

A PRIORI BOUNDS FOR CO-DIMENSION ONE ISOMETRIC EMBEDDINGS

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ABSTRACT. Let $\mathbf{X}: (\mathbb{S}^n, g) \rightarrow \mathbb{R}^{n+1}$ be a \mathcal{C}^4 isometric embedding of a \mathcal{C}^4 metric g of non-negative sectional curvature on \mathbb{S}^n into the Euclidean space \mathbb{R}^{n+1} . We prove a priori bounds for the trace of the second fundamental form H , in terms of the scalar curvature R of g , and the diameter d of the space (\mathbb{S}^n, g) . These estimates give a bound on the extrinsic geometry in terms of intrinsic quantities. They generalize estimates originally obtained by Weyl for the case $n = 2$ and positive curvature, and then by P. Guan and the first author for non-negative curvature and $n = 2$. Using $\mathcal{C}^{2,\alpha}$ interior estimates of Evans and Krylov for concave fully nonlinear elliptic partial differential equations, these bounds allow us to obtain the following convergence theorem: For any $\epsilon > 0$, the set of metrics of non-negative sectional curvature and scalar curvature bounded below by ϵ which are isometrically embeddable in Euclidean space \mathbb{R}^{n+1} is closed in the Hölder space $\mathcal{C}^{4,\alpha}$, $0 < \alpha < 1$. These results are obtained in an effort to understand the following higher dimensional version of the Weyl embedding problem which we propose: *Suppose that g is a smooth metric of non-negative sectional curvature and positive scalar curvature on \mathbb{S}^n which admits locally convex isometric embedding into \mathbb{R}^{n+1} . Does (\mathbb{S}^n, g) then admit a smooth global isometric embedding $\mathbf{X}: (\mathbb{S}^n, g) \rightarrow \mathbb{R}^{n+1}$?*

1. INTRODUCTION

In 1916, H. Weyl posed the following problem: *Given a metric g of positive Gauss curvature on the sphere \mathbb{S}^2 , is there an embedding $\mathbf{X}: \mathbb{S}^2 \rightarrow \mathbb{R}^3$ such that the metric induced on \mathbb{S}^2 by this embedding is g ?* Such an embedding $\mathbf{X}: (\mathbb{S}^2, g) \rightarrow \mathbb{R}^3$ is called *isometric*, and satisfies the following system of nonlinear partial differential equations:

$$(1) \quad \nabla_i X \cdot \nabla_j X = g_{ij}.$$

In [W], Weyl suggested the continuity method to attack the problem and obtained a priori estimates up to the second derivatives of the embedding. The a priori estimate of the second derivative is a consequence of the Weyl

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inequality which gives a bound of the mean curvature by intrinsic quantities for strictly convex closed surfaces; see Theorem 2. The main obstacle to the solution was the lack of $\mathcal{C}^{2,\alpha}$ a priori estimates for the embedding.

Later H. Lewy solved the problem under the assumption that the metric g is real analytic; see [L]. It is interesting to point out that Lewy did not use Weyl's a priori estimate. In [N2], L. Nirenberg gave a beautiful proof for any metric g of class \mathcal{C}^4 . He established, among other things, the $\mathcal{C}^{2,\alpha}$ a priori estimate of the embedding using strong a priori estimates he derived earlier for solutions of fully non-linear elliptic partial differential equations in two variables; see [N1]. An entirely different approach was taken independently by A. D. Alexandroff and A. V. Pogorelov; see [Al, P1, P2]. The Weyl estimate was later generalized to the case of non-negative curvature by P. Guan and Y. Li in [GL]. From their estimate, they obtained a $\mathcal{C}^{1,1}$ embedding result for metrics of non-negative Gauss curvature; see also [HZ] for a different approach to the $\mathcal{C}^{1,1}$ embedding result.

The main result of this paper is a Weyl-type estimate, see Theorem 1 below, which generalizes to higher dimensions the estimate of P. Guan and the first author. The theorem asserts that for any convex closed hypersurface in \mathbb{R}^{n+1} , ($n \geq 2$), one can bound the mean curvature H in terms of the scalar curvature of the induced metric g , its Laplacian, and the diameter.

Denote by $\mathcal{M}^k(\mathbb{S}^n)$ the space of metrics g of non-negative sectional curvature on \mathbb{S}^n which have k continuous derivatives, and by $\mathcal{M}_+^k(\mathbb{S}^n) \subset \mathcal{M}^k(\mathbb{S}^n)$ the subset consisting of those metrics which have positive sectional curvature. Similarly, if $0 < \alpha < 1$, we denote by $\mathcal{M}^{k,\alpha}(\mathbb{S}^n) \subset \mathcal{M}^k(\mathbb{S}^n)$ the space of metrics $g \in \mathcal{M}^k(\mathbb{S}^n)$ whose k -th derivatives are Hölder continuous, and by $\mathcal{M}_+^{k,\alpha}(\mathbb{S}^n) \subset \mathcal{M}^{k,\alpha}(\mathbb{S}^n)$ the subset consisting of those metrics which have positive sectional curvature. For convenience, we will denote $\mathcal{M}^{k,0}(\mathbb{S}^n) = \mathcal{M}^k(\mathbb{S}^n)$, $\mathcal{M}_+^{k,0}(\mathbb{S}^n) = \mathcal{M}_+^k(\mathbb{S}^n)$, and adopt the same convention for spaces of functions, i.e., $\mathcal{C}^{k,0} = \mathcal{C}^k$.

Theorem 1. *Let $g \in \mathcal{M}^4(\mathbb{S}^n)$, and let $\mathbf{X}: (\mathbb{S}^n, g) \rightarrow \mathbb{R}^{n+1}$ be a \mathcal{C}^4 isometric embedding. Let H be the trace of the second fundamental form of \mathbf{X} , and let R be the scalar curvature of g . Then the following inequality holds*

$$(2) \quad H^2 \leq C d^2 \sup_{\mathbb{S}^n} \left(2R^2 - \Delta R + \frac{(n-1)^2}{64d^2} R \right),$$

where $C = 4(n-1)^{-2}e^{(n-1)/4}$, and d is the diameter of (\mathbb{S}^n, g) .

Remark 1. The above bound of H involves four derivatives of g . It is not known whether it is possible to bound H by quantities involving only up to third derivatives of g . If the sectional curvature of g is strictly positive and $n \geq 3$, then we can bound H by quantities involving only up to second derivatives of g . See Remark 2 below and Theorem 6 for details.

As the first step in establishing Theorem 1, we show that for any convex closed surface with positive scalar curvature in \mathbb{R}^{n+1} ($n \geq 2$), one can bound

the mean curvature H in terms of the scalar curvature of the induced metric g and its Laplacian. This is a direct generalization of the Weyl estimate.

Theorem 2. *Let $g \in \mathcal{M}^4(\mathbb{S}^n)$, and let $\mathbf{X}: (\mathbb{S}^n, g) \rightarrow \mathbb{R}^{n+1}$ be a \mathcal{C}^4 isometric embedding. Let H be the trace of the second fundamental form of \mathbf{X} , and let R be the scalar curvature of g . Suppose that $R > 0$. Then the following inequality holds:*

$$(3) \quad H^2 \leq \sup_{\mathbb{S}^n} \left(2R - \frac{1}{R} \Delta R \right).$$

Remark 2. An estimate similar to (3) under the stronger hypothesis that g has positive sectional curvature was established in [Y1]. In Theorem 6 we show that when g has positive sectional curvature and the dimension $n \geq 3$, H can be bounded in terms only of the lower bound of the sectional curvature and the upper bound of the Ricci curvature.

Using $\mathcal{C}^{2,\alpha}$ interior estimates of Evans and Krylov for concave fully non-linear elliptic partial differential equations, see [Ev, K1, K2, CC], the a priori bound in Theorem 2 allows us to obtain the following convergence theorem.

Theorem 3. *For any $\epsilon > 0$, the set of metrics of non-negative sectional curvature and scalar curvature bounded below by ϵ which are isometrically embeddable in Euclidean space \mathbb{R}^{n+1} is closed in the Hölder space $\mathcal{C}^{4,\alpha}$, $0 < \alpha < 1$.*

We note that $\mathcal{C}^{2,\alpha}$ estimates up to the boundary for concave fully nonlinear elliptic partial differential equations were independently established by Caffarelli, Nirenberg, and Spruck [CNS1], and Krylov [K2].

In [Y2, Problem 53], S. T. Yau posed the following problem: *Can one generalize Weyl's embedding problem to higher dimensions? More precisely, given a compact n -dimensional Riemannian manifold (M, g) of positive sectional curvature, is there an isometric immersion $\mathbf{X}: (M, g) \rightarrow \mathbb{R}^{n(n+1)/2}$?* The Cartan-Janet dimension $n(n+1)/2$ is the smallest so that formally, e.g., in the analytic class, the problem has a local solution, regardless of curvature; see [J]. In this direction, see also [BEG, BGY], and the references therein.

Here, we wish to consider a different generalization of the Weyl embedding problem: *Can one give an intrinsic characterization of those metrics g with non-negative sectional curvatures on \mathbb{S}^n for which there is a co-dimension one isometric embedding $\mathbf{X}: (\mathbb{S}^n, g) \rightarrow \mathbb{R}^{n+1}$?* Note that a necessary condition for such an embedding to exist is that g admits locally convex isometric embedding into \mathbb{R}^{n+1} . In dimension $n \geq 3$, this is a non-trivial restriction. We formulate the following conjecture

Conjecture. *Let g be a smooth metric of non-negative sectional curvature and positive scalar curvature on \mathbb{S}^n which admits locally convex isometric embedding into \mathbb{R}^{n+1} . Then (\mathbb{S}^n, g) admits a smooth global isometric embedding $\mathbf{X}: (\mathbb{S}^n, g) \rightarrow \mathbb{R}^{n+1}$.*

The estimates we prove here are obtained in an effort to confirm the conjecture. When $n \geq 3$ and g has positive sectional curvature, any local isometric embedding is known to be rigid, see Section 5. Therefore the existence of a global immersion follows by a standard monodromy argument from local embeddability and the fact that \mathbb{S}^n is simply connected. The immersion has to be an embedding due to a theorem of Hardamard, see [S, Theorem 2.11 on page 94] for a proof in the 2-dimensional case. The argument actually applies in any dimension. Thus, in view of Theorem 3, the conjecture would be confirmed if one could approximate any metric g of non-negative curvature and positive scalar curvature which admits locally convex isometric embedding into \mathbb{R}^{n+1} , by a sequence g_i of metrics of positive curvature which are also locally isometrically embeddable in \mathbb{R}^{n+1} . Conversely, assuming the conjecture holds, any metric of non-negative curvature and positive scalar curvature which admits locally convex embedding can be approximated by embeddable metrics of positive curvature as follows from the following simple proposition:

Proposition 1. *Let $g \in \mathcal{M}^{k,\alpha}(\mathbb{S}^n)$ be isometrically embeddable in \mathbb{R}^{n+1} , then there is a sequence $g_i \in \mathcal{M}_+^{k,\alpha}(\mathbb{S}^n)$ isometrically embeddable in \mathbb{R}^{n+1} which converges to g .*

Proof. Let $\mathbf{X}: (\mathbb{S}^n, g) \rightarrow \mathbb{R}^{n+1}$ be an isometric embedding. We can write \mathbf{X} as a graph over \mathbb{S}^n :

$$\mathbf{X} = \rho \mathbf{x}, \quad \mathbf{x} \in \mathbb{S}^n, \quad \rho = \rho(\mathbf{x}) > 0.$$

Then the first and second fundamental forms are given by:

$$\begin{aligned} g_{ij} &= \rho^2 \gamma_{ij} + \rho_i \rho_j \\ \chi_{ij} &= \rho^2 \gamma_{ij} + 2\rho_i \rho_j - \rho \rho_{ij}, \end{aligned}$$

where γ_{ij} is the standard metric on \mathbb{S}^n , and the subscripts denote covariant derivatives with respect to γ , see [GS]. If we substitute $\rho = u^{-1}$, then we have for the second fundamental form:

$$\chi_{ij} = u^{-3}(u\gamma_{ij} + u_{ij}).$$

Since g has non-negative curvature, χ_{ij} is positive semi-definite, see Section 2. Hence, we have:

$$(u\gamma_{ij} + u_{ij}) \geq 0.$$

If we set $u^\epsilon = u + \epsilon$, then we get:

$$\chi_{ij}^\epsilon = (u^\epsilon)^{-3}(u\gamma_{ij} + u_{ij} + \epsilon\gamma_{ij}) > 0,$$

Hence the first fundamental forms g_{ij}^ϵ of $\mathbf{X}^\epsilon = (u^\epsilon)^{-1} \mathbf{x}$ have positive curvature. Clearly, g_{ij}^ϵ converge to g as $\epsilon \rightarrow 0$. \square

The paper is organized as follows. First, in Section 2, we prove Theorem 2. In Section 3, we prove Theorem 3. Then, in Section 4, we prove Theorem 1. Finally, in Section 5, we show that the once-contracted Gauss equations can

be solved for the second fundamental form, when the sectional curvature of g is positive. As a corollary, we obtain a priori bounds on the second fundamental form and hence also on the second derivatives of the embedding, which depend only on two derivatives of g . This also provides, when $n = 3$, a local explicit criterion for any metric g of positive sectional curvature on \mathbb{S}^3 to be locally isometrically embeddable in \mathbb{R}^4 .

2. WEYL ESTIMATES IN HIGHER DIMENSION

In this section, we will prove Theorem 2, a higher dimensional analogue of the Weyl estimate [W]. However, we first note that Theorem 2 allows us to establish a priori bounds on the second covariant derivatives of \mathbf{X} .

Corollary 1. *Let $g \in \mathcal{M}^4(\mathbb{S}^n)$, and let $\mathbf{X}: (\mathbb{S}^n, g) \rightarrow \mathbb{R}^{n+1}$ be a \mathcal{C}^4 isometric embedding. Assume that R , the scalar curvature of g , is positive. Then the following inequality holds:*

$$|\nabla^2 \mathbf{X}|^2 \leq \sup_{\mathbb{S}^n} \left(2R - \frac{1}{R} \Delta R \right).$$

Proof. We have:

$$(4) \quad \mathbf{X}_{;ij} = -\chi_{ij} \mathbf{N},$$

where \mathbf{N} is the outer unit normal, and χ_{ij} the second fundamental form of \mathbf{X} . Thus, we obtain:

$$|\nabla^2 \mathbf{X}|^2 = \mathbf{X}_{;ij} \cdot \mathbf{X}_{;ij} = \chi^{ij} \chi_{ij}.$$

Since $\chi^{ij} \chi_{ij} \leq H^2$, the corollary follows from Theorem 2. \square

We will use the Gauss and Codazzi Equations:

$$(5) \quad R_{ijkl} = \chi_{ik} \chi_{jl} - \chi_{il} \chi_{jk}$$

$$(6) \quad 0 = \chi_{ij;k} - \chi_{ik;j},$$

where R_{ijkl} is the Riemann curvature tensor of g . An immediate consequence of (5) is that if g has positive sectional curvatures, then χ definite. In view of our choice of normal (4), χ is positive definite. Furthermore, it follows from theorems of Hadamard, Chern and Lashof [CL], and R. Sacksteder [Sa] that if g has non-negative sectional curvature, and $\mathbf{X}: (\mathbb{S}^n, g) \rightarrow \mathbb{R}^{n+1}$ is an isometric immersion, then \mathbf{X} is an embedding, and $\mathbf{X}(\mathbb{S}^n)$ is the boundary of a convex body in \mathbb{R}^{n+1} . In particular, if g has non-negative sectional curvature, then χ is positive semi-definite.

We begin the proof of Theorem 2 with two lemmas.

Lemma 1. *Let $\mathbf{x} \in \mathbb{R}_+^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_i > 0\}$. Then*

$$3 \left(\sum x_i^2 \right) \left(\sum x_i \right) \leq \left(\sum x_i \right)^3 + 2 \sum x_i^3.$$

Proof. This can be proved by induction. However, we rather calculate directly:

$$\left(\sum x_i\right)^3 = \sum_{|\alpha|=3} \binom{3}{\alpha} \mathbf{x}^\alpha = \sum_{\substack{|\alpha|=3 \\ \alpha^*=3}} \mathbf{x}^\alpha + 3 \sum_{\substack{|\alpha|=3 \\ \alpha^*=2}} \mathbf{x}^\alpha + 6 \sum_{\substack{|\alpha|=3 \\ \alpha^*=1}} \mathbf{x}^\alpha,$$

Here $\alpha = (\alpha_1, \dots, \alpha_n)$ stands for a multi-index, with α_i nonnegative integers, $|\alpha| = \sum \alpha_i$, and $\alpha^* = \max \alpha_i$. The multinomial coefficients are defined for $|\alpha| = k$ as

$$\binom{k}{\alpha} = \frac{k!}{\alpha!} = \frac{k!}{\alpha_1! \dots \alpha_n!},$$

and $\mathbf{x}^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ for $\mathbf{x} \in \mathbb{R}^n$. Furthermore, we have:

$$\left(\sum x_i^2\right) \left(\sum x_i\right) = \sum_{\substack{|\alpha|=2, \alpha^*=2 \\ |\beta|=1}} \mathbf{x}^{\alpha+\beta} = \sum_{\substack{|\alpha|=3 \\ \alpha^*=3}} \mathbf{x}^\alpha + \sum_{\substack{|\alpha|=3 \\ \alpha^*=2}} \mathbf{x}^\alpha.$$

Combining these, we conclude:

$$3 \left(\sum x_i^2\right) \left(\sum x_i\right) - \left(\sum x_i\right)^3 - 2 \sum x_i^3 = -6 \sum_{\substack{|\alpha|=3 \\ \alpha^*=1}} \mathbf{x}^\alpha \leq 0. \quad \square$$

Note that for $n = 1, 2$, the lemma becomes an identity. As a corollary, we obtain the following lemma.

Lemma 2. *Let A be a symmetric, positive-definite $n \times n$ matrix. Then we have:*

$$(\operatorname{tr} A)^3 - \operatorname{tr} A^3 \leq \frac{3}{2} \left[(\operatorname{tr} A)^2 - \operatorname{tr} A^2 \right] \operatorname{tr} A.$$

Proof. Let $x_1, x_2, \dots, x_n \geq 0$ be the eigenvalues of A . Then, by Lemma 1, we have:

$$\begin{aligned} 2 \left[(\operatorname{tr} A)^3 - \operatorname{tr} A^3 \right] &= 2 \left(\sum x_i \right)^3 - 2 \sum x_i^3 \\ &\leq 3 \left[\left(\sum x_i \right)^2 - \left(\sum x_i^2 \right) \right] \sum x_i \\ &= 3 \left[(\operatorname{tr} A)^2 - \operatorname{tr} A^2 \right] \operatorname{tr} A. \quad \square \end{aligned}$$

Proof of Theorem 2. Suppose that H achieves its maximum at $p \in \mathbb{S}^n$, then at p , we have:

$$H_i = 0, \quad H_{;ij} \leq 0.$$

Since $(\chi^{ij}) \geq 0$ and $(H_{;ij}) \leq 0$, it follows with some linear algebra that

$$(7) \quad H \Delta H \leq \chi^{ij} H_{;ij}$$

Now, equation (5) implies that $R = H^2 - \text{tr} \chi^2$, where $\text{tr} \chi^2 = \chi_{ij} \chi^{ij}$. We also write $\text{tr} \chi^3 = \chi^{ij} \chi_{jk} \chi^k{}_i$. In view of (7), (6), (5), and Lemma 2, we now find that at p , the following holds:

$$\begin{aligned}
\Delta R &= 2H\Delta H + 2|\nabla H|^2 - 2|\nabla \chi|^2 - 2\chi^{ij}\Delta\chi_{ij} \\
&\leq 2\chi^{ij} \left(\chi^k{}_{k;ji} - \chi_{ij}{}^k{}_k \right) \\
&= 2\chi^{ij} \left(\chi^k{}_{j;ki} - \chi_j{}^k{}_{;ik} \right) \\
&= 2\chi^{ij} \left(R_{jlik} \chi^{lk} + R^k{}_{lik} \chi_j{}^l \right) \\
(8) \quad &= 2\chi^{ij} \left[(\chi_{ji} \chi_{lk} - \chi_{jk} \chi_{li}) \chi^{lk} + (\chi^k{}_i \chi_{lk} - \chi^k{}_k \chi_{li}) \chi_j{}^l \right] \\
&= 2 \left[(\text{tr} \chi^2)^2 - \text{tr} \chi^3 \text{tr} \chi \right] \\
&\leq 2 \left[-2RH^2 + R^2 + \frac{3}{2} \left((\text{tr} \chi)^2 - \text{tr} \chi^2 \right) (\text{tr} \chi)^2 \right] \\
&= -RH^2 + 2R^2.
\end{aligned}$$

It follows that

$$H^2 \leq H^2(p) \leq 2R(p) - \frac{1}{R(p)} \Delta R(p) \leq \sup_{\mathbb{S}^n} \left(2R - \frac{1}{R} \Delta R \right). \quad \square$$

3. A CONVERGENCE THEOREM

In this section we show how to obtain $\mathcal{C}^{2,\alpha}$ a priori bounds from Theorem 2. Bounds of this type imply Theorem 3.

Fix a finite covering $\{V_r\}$ of \mathbb{S}^3 by coordinate charts, and let $\{U_r\}$ be a refinement such that $\overline{U_r} \subset V_r$. Define for any $\mathcal{C}^{k,\alpha}$ tensor field T of rank l on \mathbb{S}^3 the norms:

$$\begin{aligned}
\|T\|_k &= \max_r \max_{\substack{0 \leq |\beta| \leq k \\ 1 \leq i_1, \dots, i_l \leq n}} \sup_{x \in U_r} \left| \partial^\beta T_{i_1 \dots i_l}(x) \right| \\
[T]_{k,\alpha} &= \max_r \sup_{\substack{x, y \in U_r \\ |\beta|=k}} \frac{|\partial^\beta T_{i_1 \dots i_l}(x) - \partial^\beta T_{i_1 \dots i_l}(y)|}{\text{dist}(x, y)^\alpha} \\
\|T\|_{k,\alpha} &= \max \left\{ \|T\|_k, [T]_{k,\alpha} \right\}.
\end{aligned}$$

where $T_{i_1 \dots i_l}$ are the components of T in the coordinate chart on U_r . Endow the space of $\mathcal{C}^{k,\alpha}$ tensor fields of rank l over \mathbb{S}^3 with this norm.

Theorem 4. *Let $g \in \mathcal{M}^{k,\alpha}(\mathbb{S}^n)$ for some $k \geq 4$ and $0 < \alpha < 1$, and let $\mathbf{X}: (\mathbb{S}^n, g) \rightarrow \mathbb{R}^{n+1}$ be a $\mathcal{C}^{k,\alpha}$ isometric embedding. Suppose that the scalar curvature R of g is positive. Then, there is $\mathbf{X}_0 \in \mathbb{R}^{n+1}$, and a constant $C > 0$ depending only on $\|g\|_{k,\alpha}$, and $\min_{\mathbb{S}^n} R$ such that*

$$(9) \quad \|\mathbf{X} - \mathbf{X}_0\|_{k,\alpha} \leq C.$$

Proof. It follows from the earlier mentioned theorems of Hadamard, Chern and Lashof, and R. Sacksteder that $\mathbf{X}(\mathbb{S}^n)$ is the boundary of an open convex set. (\mathbb{S}^n, g) has bounded diameter which implies that we have a bound on $|\mathbf{X} - \mathbf{X}_0|$ provided that \mathbf{X}_0 is chosen inside $\mathbf{X}(\mathbb{S}^n)$. Next, from (1), we get that $|\nabla_i \mathbf{X}|^2 = g_{ii}$ which clearly implies that $|\nabla \mathbf{X}|$ is bounded. Corollary 1 implies bounds on the second covariant derivatives of \mathbf{X} depending only on $\|g\|_4$ and $\min_{\mathbb{S}^n} R$. Since $\partial_i \partial_j \mathbf{X} = \nabla_i \nabla_j \mathbf{X} - \Gamma_{ij}^k \nabla_k \mathbf{X}$, we get bounds on the second coordinate derivatives of \mathbf{X} . Next, we establish Hölder bounds on the second derivatives of \mathbf{X} . To proceed, we must show that $\mathbf{X}_0 \in \mathbb{R}^{n+1}$ can be chosen so that there is a constant $C > 0$, depending only on $\|g\|_4$, such that

$$(10) \quad (\mathbf{X} - \mathbf{X}_0) \cdot \mathbf{N} \geq \frac{1}{C},$$

where \mathbf{N} is the outer unit normal to \mathbf{X} .

Let $K \subset \mathbb{R}^{n+1}$ be the closed convex set bounded by $\mathbf{X}(\mathbb{S}^n)$, and choose \mathbf{X}_0 so that $B_r(\mathbf{X}_0)$ is a ball of largest radius enclosed in K . We claim that r is bounded below by $1/C$, where C is a constant which depends only on $\|g\|_4$. Indeed, were this not the case, then there would be a family $g^\epsilon \in \mathcal{M}^{4,\alpha}(\mathbb{S}^n)$ with $\|g^\epsilon\|_4$ uniformly bounded, and isometric embeddings $\mathbf{X}^\epsilon: (\mathbb{S}^n, g^\epsilon) \rightarrow \mathbb{R}^{n+1}$ such that the largest ball B_{r^ϵ} contained in the closed convex set K^ϵ bounded by $\mathbf{X}^\epsilon(\mathbb{S}^n)$ has $r^\epsilon \rightarrow 0$, as $\epsilon \rightarrow 0$. This leads to a contradiction as follows. By the Ascoli-Arzelà Theorem, there is a subsequence such that $g^{\epsilon_j} \rightarrow g$ in $\mathcal{C}^{3,\beta}$ for any $0 < \beta < 1$. Furthermore, by our argument above, we have uniform bounds on $\|\mathbf{X}^{\epsilon_j}\|_2$, hence, perhaps along a further subsequence, $\mathbf{X}^{\epsilon_j} \rightarrow \mathbf{X}$ in $\mathcal{C}^{1,\beta}$. Note that \mathbf{N}^{ϵ_j} , the outer unit normals to \mathbf{X}^{ϵ_j} , converge in \mathcal{C}^β to \mathbf{N} , the outer unit normal to \mathbf{X} . Indeed, \mathbf{N}^{ϵ_j} is the exterior product of the n vectors $\mathbf{X}_{;1}^{\epsilon_j}, \dots, \mathbf{X}_{;n}^{\epsilon_j}$ divided by $\det(g^{\epsilon_j})$. Since, considering the normals as maps $\mathbf{N}^{\epsilon_j}: \mathbb{S}^n \rightarrow \mathbb{S}^n$, we have $\deg(\mathbf{N}^{\epsilon_j}) = 1$, we obtain $\deg(\mathbf{N}) = 1$. Now, the limit of K^{ϵ_j} is a closed convex set K contained in a hyperplane. Otherwise, there would be $n+1$ points in general position in K , which would imply that K contained the simplex spanned by these $n+1$ points, which would contradict $r^{\epsilon_j} \rightarrow 0$. However, $\mathbf{X}(\mathbb{S}^n)$ is contained in the same hyperplane, hence the image of $\mathbf{N}: \mathbb{S}^n \rightarrow \mathbb{S}^n$ consists of at most 2 points, which contradicts $\deg(\mathbf{N}) = 1$. Inequality (10) now follows easily, for if Π_p is the tangent plane at $p \in \mathbb{S}^n$, then

$$\text{dist}(p, \Pi_p) = (\mathbf{X} - \mathbf{X}_0) \cdot \mathbf{N}|_p.$$

Since K lies on one side of Π_p , and $B_{1/C}(\mathbf{X}_0) \subset K$, we have $\text{dist}(p, \Pi_p) \geq 1/C$, and (10) is established. We now assume without loss of generality that $\mathbf{X}_0 = \mathbf{0}$.

Define the function $\rho = \frac{1}{2} \mathbf{X} \cdot \mathbf{X}$. Then we have:

$$(11) \quad \rho_{;ij} = g_{ij} - (\mathbf{X} \cdot \mathbf{N}) \chi_{ij},$$

and consequently:

$$(12) \quad \chi_{ij} = \frac{1}{\mathbf{X} \cdot \mathbf{N}} (g_{ij} - \rho_{;ij}).$$

Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ be the eigenvalues of $g_{ij} - \rho_{;ij}$ (with respect to g_{ij}), and define for $\mathbf{x} \in \mathbb{R}^n$:

$$f(\mathbf{x}) = \sum_{1 \leq i < j \leq n} x_i x_j.$$

The function f is the second symmetric elementary function $\sigma_{2,n}$; see Section 5. It follows from equation (12) that

$$(13) \quad f(\boldsymbol{\lambda})^{1/2} = 2^{-1/2} (\mathbf{X} \cdot \mathbf{N}) R^{1/2}.$$

Let Γ_2 denote the connected component of the set $\{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) > 0\}$ containing the positive cone. It follows from [CNS2] that Γ_2 is a cone with the property that:

$$\begin{aligned} \frac{\partial}{\partial x_i} (f(\mathbf{x})^{1/2}) &> 0, \quad \forall \mathbf{x} \in \Gamma_2, \forall 1 \leq i \leq n; \\ \left(\frac{\partial^2}{\partial x_i \partial x_j} (f(\mathbf{x})^{1/2}) \right) &\leq 0, \quad \forall \mathbf{x} \in \Gamma_2, \end{aligned}$$

see also [G]. Now, in view of our \mathcal{C}^0 bound, (10), and our hypothesis on R , we have an estimate:

$$\frac{1}{C} \leq (\mathbf{X} \cdot \mathbf{N}) R^{1/2} \leq C,$$

hence the eigenvalues $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ of $g_{ij} - \rho_{;ij}$ with respect to g_{ij} remain within a fixed compact set of Γ_2 . We conclude that Equation (13), as an equation in the Hessian $\rho_{;ij}$ of ρ , is uniformly elliptic and convex in $\rho_{;ij}$. As we already have estimates on $\|\rho\|_2$, it now follows from [CC, Theorem 6.6 and Theorem 8.1] that there is a constant C depending only on $\|g\|_{4,\alpha}$ and $\min_{\mathbb{S}^2} R$ such that:

$$(14) \quad \|\rho\|_{2,\alpha} \leq C;$$

see also [Ev] and [K1]. Once we have established (14), we can apply Schauder estimates to (13) to get

$$(15) \quad \|\rho\|_{k,\alpha} \leq C.$$

In view of (4) and (12), we have:

$$\mathbf{X}_{;ij} = -\frac{1}{\mathbf{X} \cdot \mathbf{N}} (g_{ij} - \rho_{;ij}) \mathbf{N}.$$

Thus, (15) implies (9). □

Proof of Theorem 3. Follows directly from Theorem 4. □

4. WEYL-TYPE ESTIMATES FOR NON-NEGATIVE SCALAR CURVATURE

In this section, we prove Theorem 1. The proof of Theorem 1 is similar to that of the Proposition in [GL], using (8) instead of the Weyl estimate. We first point out a consequence of Theorem 1 which is similar to Corollary 1.

Corollary 2. *Let $g \in \mathcal{M}^4(\mathbb{S}^n)$, and let $\mathbf{X}: (\mathbb{S}^n, g) \rightarrow \mathbb{R}^{n+1}$ be a \mathcal{C}^4 isometric embedding. Let R be the scalar curvature of g . Then the following inequality holds:*

$$|\nabla^2 \mathbf{X}|^2 \leq C d^2 \sup_{\mathbb{S}^n} \left(2R^2 - \Delta R + \frac{(n-1)^2}{64d^2} R \right),$$

where $C = 4(n-1)^{-2} e^{(n-1)/4}$, and d is the diameter of (\mathbb{S}^n, g) .

The proof of Corollary 2 is the same as that of Corollary 1, using Theorem 1 instead of Theorem 2.

Proof of Theorem 1. Define $f = e^{\alpha\rho} H$, where $\alpha > 0$ is a constant to be determined later, and as before $\rho = \frac{1}{2} \mathbf{X} \cdot \mathbf{X}$. Suppose that f achieves its maximum at $p \in \mathbb{S}^n$, then at p , we have:

$$(16) \quad f_i = \alpha \rho_i H + H_i = 0,$$

$$(17) \quad f_{;ij} = e^{\alpha\rho} (\alpha H \rho_{;ij} - \alpha^2 H \rho_i \rho_j + H_{;ij}) \leq 0.$$

Furthermore, note that

$$\mathbf{X} = g^{ij} \rho_i \mathbf{X}_j + (\mathbf{X} \cdot \mathbf{N}) \mathbf{N}.$$

This is verified by taking inner product with the $n+1$ independent vectors $\mathbf{X}_{;1}, \dots, \mathbf{X}_{;n}, \mathbf{N} \in \mathbb{R}^{n+1}$. On the other hand, taking inner product with \mathbf{X} we obtain:

$$(18) \quad 2\rho = |\nabla\rho|^2 + (\mathbf{X} \cdot \mathbf{N})^2.$$

In particular, since $2\rho \leq d^2$, we have the following inequalities:

$$(19) \quad |\nabla\rho|^2 \leq d^2,$$

$$(20) \quad \mathbf{X} \cdot \mathbf{N} \leq d.$$

As in (8), we now calculate at p , taking (16) and (19) into account:

$$(21) \quad \begin{aligned} \Delta R &= 2H \Delta H + 2|\nabla H|^2 - 2|\nabla \chi|^2 - 2\chi^{ij} \Delta \chi_{ij} \\ &\leq 2H \Delta H - 2\chi^{ij} H_{;ij} + 2\alpha^2 |\nabla\rho|^2 H^2 + 2\chi^{ij} \left(R_{jlik} \chi^{lk} + R^k_{lik} \chi_j^l \right) \\ &\leq 2(H g^{ij} - \chi^{ij}) H_{;ij} + 2\alpha^2 d^2 H^2 - R H^2 + 2R^2. \end{aligned}$$

Since $(\chi^{ij}) \geq 0$ and $(f_{;ij}) \leq 0$ at p , we have,

$$\chi^{ij} f_{;ij} \geq H g^{ij} f_{;ij},$$

namely,

$$(H g^{ij} - \chi^{ij}) f_{;ij} \leq 0.$$

Thus, in view of (17), (11), and (20):

$$\begin{aligned}
(22) \quad (Hg^{ij} - \chi^{ij})H_{;ij} &\leq (Hg^{ij} - \chi^{ij})(-\alpha H \rho_{ij} + \alpha^2 H \rho_i \rho_j) \\
&\leq -\alpha H (Hg^{ij} - \chi^{ij})(g_{ij} - (\mathbf{X} \cdot \mathbf{N}) \chi_{ij}) + \alpha^2 H^2 |\nabla \rho|^2 \\
&\leq -(n-1)\alpha H^2 + \alpha R H (\mathbf{X} \cdot \mathbf{N}) + \alpha^2 d^2 H^2 \\
&\leq -(n-1)\alpha H^2 + \alpha^2 d^2 H^2 + \alpha d R H.
\end{aligned}$$

Thus, combining (21) and (22), we obtain that the following inequality holds at p :

$$\Delta R \leq -2(n-1)\alpha H^2 + 4\alpha^2 d^2 H^2 + R H (\alpha d - H) + 2R^2.$$

Taking $\alpha = (n-1)/(4d^2)$, and using $H(\alpha d - H) \leq \alpha^2 d^2/4$, we get that the following inequality holds at p :

$$\Delta R \leq -\frac{(n-1)^2}{4d^2} H^2 + 2R^2 + \frac{(n-1)^2}{64d^2} R.$$

Thus, we conclude

$$(23) \quad H^2(p) \leq \frac{4d^2}{(n-1)^2} \sup_{\mathbb{S}^n} \left(2R^2 - \Delta R + \frac{(n-1)^2}{64d^2} R \right).$$

If q is any point on \mathbb{S}^n , then we conclude:

$$\begin{aligned}
(24) \quad H^2(q) &= e^{-2\alpha\rho(q)} f^2(q) \leq e^{-2\alpha\rho(q)} f^2(p) \\
&= e^{2\alpha(\rho(p)-\rho(q))} H^2(p) \leq e^{\alpha d^2} H^2(p) \leq e^{(n-1)/4} H^2(p).
\end{aligned}$$

Theorem 1 now follows from (23) and (24). \square

5. SOLVING THE ONCE-CONTRACTED GAUSS EQUATION

In this section, we show that when the sectional curvature of g is strictly positive, one can bound H in terms of the supremum of the Ricci tensor and the minimum of the sectional curvature, i.e., in terms of only two derivatives of g . This is done by showing that the once-contracted Gauss Equation:

$$(25) \quad R_{ij} = \text{tr } \chi \chi_{ij} - \chi_i^k \chi_{kj},$$

can be solved for the second fundamental form χ ; cf. Theorem 5. It is well known that the full Gauss equation has at most one solution when $n \geq 3$ and the sectional curvature is positive, see for example [Ei, Section 60] and [All]. Here we show that the subset consisting of (25) can always be solved under this assumption. As a consequence, we obtain an estimate on the second derivatives of \mathbf{X} which depends only on the second derivatives of the metric g , provided the sectional curvatures of g are positive; cf. Theorem 6. When $n = 3$, this also gives, in conjunction with the Codazzi equation (6), an explicit local necessary and sufficient intrinsic condition for a metric g of

positive sectional curvature on \mathbb{S}^3 to be locally embeddable in \mathbb{R}^4 , and consequently also globally embeddable; cf. Theorem 7 and the remark following it.

We will use the elementary symmetric functions $\sigma_{k,n}(\mathbf{x})$ defined for integers $k \geq 0$ and $n \geq 1$, and for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, by:

$$\begin{aligned} \sigma_{0,n}(\mathbf{x}) &= 1, & \sigma_{k,n}(\mathbf{x}) &= 0, \text{ if } k > n, \\ \sigma_{k,n}(\mathbf{x}) &= \sum_{i_1 < \dots < i_k} x_{i_1} \dots x_{i_k}, \text{ for } 1 \leq k \leq n. \end{aligned}$$

Let $\mathbf{x}_i = (x_1, \dots, \hat{x}_i, \dots, x_n) \in \mathbb{R}^{n-1}$ be the vector obtained from \mathbf{x} by deleting its i -th coordinate. We have for $k \geq 1$, and $1 \leq i \leq n$:

$$(26) \quad \sigma_{k,n}(\mathbf{x}) = x_i \sigma_{k-1,n-1}(\mathbf{x}_i) + \sigma_{k,n-1}(\mathbf{x}),$$

from which it follows by induction on n that:

$$(27) \quad \sum_{i=1}^n \sigma_{k,n-1}(\mathbf{x}_i) = (n-k) \sigma_{k,n}(\mathbf{x}),$$

for $k \geq 0$.

Lemma 3. *Let $n \geq 3$, $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $x = \sigma_{1,n}(\mathbf{x})$, and define the $n \times n$ matrix $G_n(\mathbf{x})$ by:*

$$G_n(\mathbf{x}) = \begin{pmatrix} x - x_1 & x_1 & \dots & x_1 \\ x_2 & x - x_2 & \dots & x_2 \\ \vdots & & & \\ x_n & x_n & \dots & x - x_n \end{pmatrix}.$$

Then:

$$\det G_n(\mathbf{x}) = \sum_{|\gamma|=n} a_{\gamma,n} \mathbf{x}^\gamma,$$

where

$$(28) \quad a_{\gamma,n} = \sum_{k=3}^n \frac{(-2)^{k-1} (k-2)(n-k)!}{\gamma!} \sigma_{k,n}(\gamma).$$

Proof. Define for $\mathbf{x} \in \mathbb{R}^n$, and $s \in \mathbb{R}$:

$$F_n(s, \mathbf{x}) = \det \begin{pmatrix} s - x_1 & x_1 & \dots & x_1 \\ x_2 & s - x_2 & \dots & x_2 \\ \vdots & & & \\ x_n & x_n & \dots & s - x_n \end{pmatrix},$$

$$f_n(s, \mathbf{x}) = \det F_n(s, \mathbf{x}).$$

We will show that:

$$(29) \quad f_n(s, \mathbf{x}) = s^n - \sigma_{1,n}(\mathbf{x})s^{n-1} + \sum_{k=3}^n (-2)^{k-1} (k-2) \sigma_{k,n}(\mathbf{x}) s^{n-k}.$$

This is shown by induction on $n \geq 3$. Equation (29) is easily verified for $n = 3$. Assume (29) holds for $n-1 \geq 3$, and let $g_n(s, \mathbf{x})$ denote the right hand side of (29). Now, using

$$\frac{d}{ds} \log \det F_n(s, \mathbf{x}) = \operatorname{tr} \left(F_n(s, \mathbf{x})^{-1} \frac{d}{ds} F_n(s, \mathbf{x}) \right),$$

the induction hypothesis, and (27), we obtain:

$$\frac{d}{ds} f_n(s, \mathbf{x}) = \sum_{i=1}^n f_{n-1}(s, \mathbf{x}_i) = \sum_{i=1}^n g_{n-1}(s, \mathbf{x}_i) = \frac{d}{ds} g_n(s, \mathbf{x}).$$

It is easily checked that $f_n(0, \mathbf{x}) = (-2)^{n-1} (n-2) \sigma_{n,n}(\mathbf{x}) = g_n(0, \mathbf{x})$, hence $f_n(s, \mathbf{x}) = g_n(s, \mathbf{x})$, which proves (29). Substituting $s = \sigma_{1,n}(\mathbf{x})$ in (29), we now obtain:

$$\det G_n(\mathbf{x}) = \sum_{k=3}^n (-2)^{k-1} (k-2) \sigma_{k,n}(\mathbf{x}) \sigma_{1,n}(\mathbf{x})^{n-k}.$$

We have

$$\sigma_{1,n}(\mathbf{x})^{n-k} = \sum_{|\alpha|=n-k} \binom{n-k}{\alpha} \mathbf{x}^\alpha, \quad \sigma_{k,n}(\mathbf{x}) = \sum_{\substack{|\beta|=k \\ \beta^*=1}} \mathbf{x}^\beta,$$

see the proof of Lemma 1 on page 5. Thus, substituting $\gamma = \alpha + \beta$, we find

$$\det G_n(\mathbf{x}) = \sum_{k=3}^n (-2)^{k-1} (k-2) \sum_{|\gamma|=n} \sum_{\substack{|\beta|=k \\ \beta \leq \gamma, \beta^*=1}} \binom{n-k}{\gamma-\beta} \mathbf{x}^\gamma = \sum_{|\gamma|=n} a_{\gamma,n} \mathbf{x}^\gamma,$$

where

$$a_{\gamma,n} = \sum_{k=3}^n (-2)^{k-1} (k-2) \sum_{\substack{|\beta|=k \\ \beta \leq \gamma, \beta^*=1}} \binom{n-k}{\gamma-\beta}.$$

To obtain (28), note that the last sum in this equation can be rewritten as:

$$\sum_{\substack{|\beta|=k \\ \beta \leq \gamma, \beta^*=1}} \binom{n-k}{\gamma-\beta} = \frac{(n-k)!}{\gamma!} \sigma_{k,n}(\gamma).$$

□

Lemma 4. *Let $n \geq 3$, and $\mathbf{x} \in \mathbb{R}_+^n$, then $\det G_n(\mathbf{x}) > 0$.*

Proof. Let

$$(30) \quad b_{\gamma,k,n} = \frac{2^{k-1}(k-2)(n-k)!}{\gamma!} \sigma_{k,n}(\gamma).$$

Then, we have:

$$(31) \quad a_{\gamma,n} = \sum_{k=3}^n (-1)^{k-1} b_{\gamma,k,n}.$$

We now claim that for all $|\gamma| = n$, and all $k \geq 3$, there holds:

$$(32) \quad 2(k-1)\sigma_{k+1,n}(\gamma) \leq (n-k)(k-2)\sigma_{k,n}(\gamma),$$

It follows from (32) and (30) that $b_{\gamma,k+1,n} \leq b_{\gamma,k,n}$, for all $|\gamma| = n$, and $3 \leq k \leq n-1$, and hence, in view of (31), we obtain $a_{\gamma,n} \geq 0$, for all $|\gamma| = n$. For $n = 3$, we have $\det G(x_1, x_2, x_3) = 4x_1x_2x_3$, that is $a_{\gamma,3} = 0$ unless $\gamma = (1, 1, 1)$, in which case $a_{\gamma,3} = 4$. Since for $n > 3$, we have

$$\det G(x_1, x_2, x_3, 0, \dots, 0) = (x_1 + x_2 + x_3)^{n-3} \det G(x_1, x_2, x_3),$$

it is clearly impossible that $a_{\gamma,n} = 0$ for all $|\gamma| = n$, and the lemma follows. It remains to prove (32). In fact, we will prove the more general inequality:

$$(33) \quad (k+1)\sigma_{k+1,n}(\gamma) \leq (\sigma_{1,n}(\gamma) - k)\sigma_{k,n}(\gamma),$$

for $k \geq 0$, and any multi-index γ . This clearly implies (32) when $|\gamma| = n$, and $k \geq 3$. We will use the identity:

$$\sum_{i=1}^n \gamma_i \sigma_{k,n-1}(\gamma_i) = (k+1)\sigma_{k+1,n}(\gamma),$$

which holds for $k \geq 0$, and which follows easily from (26) by induction on n . Using this identity, we find that:

$$\begin{aligned} \sigma_{1,n}(\gamma)\sigma_{k,n}(\gamma) &= \sum_{i=1}^n \gamma_i (\gamma_i \sigma_{k-1}(\gamma_i) + \sigma_{k,n-1}(\gamma_i)) \\ &\geq \sum_{i=1}^n (\gamma_i \sigma_{k,n}(\gamma_i) + \gamma_i \sigma_{k,n-1}(\gamma_i)) \\ &= (k+1)\sigma_{k+1,n}(\gamma) + k\sigma_{k,n}(\gamma), \end{aligned}$$

which proves (33). \square

Lemma 5. *Let $n \geq 3$, and let \mathbf{S}_n be the set of positive-definite symmetric $n \times n$ matrices over \mathbb{R} . The map $\Phi: \mathbf{S}_n \rightarrow \mathbf{S}_n$, defined by*

$$\Phi: A \mapsto (\operatorname{tr} A)A - A^2$$

is one-to-one.

Proof. Suppose that $\Phi(A) = \Phi(B)$, and let $E = A + B$, $F = A - B$. Then, E and F are symmetric, and E is positive-definite. We have:

$$(34) \quad \operatorname{tr}(E)F + \operatorname{tr}(F)E - (EF + FE) = 0.$$

This implies that E and F can be simultaneously diagonalized. Indeed, let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ be the eigenvalues of E , and let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ be the eigenvalues of F , then there is an orthogonal matrix Q such that $Q^T E Q = D = \operatorname{diag}(\boldsymbol{\lambda})$. Let $C = Q^T F Q$, then we have

$$\operatorname{tr}(D)C + \operatorname{tr}(C)D - (DC + CD) = 0.$$

Writing $C\mathbf{e}_j = \sum_{i=1}^n c_{ij}\mathbf{e}_i$, where $\mathbf{e}_1, \dots, \mathbf{e}_n$ is the standard basis of \mathbb{R}^n , this implies:

$$\sum_{i=1}^n (\sigma_{1,n}(\boldsymbol{\lambda}) - (\lambda_i + \lambda_j))c_{ij}\mathbf{e}_i = -\lambda_j\sigma_{1,n}(\boldsymbol{\mu})\mathbf{e}_j.$$

Since $\sigma_{1,n}(\boldsymbol{\lambda}) - (\lambda_i + \lambda_j) > 0$ when $i \neq j$, this shows that C is diagonal. It now follows from (34) that

$$\mu_i \sum_{j \neq i} \lambda_j + \lambda_i \sum_{j \neq i} \mu_j = 0.$$

In view of Lemma 4, we obtain that $\mu_i = 0$ for $i = 1, \dots, n$. Thus, we conclude that $F = 0$, and $A = B$. \square

Denote by $\mathbf{T}_n \subset \mathbf{S}_n$ the set of those matrices $B \in \mathbf{S}_n$ with the following property: if μ_1, \dots, μ_n are the eigenvalues of B , then $\mu_i < \sum_{j \neq i} \mu_j$ for each $1 \leq i \leq n$. For $A \in \mathbf{S}_n$, let $|A|^2 = \sum_{i=1}^n \lambda_i^2$, where λ_i are the eigenvalues of A , and for $B \in \mathbf{T}_n$, let

$$\varepsilon(B) = \min_{1 \leq i \leq n} (\sigma_{1,n}(\boldsymbol{\mu}) - 2\mu_i) > 0,$$

where $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ are the eigenvalues of B .

Lemma 6. *Let $n \geq 3$, then Φ maps \mathbf{S}_n onto \mathbf{T}_n . Furthermore if $A \in \mathbf{S}_n$, then*

$$(35) \quad |A| \leq \frac{n}{2} |\Phi(A)| \varepsilon(\Phi(A))^{-1/2}.$$

Proof. We first prove that $\Phi(\mathbf{S}_n) \subset \mathbf{T}_n$. Let $A \in \mathbf{S}_n$, and let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ be its eigenvalues. Then, in view of (26), the eigenvalues of $\Phi(A)$ are:

$$\mu_i = \sum_{j \neq i} \lambda_j \lambda_i = \lambda_i \sigma_{1,n-1}(\boldsymbol{\lambda}_i) = \sigma_{2,n}(\boldsymbol{\lambda}) - \sigma_{2,n-1}(\boldsymbol{\lambda}_i).$$

Thus, using (27), we find that $\sigma_{1,n}(\boldsymbol{\mu}) = 2\sigma_{2,n}(\boldsymbol{\lambda})$, and hence:

$$(36) \quad \sigma_{1,n}(\boldsymbol{\mu}) - 2\mu_i = 2\sigma_{2,n-1}(\boldsymbol{\lambda}_i) > 0.$$

We conclude that $\Phi(A) \in \mathbf{T}_n$. The converse will be proved by continuity, i.e., we will show that $\Phi(\mathbf{S}_n)$ is open and closed in \mathbf{T}_n . Since \mathbf{T}_n is clearly

connected, this implies that $\Phi(\mathbf{S}_n) = \mathbf{T}_n$. To show that $\Phi(\mathbf{S}_n)$ is open suppose that $A \in \mathbf{S}_n$, and that

$$(37) \quad \Phi'(A)C = \text{tr}(A)C + \text{tr}(C)A - (AC + CA) = 0,$$

for some symmetric $n \times n$ matrix C . Then, as in the proof of Lemma 5, we get that $C = 0$, and hence $\Phi'(A)$ is non-singular. We conclude, by the Inverse Function Theorem, that there is a neighborhood of $\Phi(A)$ contained in $\Phi(\mathbf{S}_n)$, and thus, $\Phi(\mathbf{S}_n)$ is open. We will now prove (35). Let $A \in \mathbf{S}_n$, and let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ be its eigenvalues. Let $\lambda_i = \max_j \lambda_j$, so that $|A|^2 \leq n\lambda_i^2$. Then, in view of (26) and (36), we have:

$$\sigma_{2,n}(\boldsymbol{\lambda})^2 \geq \lambda_i^2 \sigma_{1,n-1}(\boldsymbol{\lambda}_i)^2 \geq \frac{2}{n} |A|^2 \sigma_{2,n-1}(\boldsymbol{\lambda}_i) \geq \frac{1}{n} |A|^2 \varepsilon(\Phi(A)).$$

If $B = \Phi(A)$, we can now estimate:

$$|B|^2 \geq \frac{1}{n} (\text{tr } B)^2 = \frac{1}{n} ((\text{tr } A)^2 - \text{tr}(A^2))^2 = \frac{4}{n} \sigma_{2,n}(\boldsymbol{\lambda})^2 \geq \frac{4}{n^2} |A|^2 \varepsilon(\Phi(A)),$$

which proves (35). This shows that $\Phi(\mathbf{S}_n)$ is closed in \mathbf{T}_n . Indeed, if $A_j \in \mathbf{S}_n$ is such that $\Phi(A_j) \rightarrow B \in \mathbf{T}_n$, then (35) shows that the eigenvalues $\boldsymbol{\lambda}(A_j)$ of A_j are uniformly bounded above. Therefore, by passing to a subsequence if necessary, we see that $A_j \rightarrow A$ for some symmetric $A \geq 0$. By continuity $\Phi(A) = B$, hence $A > 0$, i.e., $A \in \mathbf{S}_n$, and $B \in \Phi(\mathbf{S}_n)$. \square

We note here that when $n = 3$, the inverse of Φ can be written explicitly. Indeed, let $B \in \mathbf{T}_3$, let $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ be its eigenvalues, and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be its eigenvectors. Define:

$$(38) \quad \lambda_i = \frac{\sqrt{\prod_{j=1}^3 (\sigma_{1,3}(\boldsymbol{\mu}) - 2\mu_j)}}{\sqrt{2}(\sigma_{1,3}(\boldsymbol{\mu}) - 2\mu_i)}, \quad i = 1, 2, 3; \quad A = \sum_{i=1}^3 \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i.$$

It is easy to check that $\lambda_i \sum_{j \neq i} \lambda_j = \mu_i$, hence $\Phi(A) = B$.

We note, furthermore, that since $\Phi: \mathbf{S}_n \rightarrow \mathbf{T}_n$ is \mathcal{C}^∞ (in fact analytic), it follows that $\Phi^{-1}: \mathbf{T}_n \rightarrow \mathbf{S}_n$ is also of class \mathcal{C}^∞ .

Theorem 5. *Let $n \geq 3$, and let $g \in \mathcal{M}_+^2(\mathbb{S}^n)$. Then there exists a unique symmetric positive twice-covariant tensor χ on \mathbb{S}^n which satisfies the once-contracted Gauss Equations (25). Furthermore, if $g \in \mathcal{C}^{k,\alpha}$ for some $k \geq 2$ and $0 \leq \alpha < 1$, then $\chi \in \mathcal{C}^{k-2,\alpha}$.*

Proof. Let R_{ij} be the Ricci tensor of g in an orthonormal basis, and let $p \in \mathbb{S}^n$. Then, we have $R_{ij}(p) \in \mathbf{T}_n$. Indeed, let μ_i be the eigenvalues of R_{ij} , let \varkappa_{ij} be the sectional curvature of g in the plane spanned by the i -th and j -th eigenvector of R_{ij} , and write $\varkappa_{ii} = 0$. Then, $\mu_i = \sum_{j=1}^n \varkappa_{ij}$ for each i , hence:

$$(39) \quad \sum_{j \neq i} \mu_j = \sum_{j \neq i} \sum_{l=1}^n \varkappa_{jl} = \sum_{j \neq i} \sum_{l \neq i} \varkappa_{jl} + \mu_i > \mu_i.$$

Thus, the first assertion of the theorem follows from Lemma 6 and Lemma 5. The second assertion follows from the remark just preceding this theorem. \square

This theorem, in conjunction with (35) and (39), immediately implies the following theorem:

Theorem 6. *Let $n \geq 3$, let $g \in \mathcal{M}_+^2(\mathbb{S}^n)$, and let $\mathbf{X}: (\mathbb{S}^n, g) \rightarrow \mathbb{R}^{n+1}$ be a \mathcal{C}^2 isometric embedding. Let χ be the second fundamental form of \mathbf{X} , let \varkappa be the minimum of all the sectional curvatures of g on \mathbb{S}^n , and let Λ be the maximum of the norm $|\text{Ric}| = (R_{ij}R^{ij})^{1/2}$ of the Ricci tensor of g over \mathbb{S}^n . Then, we have:*

$$(40) \quad |\chi| \leq C_n \Lambda \varkappa^{-1/2},$$

where $C_n = n/(2\sqrt{(n-1)(n-2)})$.

This last theorem should be compared with Theorems 2 and 1 when $n \geq 3$. Here we require positive sectional curvature. However, we obtain an estimate which depends only on two derivatives of g . In Theorem 2, we only required non-negative sectional curvature and positive scalar curvature, but our estimate relied on four derivatives of g .

Definition 1. Let (M, g) be a Riemannian manifold of class $\mathcal{C}^{k,\alpha}$ for some $k \geq 2$ and $0 \leq \alpha < 1$. We say that (M, g) is $\mathcal{C}^{k,\alpha}$ locally isometrically embeddable in \mathbb{R}^N if for each $p \in M$ there is a neighborhood $U \subset M$ of p and a $\mathcal{C}^{k,\alpha}$ isometric embedding $\mathbf{X}: (U, g) \rightarrow \mathbb{R}^N$. If (M, g) is locally isometrically embeddable in \mathbb{R}^N , we say that (M, g) is locally rigid in \mathbb{R}^N if whenever $\mathbf{X}, \mathbf{X}': (U, g) \rightarrow \mathbb{R}^N$ are local \mathcal{C}^2 isometric embeddings of some open set $U \subset M$, then $\mathbf{X}' = \Psi \circ \mathbf{X}$, where $\Psi: \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a rigid motion possibly composed with a reflection.

We now have:

Theorem 7. *Let $g \in \mathcal{M}_+^{k,\alpha}(\mathbb{S}^3)$ for some $k \geq 3$ and $0 \leq \alpha < 1$. Let χ be the solution of the once-contracted Gauss Equation (25) given by Theorem 5. Then (\mathbb{S}^3, g) is $\mathcal{C}^{k,\alpha}$ locally isometrically embeddable in \mathbb{R}^4 if and only if χ satisfies the Codazzi Equations:*

$$(41) \quad \chi_{ij;k} - \chi_{ik;j} = 0$$

In this case, (\mathbb{S}^3, g) is also locally rigid.

Proof. Let $\mathbf{X}: (U, g) \rightarrow \mathbb{R}^4$ be a $\mathcal{C}^{k,\alpha}$ local isometric embedding. Then, by Theorem 5, χ is the second fundamental form of \mathbf{X} , and $\chi \in \mathcal{C}^{k-2,\alpha}$. Hence χ satisfies the Codazzi Equations. Conversely, the system

$$(42) \quad \begin{cases} \mathbf{X}_{;ij} = -\chi_{ij}\mathbf{N} \\ \mathbf{N}_i = \chi_i^j \mathbf{X}_{;j} \end{cases}$$

is an overdetermined system for the 4 vector fields $\mathbf{X}_{;1}, \mathbf{X}_{;2}, \mathbf{X}_{;3}, \mathbf{N}$ along \mathbb{S}^3 , whose integrability conditions are the Gauss and Codazzi Equations (5)–(6).

The Codazzi Equations are satisfied by hypothesis, and the once-contracted Gauss Equations (25) imply the full Gauss Equations (5) when $n = 3$. Thus, if $p \in \mathbb{S}^3$, we can integrate (42) in a neighborhood U of p in \mathbb{S}^3 with $\mathbf{X}_{;i} \cdot \mathbf{X}_{;j}|_p = g_{ij}(p)$. Furthermore, since $\mathbf{X}_{;ij} = \mathbf{X}_{;ji}$ we can integrate again to obtain \mathbf{X} . It then follows that $\mathbf{X}_{;i} \cdot \mathbf{X}_{;j} = g_{ij}$ throughout U . Since $\chi \in \mathcal{C}^{k-2,\alpha}$, we obtain from (42) that $\mathbf{X} \in \mathcal{C}^{k,\alpha}$. The rigidity statement follows from the uniqueness of χ . \square

Remark 3. As mentioned in the introduction, local embeddability implies global embeddability under the hypothesis of positive sectional curvature. Therefore as a corollary of the previous theorem we can replace *local* by *global* in the previous theorem.

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