and then take L large, we achieve, for large α , that

$$-\Delta_{g_\alpha}(v_\alpha - H_\alpha) \geq 0 \quad \text{in } \Omega_\alpha \setminus B_1.$$

By the maximum principle,

$$v_\alpha \geq H_\alpha \quad \text{in } \Omega_\alpha \setminus B_1.$$

To conclude, we observe that for any fixed $1/2 < \gamma < 1$ we can find a $C > 0$ such that:

$$H_\alpha(y) \geq C^{-1}|y|^{-4} \quad \text{in } B_{\delta_\alpha\mu_\alpha^{\gamma-1}} \setminus B_1.$$

□

Proof of Theorem 1.1 for $n = 6$. When $n = 6$ and $r = \bar{r} = 3/2$, we have:

$$\begin{aligned} \|U\|_{2^{*'}, \mu_\alpha^{-1}} &\leq C \left(\log \frac{1}{\mu_\alpha}\right)^{2/3}, \\ \|U^{r-1}\|_{2^{*'}, \mu_\alpha^{-1}} &\leq C \mu_\alpha^{-2}, \\ \|U^{2^* - 2}\|_{2^{*'}, \mu_\alpha^{-1}} &= \|U\|_{2^{*'}, \mu_\alpha^{-1}} \leq C \left(\log \frac{1}{\mu_\alpha}\right)^{2/3}. \end{aligned}$$

We know

$$|Y_{\tilde{g}}(\tilde{\xi}_{\tilde{x}_\alpha, \lambda_\alpha}^g) - K^{-2}| \leq C \mu_\alpha^4 \log \frac{1}{\mu_\alpha}.$$

Hence, (ii) in Corollary 5.2 implies:

$$\alpha \|u_\alpha\|_{L^{\bar{r}}(M)}^2 \leq C \left(\mu_\alpha^4 \left(\log \frac{1}{\mu_\alpha}\right)^{4/3} + \varepsilon_\alpha \left(\log \frac{1}{\mu_\alpha}\right)^{2/3} + \mu_\alpha^6 \left(\log \frac{1}{\mu_\alpha}\right)^{4/3} \right) \quad (6.2)$$

From the uniform estimate (3.2) we derive:

$$\varepsilon_\alpha \leq C \alpha \mu_\alpha^4 \left(\log \frac{1}{\mu_\alpha}\right)^{1/3},$$

and by Proposition 6.1 we have

$$\|u_\alpha\|_{L^{\bar{r}}(B_\alpha)} \geq C^{-1} \mu_\alpha^2 \left(\log \frac{1}{\mu_\alpha}\right)^{2/3}.$$

Inserting into (6.2), we obtain

$$\alpha \mu_\alpha^4 \left(\log \frac{1}{\mu_\alpha}\right)^{4/3} \leq C \left(\mu_\alpha^4 \left(\log \frac{1}{\mu_\alpha}\right)^{4/3} + \alpha \mu_\alpha^4 \log \frac{1}{\mu_\alpha} \right).$$

Once again we obtain $\alpha \leq C$, a contradiction. Theorem 1.1 is thus established in the remaining limit case $n = 6$. □

7. Appendix: A Local to Global Argument. In this Appendix we provide a proof of the local to global argument used in Theorem 1.3. We adapt some ideas from [6]. Let (M, g) be a smooth compact Riemannian manifold without boundary, $n \geq 3$.

LEMMA 7.1. *Suppose that there exist $\bar{\varepsilon} > 0$ and $A_{\bar{\varepsilon}} > 0$ such that*

$$\|u\|_{L^{2^*}(M, g)}^2 \leq K^2 \int_M \{|\nabla_g u|^2 + c(n) R_g u^2\} dv_g + A_{\bar{\varepsilon}} \|u\|_{L^1(M, g)}^2, \quad (7.1)$$

for all $u \in H^1(M)$ such that $\text{diam}_g(\text{supp}) u < \bar{\varepsilon}$. Then there exists a constant $A > 0$ such that

$$\|u\|_{L^{2^*}(M, g)}^2 \leq K^2 \int_M \{|\nabla_g u|^2 + c(n) R_g u^2\} dv_g + A \|u\|_{L^1(M, g)}^2, \quad \forall u \in H^1(M). \quad (7.2)$$

Proof. By contradiction. Suppose (7.2) is not true. Then by density of smooth functions in $H^1(M)$, for all $\alpha > 0$ there exists $r_\alpha \in (1, 2)$ such that

$$\ell_\alpha := \inf_{u \in H^1(M) \setminus \{0\}} \frac{\int_M \{|\nabla_g u|^2 + c(n) R_g u^2\} dv_g + \alpha \|u\|_{L^{r_\alpha}(M)}^2}{\|u\|_{L^{2^*}(M)}^2} < K^{-2}. \quad (7.3)$$

By the results in Section 2 with $r = r_\alpha$, there exists $u_\alpha \in H^1(M)$, $u_\alpha \geq 0$, $\int_M u_\alpha^{2^*} dv_g = 1$ such that $\ell_\alpha = I_\alpha(u_\alpha)$. Moreover, u_α satisfies the Euler-Lagrange equation:

$$-\Delta_g u_\alpha + c(n)R_g u_\alpha + \alpha \|u_\alpha\|_{L^{r_\alpha}(M)}^{2-r_\alpha} u_\alpha^{r_\alpha-1} = \ell_\alpha u_\alpha^{2^*-1} \quad \text{on } M. \quad (7.4)$$

Denote by x_α a maximum point of u_α . By Corollary 2.5, u_α concentrates in energy at x_α . In particular, for any fixed $\varepsilon > 0$,

$$\lim_{\alpha \rightarrow +\infty} \int_{M \setminus B_\varepsilon(x_\alpha)} \{|\nabla_g u_\alpha|^2 + u_\alpha^{2^*}\} dv_g = 0.$$

For a fixed $0 < \varepsilon < \bar{\varepsilon}/9$, denote by η a smooth cutoff function such that $\eta \equiv 1$ in $B_{2\varepsilon}(x_\alpha)$, $\eta \equiv 0$ in $M \setminus B_{4\varepsilon}(x_\alpha)$, $0 \leq \eta \leq 1$, $|\nabla_g \eta| \leq \varepsilon^{-1}$ in M . Then, by (7.1) and the Hölder inequality,

$$\begin{aligned} \|\eta u_\alpha\|_{L^{2^*}(M,g)}^2 &\leq K^2 \int_M \{|\nabla_g(\eta u_\alpha)|^2 + c(n)R_g(\eta u_\alpha)^2\} dv_g \\ &\quad + A_{\bar{\varepsilon}}(\text{vol}_g M)^{2-2/r_\alpha} \|\eta u_\alpha\|_{L^{r_\alpha}(M)}^2, \end{aligned}$$

and consequently,

$$\begin{aligned} \|u_\alpha\|_{L^{2^*}(B_{2\varepsilon}(x_\alpha))}^2 &\leq K^2 \int_M \{|\nabla_g u_\alpha|^2 + c(n)R_g u_\alpha^2\} dv_g + C \|u_\alpha\|_{L^{r_\alpha}(M)}^2 \\ &\quad + C \int_{B_{4\varepsilon}(x_\alpha) \setminus B_{2\varepsilon}(x_\alpha)} \{|\nabla_g u_\alpha|^2 + u_\alpha^2\} dv_g. \end{aligned}$$

In turn, using the contradiction assumption (7.3), we have

$$\begin{aligned} \|u_\alpha\|_{L^{2^*}(B_{2\varepsilon}(x_\alpha))}^2 &\leq K^2 \ell_\alpha - (\alpha K^2 - C) \|u_\alpha\|_{L^{r_\alpha}(M)}^2 \\ &\quad + C \int_{B_{4\varepsilon}(x_\alpha) \setminus B_{2\varepsilon}(x_\alpha)} \{|\nabla_g u_\alpha|^2 + u_\alpha^2\} dv_g. \end{aligned}$$

Using the expansion

$$\|u_\alpha\|_{L^{2^*}(B_{2\varepsilon}(x_\alpha))}^2 = 1 - O(1) \|u_\alpha\|_{L^{2^*}(M \setminus B_{2\varepsilon}(x_\alpha))}^2,$$

and recalling that $\ell_\alpha K < 1$, we obtain

$$\alpha \|u_\alpha\|_{L^{r_\alpha}(M)}^2 \leq C \|u_\alpha\|_{L^{2^*}(M \setminus B_{2\varepsilon}(x_\alpha))}^2 + C \int_{B_{4\varepsilon}(x_\alpha) \setminus B_{2\varepsilon}(x_\alpha)} \{|\nabla_g u_\alpha|^2 + u_\alpha^2\} dv_g.$$

Now let η be a cutoff function supported in $M \setminus B_\varepsilon(x_\alpha)$. Multiplying (7.4) by $\eta^2 u_\alpha$ and integrating by parts, we find

$$\int_M \eta^2 |\nabla_g u_\alpha|^2 dv_g \leq C \int_{\text{supp} \eta} (u_\alpha^2 + u_\alpha^{2^*}) dv_g.$$

Therefore,

$$\alpha \|u_\alpha\|_{L^{r_\alpha}(M)}^2 \leq C (\|u_\alpha\|_{L^2(M \setminus B_\varepsilon(x_\alpha))}^2 + \|u_\alpha\|_{L^{2^*}(M \setminus B_\varepsilon(x_\alpha))}^2). \quad (7.5)$$

Finally, by Moser iterations,

$$\|u_\alpha\|_{L^\infty(M \setminus B_\varepsilon)} \leq C \|u_\alpha\|_{L^1(M)} \leq C (\text{vol}_g M)^{1-1/r_\alpha} \|u_\alpha\|_{L^{r_\alpha}(M)}, \quad (7.6)$$

see Corollary 2.6. The estimates (7.5)–(7.6) imply $\alpha \leq C$, a contradiction, and (7.2) is established. \square

Acknowledgments. The second author is grateful to the Rutgers University Mathematics Department for hospitality, during her leave of absence from Università di Napoli Federico II.

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Final version received for publication November 2002.

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