Colloquium

Some nonlinear elliptic equations from geometry

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We describe some recent work on certain nonlinear elliptic equations from geometry. These include the problem of prescribing scalar curvature on $\mathbb{S}^n$, the Yamabe problem on manifolds with boundary, and the best Sobolev inequality on Riemannian manifolds.

Yamabe problem | scalar curvature | best Sobolev inequality

1. Prescribing Scalar Curvature on S^n

Let $(\mathbb{S}^n, g_0)$ be the standard n-sphere. The following question was raised by L. Nirenberg. Which function $K(x)$ on $\mathbb{S}^2$ is the Gauss curvature of a metric g on $\mathbb{S}^2$ conformally equivalent to g_0? Naturally one may ask a similar question in a higher dimensional case, namely, which function $K(x)$ on $\mathbb{S}^n$ is the scalar curvature of a metric g on $\mathbb{S}^n$ conformally equivalent to g_0?

For $n \geq 3$, we write $g = u^{(4/n-2)}g_0$; the problem is equivalent to finding a solution of

$$-\Delta_{g_0}u + \frac{n(n-2)}{4}u = \frac{n-2}{4(n-1)}K(x)u^{\frac{n+2}{n-2}}, \quad u > 0, \quad \text{on } \mathbb{S}^n, \qquad \textbf{[1.1]}$$

where Δ_{g_0} denotes the Laplace–Beltrami operator associated with the metric g_0. For $n = 2$, we write $g = e^{2u}g_0$, and the equation takes the form

$$-\Delta_{g_0}u + 1 = K(x)e^{2u}, \quad \text{on } \mathbb{S}^2. \qquad \textbf{[1.2]}$$

A necessary condition for solving **1.1** or **1.2** is that K is positive somewhere. A more subtle and important necessary condition was given by Kazdan and Warner (1) (see also ref. 2 for a generalization). These necessary conditions are not sufficient for the existence (see refs. 3 and 4). Much effort has been put forth to find various sufficient conditions. In particular, the sufficient condition on $\mathbb{S}^2$ given by Moser (5) is that $K(-x) \equiv K(x)$ and $K(x) > 0$ somewhere. This was extended by Escobar and Schoen (6) to higher dimensions: $K(-x) \equiv K(x)$ and there exists a point $\bar{x}$ such that $K(\bar{x}) = \max K > 0$ and all derivatives of K up to $n - 2$ order at $\bar{x}$ vanish. In dimension $n = 3$, this is a complete generalization of the result of Moser. Such complete generalization is no longer valid in dimension $n \geq 4$, see Bianchi and Engnell (7) and Bianchi (8). Further extensions were made by Chen (9) and Hebey (10) to functions K which are invariant under more general groups of isometries. Another type of sufficient condition, discovered by Chang and Yang (11, 12) on $\mathbb{S}^2$ and Bahri and Coron (13) on $\mathbb{S}^3$ is as follows: K is a C^2 positive Morse function having nonzero ΔK at critical points of K and satisfying

$$\sum_{\nabla K(x)=0, \Delta K(x)<0} (-1)^{i(x)} - (-1)^n \neq 0. \qquad \textbf{[1.3]}$$

Due to some more recent work of Schoen (courses at Stanford University, 1988, and the Courant Institute, 1989); Chang, Gursky, and Yang (14); and Schoen and Zhang (15), C^3 norms of all solutions are bounded and the number on the left-hand side of **1.3** is the Leray–Schauder degree of all solutions. Chang and Yang (16) showed that **1.3** is sufficient in all dimensions if we further assume that K is close to 1 in $L^\infty(\mathbb{S}^n)$ norm. Without this closeness hypothesis **1.3** is not sufficient for the existence on $\mathbb{S}^4$, a fact which can be deduced, by elementary consideration, from the results of Bianchi and Engnell (7, 8) and Li (17). It is reasonable to believe that **1.3** is not sufficient for $n \geq 5$ either. A third type of sufficient conditions (Mountain Path type) was given by Chen and Ding (18). There are more existence results for dimension $n = 2, 3$. For higher dimensions, see refs. 6, 16, 17, 19–23, and 65 and the references therein. In the following, we present one of our results in ref. 17 for $n = 4$. Let $\mathcal{M}_K$ denote the set of positive solutions of **1.1** and set $C^2(\mathbb{S}^4)^+ = \{K \in C^2(\mathbb{S}^4) \mid K(x) > 0 \ \forall\, x \in \mathbb{S}^4\}$. We have defined explicitly a subset $\mathcal{A} \subset C^2(\mathbb{S}^4)^+$, which is open and dense with respect to the C^2 topology, and on which we have defined an integer valued, continuous function $\text{Index}: \mathcal{A} \to \mathbb{Z}$. Though there is an explicit formula of Index(K) for Morse functions K in $\mathcal{A}$, it is by no means obvious that the Index mapping can be extended as a continuous function on $\mathcal{A}$. The next theorem gives optimal compactness results, as well as existence results and a degree counting formula of solutions.

Theorem 1.1 (17). (a) *For any $K \in \mathcal{A}$, $K_i \to K$ in $C^2(\mathbb{S}^4)$, $\mathcal{M}_{K_i}$ is precompact in $C^3(\mathbb{S}^4)$.* (b) *For any $K \in C^2(\mathbb{S}^4)^+ \setminus \mathcal{A} = \partial\mathcal{A}$, there exists $K_i \to K$ in $C^2(\mathbb{S}^4)$, and $u_i \in \mathcal{M}_{K_i}$ such that*

$$\lim_{i\to\infty}(\max_{\mathbb{S}^4} u_i) = \infty, \quad \lim_{i\to\infty}(\min_{\mathbb{S}^4} u_i) = 0.$$

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(c) *For any* $K \in \mathcal{A}$, $0 < \alpha < 1$, *there exists constant* $C = C(K)$ *such that for all* $R \geq C$,

$$\deg(u - \tfrac{1}{6}(-\Delta_{g_0} + 2)^{-1}(Ku^3), \mathbb{O}_R, 0) = \text{Index}(K), \quad [1.4]$$

where $\mathbb{O}_R = \{u \in C^{2,\alpha}(\mathbb{S}^4) \mid 1/R < u < R, \|u\|_{C^{2,\alpha}(\mathbb{S}^4)} < R\}$, *and* deg *denotes the Leray–Schauder degree in* $C^{2,\alpha}(\mathbb{S}^4)$. *As a consequence,* $\mathcal{M}_K \neq \phi$ *provided* $\text{Index}(K) \neq 0$.

We only briefly describe what is needed in the proof of Part c. For any Morse function $K \in \mathcal{A}$, let $K_\lambda = (1 - \lambda) + \lambda K$ for $0 \leq \lambda \leq 1$. Because of Part a, there exist $0 < \lambda_1 < \cdots < \lambda_l < 1$ such that for all $\lambda \in (0, 1]\backslash\{\lambda_1, \ldots, \lambda_l\}$ the total Leray–Schauder degree of all possible solutions with respect to K_λ, denoted as d_λ, is well defined and is a constant function of λ in each component of $(0, 1]\backslash\{\lambda_1, \ldots, \lambda_l\}$. The left-hand side of **1.4** is d_1, which is what we want to calculate. The value of d_λ for small λ is known due to the work of Chang and Yang (16). So it is sufficient to calculate the jump-values of d_λ at λ_m for $1 \leq m \leq l$. Indeed we know these jump-values from ref. 17 which yields **1.4** for a Morse function K in $\mathcal{A}$. The validity of **1.4** for all K in $\mathcal{A}$ then follows from Part a. This is different from the situation in dimension $n = 2, 3$, where d_λ is a constant in the whole interval $(0, 1]$; no jumps occur.

The above compactness results are new and optimal. The existence result extends that of Bahri and Coron (13), Benayed, Chen, Chtioui, and Hammami (19), and Zhang (24). It follows from our results that solutions to the problem may have more than one point of blow up, a new phenomena which is very different from the situation in dimension $n = 2, 3$. The phenomena of more than one point of blow up for $n \geq 5$ has also been exhibited in ref. 17.

The main ingredient in establishing *Theorem 1.1* is some fine analysis of blow up to solutions of **1.1**. Such analysis, local in nature, was carried out by Schoen (courses at Stanford University, 1988, and the Courant Institute, 1989) and Schoen and Zhang (15) in dimension $n = 3$ for any positive C^2 functions K, and in dimension $n \geq 4$ for constant K. This was extended by the author to dimension $n = 4$ for any positive C^2 functions K, and in dimension $n \geq 5$ for those K which further satisfy suitable $(n - 2)$-flatness hypothesis near critical points of K. Such analysis in particular implies, under the above hypothesis, the following Harnack type inequality for solutions u of **1.1**:

$$(\sup_{B_\varepsilon} u)(\inf_{B_{2\varepsilon}} u) \leq C\varepsilon^{2-n},$$

where B_ε and $B_{2\varepsilon}$ are concentric balls and C is independent of u. The flatness order $n - 2$ is the borderline for various existence and compactness results to hold. The relevance to the existence of the flatness order $n - 2$ was exhibited in ref. 6. Our notion of the flatness order, introduced in ref. 20, is stronger than merely requiring derivatives up to $n - 2$ order vanish at critical points, as in ref. 6. As an application, existence and degree counting formula are obtained for such K. A consequence of the results in refs. 17 and 20 is the following density result: C^∞ scalar curvature functions are dense in $C^{1,\alpha}$ $(0 < \alpha < 1)$ norms among positive functions. This is not true in the C^2 norm.

In a series of papers, Chen and Lin have made further investigation on the scalar curvature equations. Among other things, they used the method of moving planes to establish such Harnack-type inequalities in ref. 25 under a suitable $(n - 2)$-flatness hypothesis. Moreover, they have obtained delicate estimates when the flatness order is $\beta \in (1, n - 2)$. In particular they exhibited that $(n - 2)/2$ is the borderline flatness order for the existence of a sequence of solutions with energy tending to infinity. Their analysis should be useful in establishing strong existence results in high dimensions for those K with flatness order $\beta \in (1, n - 2)$. Indeed they have already established a degree counting formula in dimension $n = 5$ for such functions (65).

In the above discussion, we have required that K be positive everywhere. If K changes signs, what was missing is the control of solutions u near $K^{-1}(0)$. This has been settled for $n \geq 3$ by Chen and Li (26), who showed that if 0 is a regular value of K, the L^∞ norms of u near $K^{-1}(0)$ is uniformly bounded for all solutions u. Though the estimates were established on $\mathbb{S}^n$, the proof applies to locally conformally flat Riemannian manifolds. With the help of this result, one can extend a number of previously known results for positive K to include those K that change signs. For instance, one can extend the previously mentioned density result in ref. 20 to its full generality: C^∞ scalar curvature functions are dense in $C^{1,\alpha}$ $(0 < \alpha < 1)$ norms among functions that are positive somewhere. This is a generalization of the L^p density result in ref. 2 and the C^0 density result in ref. 20.

Another way to obtain the estimates of Chen and Li is to show that there is no positive solution to

$$-\Delta u = x_n u^{\frac{n+2}{n-2}}, \quad \text{in } \mathbb{R}^n.$$

This was established independently by Zhu (27) for even n, and by Lin (28) for all n.

2. The Yamabe Problem on Manifolds with Boundary

Let (M, g) be an n-dimensional $(n \geq 3)$ compact, smooth, Riemannian manifold without boundary. The well known Yamabe conjecture states that there exist metrics on M that are pointwise conformal to g and have constant scalar curvature. The Yamabe conjecture was proved through the work of Yamabe (29), Trudinger (30), Aubin (31), and Schoen (32). See Lee and Parker (33) for a survey. See also Bahri and Brezis (34), Bahri (35), and Schoen (36, 37) for works on the problem and related ones. Analogues of the Yamabe problem for compact Riemannian manifolds with boundary have been studied by Cherrier, Escobar, and others. In particular, Escobar proved that a large class of compact Riemannian manifolds with boundary are conformally equivalent to one with constant scalar curvature and zero mean curvature on the boundary. In the following, (M, g) denotes some smooth, compact, oriented, n-dimensional $(n \geq 3)$ Riemannian manifold with boundary, ν denotes the outward unit normal on ∂M, and h_g denotes the mean curvature of ∂M with respect to ν. We restrict ourselves to manifolds of positive type. This means that the sign of the first eigenvalue λ_1 of the conformal Laplacian with some natural geometric boundary condition is positive. Escobar proved (38) that for a large class of manifolds, there exists $\tilde{g}$, conformally equivalent to g, such that $R_{\tilde{g}} = 1$ in M and $h_{\tilde{g}} = 0$ on ∂M. He also showed (39) that for the same class of manifolds there exist $c_+ > 0$ and $c_- < 0$ and two conformal metrics $\tilde{g}_\pm$ such that $R_{\tilde{g}_\pm} = 1$ in M and $h_{\tilde{g}_\pm} = c_\pm$ on ∂M. Together with Z. C. Han, I proposed (40) the following:

Conjecture 2.1. *For all $c \in \mathbb{R}$, there exists $\tilde{g}$ conformal to g such that*

$$R_{\tilde{g}} = 1 \text{ in } M, \quad h_{\tilde{g}} = c \text{ on } \partial M.$$

Conjecture 2.2. *All solutions to the above stay in a compact set of $C^3(M)$, unless (M, g) is conformally equivalent to standard half spheres.*

Theorem 2.1 (40). Conjecture 2.1 *and* Conjecture 2.2 *hold when (M, g) is locally conformally flat with umbilic boundary.*

Theorem 2.2 (41). Conjecture 2.1 *holds when $n \geq 5$ and the boundary of M has at least one non-umbilic point.*

The boundary of M is called umbilic if the traceless part of the second fundamental form is identically zero on the boundary. A boundary point of M is called non-umbilic if the traceless part of the second fundamental form is nonzero at the point. One ingredient in the proof of *Theorem 1.1* is a Liouville-type theorem in $\mathbb{R}^n_+$ established in ref. 42. Such Liouville-type theorem in $\mathbb{R}^n$ is due to Caffarelli, Gidas, and Spruck (43), and, under an additional hypothesis $u(x) = O(|x|^{2-n})$ for large $|x|$, is due to Obata (44) and Gidas, Ni, and Nirenberg (45). Under the same additional decay hypothesis, the result in $\mathbb{R}^n_+$ is due to Escobar (46).

We have also proved *Conjecture 2.1* for $n = 3, 4, 5$ (Z. C. Han and Y.Y.L., unpublished work). For the Yamabe problem, the solutions of Schoen in low dimensions rely on a consequence of the Positive Mass theorem of Schoen and Yau (47, 48): In appropriate local coordinates, the expansion of the Green's function is of the form $|x|^{2-n} + A + O(|x|^\alpha)$ with constant $A \geq 0$ and $A = 0$ if and only if the manifold is conformally equivalent to standard spheres. For the Yamabe problem with boundary, new difficulties occur in low dimensions when the boundary has no umbilic points. In any local coordinates with $g_{ij}(0) = \delta_{ij}$ near a non-umbilic boundary point (identified as $x = 0$), the term after $|x|^{2-n}$ in the expansion of the Green's function is singular. For instance, in dimension $n = 3$, the expansion of the Green's function is of the form $|x|^{-1} + A(x) + O(|x|^\alpha)$, where $A(x)$ is homogeneous of degree 0, nonconstant, with no definite sign. Nevertheless, we obtain some kind of average sign condition on $A(x)$ and prove *Conjecture 2.1.*

3. The Best Sobolev Inequality on Riemannian Manifolds

It is well known that sharp Sobolev inequalities are important in the study of partial differential equations, especially in the study of those arising from geometry and physics. There has been much work on such inequalities and their applications. See, for example, the references in ref. 49.

For $n \geq 2$, $1 \leq p < n$, and $p^* = np/(n - p)$, let

$$\frac{1}{K(n, p)} = \inf\left\{\frac{\|\nabla u\|_{L^p(\mathbb{R}^n)}}{\|u\|_{L^{p^*}(\mathbb{R}^n)}} \;\middle|\; u \in L^{p^*}(\mathbb{R}^n)\backslash\{0\}, \nabla u \in L^p(\mathbb{R}^n)\right\}.$$

Aubin (50) and Talenti (51) calculated the value of the above Sobolev constant $K(n, p)$ and found the extremal functions.

Let (M^n, g) be a compact Riemannian manifold; it was shown (50) that for any $\varepsilon > 0$ there exists a constant $A_p(\varepsilon)$ [depending also on the manifold (M^n, g)] such that

$$\|\varphi\|_{L^{p^*}} \leq [K(n, p) + \varepsilon]\|\nabla\varphi\|_{L^p} + A_p(\varepsilon)\|\varphi\|_{L^p}, \quad \forall\, \varphi \in W^{1,p}(M^n),$$

and $K(n, p)$ is the smallest constant having this property. Aubin made a conjecture in ref. 52 concerning the following inequalities:

Conjecture. *There exist constants $A(p)$ such that any $\varphi \in W^{1,p}(M_n)$ satisfies*

$$\|\varphi\|^p_{L^{p^*}} \leq K(n, p)^p\|\nabla\varphi\|^p_{L^p} + A(p)\|\varphi\|^p_{L^p} \quad \text{if } 1 \leq p \leq 2, \qquad \textbf{[3.1]}$$

and

$$\|\varphi\|^{\frac{p}{p-1}}_{L^{p^*}} \leq K(n,p)^{\frac{p}{p-1}}\|\nabla\varphi\|^{\frac{p}{p-1}}_{L^p} + A(p)\|\varphi\|^{\frac{p}{p-1}}_{L^p} \quad \text{if } 2 < p < n. \qquad \textbf{[3.2]}$$

A stronger form of **3.2** is

$$\|\varphi\|^2_{L^{p^*}} \leq K(n, p)^2\|\nabla\varphi\|^2_{L^p} + A(p)\|\varphi\|^2_{L^p} \quad \text{if } 2 < p < n. \qquad \textbf{[3.3]}$$

The above conjecture was made because he proved these inequalities when the manifold is the standard n-sphere $\mathbb{S}^n$. He also proved that the best constant is achieved for manifolds of dimension two, and for manifolds of constant sectional curvature. Related problems on domains of $\mathbb{R}^n$ were studied by Brezis and Nirenberg (53), Brezis and Lieb (54), and Adimurthi and Yadava (55). Hebey and Vaugon (56, 57) proved inequality **3.1**.

Results on compact manifolds with boundaries, also for $p = 2$, were obtained by Li and Zhu (58, 59). Further results were given by Zhu (60, 61). Recently, Druet (62) has shown that inequality **3.1** is false for $4 < p^2 < n$ if the scalar curvature is positive somewhere. Then Aubin, Druet, and Hebey (63) proved that inequality **3.1** holds for all $p \in (1, n)$ on compact manifolds of dimension 2, 3, or 4 with nonpositive sectional curvature. Together with Aubin we established *Theorem 3.1.*

Theorem 3.1 (49). *Let (M_n, g) be a C^∞ compact Riemannian manifold. Then there exist constants $A(p)$, depending also on (M_n, g), such that for all $\varphi \in W^{1,p}(M_n, g)$, inequality* **3.1** *holds for all $1 < p \leq 2$, and inequality* **3.3** *holds for all $2 < p < n$.*

The larger the exponent of the norms is, the stronger is the inequality, so the conjecture for $1 < p < n$ follows from the above

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theorem. In fact we established stronger inequalities that require much more delicate analysis (see ref. 49 for details). *Theorem 3.1* was also independently obtained by Druet (64).

1. Kazdan, J. & Warner, F. (1975) *J. Differ. Geom.* **10,** 113–134.
2. Bourguignon, J. P. & Ezin, J. P. (1987) *Trans. Am. Math. Soc.* **301,** 723–736.
3. Chen, W. & Li, C. (1995) *Commun. Pure Appl. Math.* **48,** 657–667.
4. Han, Z. C. & Li, Y.Y. (1996) *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **13,** 541–597.
5. Moser, J. (1973) in *Dynamical Systems*, ed. Peixoto, M. (Academic, New York).
6. Escobar, J. F. & Schoen, R. (1986) *Invent. Math.* **86,** 243–254.
7. Bianchi, G. & Egnell, H. (1992) *Math. Z.* **210,** 137–166.
8. Bianchi, G. (1996) *Commun. Partial Differ. Equations* **21,** 229–234.
9. Chen, W. (1989) *Math. Ann.* **283,** 353–365.
10. Hebey, E. (1990) *Bull. Sci. Math.* **114,** 215–242.
11. Chang, S. Y. & Yang, P. (1988) *J. Differ. Geom.* **27,** 256–296.
12. Chang, S. Y. & Yang, P. (1987) *Acta Math.* **159,** 215–259.
13. Bahri, A. & Coron, J. M. (1991) *J. Funct. Anal.* **95,** 106–172.
14. Chang, S. Y., Gursky, M. J. & Yang, P. (1993) *Calc. Var. Partial Differ. Equations* **1,** 205–229.
15. Schoen, R. & Zhang, D. (1996) *Calc. Var. Partial Differ. Equations* **4,** 1–25.
16. Chang, S. Y. & Yang, P. (1991) *Duke Math. J.* **64,** 27–69.
17. Li, Y.Y. (1996) *Commun. Pure Appl. Math.* **49,** 541–597.
18. Chen, W. & Ding, W. (1987) *Trans. Am. Math. Soc.* **303,** 365–382.
19. Benayed, M., Chen, Y., Chtioui, H. & Hammami, M. (1996) *Duke Math. J.* **85,** 633–677.
20. Li, Y.Y. (1995) *J. Differ. Equations* **120,** 319–410.
21. Bahri, A. (1996) *Duke Math. J.* **81,** 323–466.
22. Aubin, T. & Bahri, A. (1997) *J. Math. Pures Appl.* **76,** 525–549.
23. Ambrosetti, A., Garcia Azorero, J. & Peral, I. (1999) *J. Funct. Anal.* **165,** 117–149.
24. Zhang, D. (1990) Ph.D. thesis (Stanford Univ., Stanford, CA).
25. Chen, C. C. & Lin, C. S. (1997) *Commun. Pure Appl. Math.* **50,** 971–1017.
26. Chen, W. & Li, C. (1997) *Ann. Math.* **145,** 547–564.
27. Zhu, M. (1999) *Proc. R. Soc. Edinburgh Sect. A.* **129,** 649–661.
28. Lin, C. S. (1998) *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **27,** 107–130.
29. Yamabe, H. (1960) *Osaka Math. J.* **12,** 21–37.
30. Trudinger, N. (1968) *Ann. Sc. Norm. Super. Cl. Sci.* **22,** 265–274.
31. Aubin, T. (1976) *J. Math. Pures Appl.* **55,** 269–296.
32. Schoen, R. (1984) *J. Differ. Geom.* **20,** 479–495.
33. Lee, J. & Parker, T. (1987) *Bull. Am. Math. Soc.* **17,** 37–91.
34. Bahri, A. & Brezis, H. (1996) *Prog. Nonlinear Differ. Equations Appl.* **20,** 1–100.
35. Bahri, A. (1995) *Pitman Res. Notes Math. Ser.* **320,** 13–43.
36. Schoen, R. (1989) *Lect. Notes Math.* **1365,** 120–154.
37. Schoen, R. (1991) in *Differential Geometry: A Symposium in Honor of Manfredo Do Carmo*, eds. Lawson, H. B. & Tenenblat, K. (Wiley, New York), pp. 311–320.
38. Escobar, J. F. (1992) *J. Differ. Geom.* **35,** 21–84.
39. Escobar, J. F. (1996) *Indiana Univ. Math. J.* **45,** 917–943.
40. Han, Z. C. & Li, Y.Y. (1999) *Duke Math. J.* **99,** 489–542.
41. Han, Z. C. & Li, Y.Y. (2000) *Commun. Anal. Geom.* **8,** 809–869.
42. Li, Y.Y. & Zhu, M. (1995) *Duke Math. J.* **80,** 383–417.
43. Caffarelli, L., Gidas, B. & Spruck, J. (1989) *Commun. Pure Appl. Math.* **42,** 271–297.
44. Obata, M. (1971) *J. Differ. Geom.* **6,** 247–258.
45. Gidas, B., Ni, W. M. & Nirenberg, L. (1979) *Commun. Math. Phys.* **68,** 209–243.
46. Escobar, J. F. (1990) *Commun. Pure Appl. Math.* **43,** 857–883.
47. Schoen, R. & Yau, S. T. (1979) *Commun. Math. Phys.* **65,** 45–76.
48. Schoen, R. & Yau, S. T. (1988) *Invent. Math.* **92,** 47–71.
49. Aubin, T. & Li, Y.Y. (1999) *J. Math. Pures Appl.* **78,** 353–387.
50. Aubin, T. (1976) *Bull. Sci. Math.* **100,** 149–173.
51. Talenti, G. (1976) *Ann. Mat. Pura Appl.* **110,** 353–372.
52. Aubin, T. (1976) *J. Differ. Geom.* **11,** 573–598.
53. Brezis, H. & Nirenberg, L. (1983) *Commun. Pure Appl. Math.* **36,** 437–477.
54. Brezis, H. & Lieb, E. H. (1985) *J. Funct. Anal.* **62,** 73–86.
55. Adimurthi & Yadava, S. L. (1994) *Calc. Var. Partial Differ. Equations* **2,** 427–442.
56. Hebey, E. & Vaugon, M. (1996) *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **13,** 57–93.
57. Hebey, E. & Vaugon, M. (1995) *Duke Math. J.* **79,** 235–279.
58. Li, Y.Y. & Zhu, M. (1997) *Commun. Pure Appl. Math.* **50,** 449–487.
59. Li, Y.Y. & Zhu, M. (1998) *Geom. Funct. Anal.* **8,** 59–87.
60. Zhu, M. (1998) *J. Funct. Anal.* **156,** 75–120.
61. Zhu, M. (1999) *Calc. Var. Partial Differ. Equations* **8,** 27–43.
62. Druet, O. (1998) *J. Funct. Anal.* **159,** 217–242.
63. Aubin, T., Druet, O. & Hebey, E. (1998) *C. R. Acad. Sci. Paris* **326,** 1117–1121.
64. Druet, O. (1999) *Math. Ann.* **314,** 327–346.
65. Chen, C. C. & Lin, C. S., *On Prescribing Scalar Curvature on S^n, Part 2: Counting the Leray–Schauder Degree*, preprint.