Adjoint functors; categories in topology

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0. Introduction

The concept of adjoint functors was first defined by Daniel Kan in 1956 [K-1958], and since then it has proven to be quite useful. The history leading up to Kan's breakthrough is concisely described in [ML-1971, p. 103]. In this paper, we investigate technical properties of adjoint functors, and their appearance in algebraic topology.

1. Basic Notions of Category Theory

In this section, we develop the some important categorical definitions and ideas which will be used throughout this paper. For a more complete treatment, the interested reader should consult either [ML-1971], [H-1970] or [M-1967].

Definition 1.1: A *metacategory* (which we typically denote as $\mathfrak{C}$ or $\mathfrak{D}$) is a pair $\mathfrak{C} = (\mathcal{O}\mathfrak{C}, \mathcal{M}\mathfrak{C})$ where $\mathcal{O}\mathfrak{C}$ is considered to be the collection of objects of $\mathfrak{C}$ and $\mathcal{M}\mathfrak{C}$ is considered a collection of morphisms (or arrows) between the objects of $\mathfrak{C}$ that are subject to a few rules.

1. For a morphism f of $\mathfrak{C}$, there are two associated objects: $\operatorname{dom}(f)$ and $\operatorname{cod}(f)$. If $\operatorname{dom}(f) = X$ and $\operatorname{cod}(f) = Y$, we will typically depict this in a diagram as:

$$f : X \longrightarrow Y \qquad \text{or} \qquad X \xrightarrow{f} Y$$

2. For any object X of $\mathfrak{C}$, there is a morphism $\mathrm{id}_X : X \to X$ which makes the following diagram commute for any $f, g \in \mathcal{M}\mathfrak{C}$ with $\operatorname{cod}(f) = \operatorname{dom}(g) = X$

$$\begin{tikzcd}
\operatorname{dom}(f) \arrow[r, "f"] \arrow[dr, "f"'] & X \arrow[d, "\mathrm{id}_X"] \arrow[dr, "g"] & \\
& X \arrow[r, "g"] & \operatorname{cod}(g)
\end{tikzcd}$$

3. Every composable pair of morphisms (f, g), i.e. $\operatorname{cod}(f) = \operatorname{dom}(g)$, has the composition $f \circ g \in \mathcal{M}\mathfrak{C}$.

4. If (f, g) and (g, h) are composable pairs of morphisms, then $(f \circ g) \circ h = f \circ (g \circ h)$.

Definition 1.2: A *category* $\mathfrak{C}$ is a metacategory in which $\mathcal{O}\mathfrak{C}$, $\mathcal{M}\mathfrak{C}$ are classes and for all $X, Y \in \mathfrak{C}$ the morphism sets $\mathfrak{C}(X, Y)$ are sets.

Remark 1.3: We use this definition of category in order to speak about adjunctions in a straightforward manner. The above definitions follow [ML-1971, 7–10], [M-1967] and [A-2004]. A more general approach is to define category to be what we define a metacategory to be. There are obvious benefits to the more general viewpoint, but it complicates our discussion of adjoints. We thus recommend the interested reader to *http://ncatlab.net* for more details on the abstract approach.

Definition 1.4: A *small category* is a category $\mathfrak{C}$ for which $\mathcal{O}\mathfrak{C}$ and $\mathcal{M}\mathfrak{C}$ are sets. In general, by appending small to a noun, we imply that we restrict our attention to sets. In this case, the relevant objects are $\mathcal{O}\mathfrak{C}$ and $\mathcal{M}\mathfrak{C}$ which we require to be small.

Notation 1.5: Let $\mathfrak{C}$ be a category. For objects $X, X' \in \mathfrak{C}$, we denote the collection of all morphisms $f \in \mathfrak{C}$ with $\mathrm{dom}(f) = X$ and $\mathrm{cod}(f) = X'$ by $\mathfrak{C}(X, X')$.

Definition 1.6: Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories. A *functor* is a transformation $F : \mathfrak{C} \to \mathfrak{D}$ which assigns to each object $X \in \mathfrak{C}$ an object $FX \in \mathfrak{D}$, to each morphism $f \in \mathfrak{C}(X, X')$ a morphism $Ff \in \mathfrak{D}(FX, FX')$ and satisfies the following properties:

1. $F(\mathrm{id}_X) = \mathrm{id}_{FX}$;
2. $F(f \circ g) = Ff \circ Fg$.

Example 1.7:

1. Define $\underline{\mathrm{Cat}}$ to be the category with objects all small categories and morphisms all functors between small categories.
2. Take $\underline{\mathrm{Ab}}$ to be the category of Abelian groups with morphisms being group homomorphisms.
3. Define $\underline{\mathrm{Gp}}$ to be the category of groups with group homomorphisms.
4. Define $\underline{\mathrm{Top}}$ to be the category of topological spaces with morphisms continuous maps.
5. Define $\underline{\mathrm{Top}}_*$ to be the category of pointed topological spaces with continuous maps preserving the basepoint.
6. Define $\underline{\mathrm{TopPair}}$ to be the category of pairs of spaces (X, A) with $A \subseteq X$ a subspace, and continuous maps of pairs, i.e. $f : (X, A) \to (Y, B)$ is a map $f : X \to Y$ in $\underline{\mathrm{Top}}$ with $f(A) \subseteq B$.

7. Define $\underline{\text{Haus}}$ to be the full subcategory of $\underline{\text{Top}}$ consisting of Hausdorff topological spaces.

8. Define $\underline{\text{CGTop}}$ to be the full sub-category of $\underline{\text{Top}}$ consisting of compactly generated topological spaces. Recall that a space $X \in \underline{\text{Top}}$ is compactly generated iff a subset $A \subset X$ is closed iff for every compact $K \subseteq X$, the set $A \cap K$ is closed in K. Define $\underline{\text{CGHaus}}$ to be the full subcategory of $\underline{\text{Haus}}$ consisting of compactly generated topological spaces.

9. Define $\underline{\text{CW}}$ to be the category of CW complexes with cellular, continuous maps. A map $f : X \to Y$ between CW complexes is cellular if for all $n \geq 0$, $f(X^n) \subseteq Y^n$.

10. Define $\underline{\text{Set}}^s$ to be the category of simplicial sets.

11. Define $\underline{\text{Man}}$ to be the full subcategory of $\underline{\text{Top}}$ consisting of Hausdorff spaces which are locally Euclidean and paracompact.

12. Define $\underline{\text{SmMan}}$ to be the category of smooth manifolds with smooth functions.

13. Define $\mathcal{A}\underline{\text{Man}}$ to be the category of analytic real manifolds with analytic functions.

14. For a concrete category $\mathfrak{T}$, let $\mathfrak{T}_*$ be the corresponding category of pointed spaces and basepoint preserving maps. So we have categories $\underline{\text{Top}}_*$, CW_*, etc.

Definition 1.8: A *full* functor is a functor $F : \mathfrak{C} \to \mathfrak{D}$ which is surjective on morphism sets. That is, for any $f \in \mathfrak{D}(FX, FX')$ there exists $g \in \mathfrak{C}(X, X')$ such that $f = Fg$.

Definition 1.9: A *faithful* functor, or an *embedding* is a functor $F : \mathfrak{C} \to \mathfrak{D}$ which is injective on morphism sets, that is for $f, f' \in \mathfrak{C}(X, X')$, if $Tf = Tf'$, then $f = f'$. Note that a functor can be faithful without being injective on objects.

Definition 1.10: Let $F, G : \mathfrak{C} \to \mathfrak{D}$ be functors. A *natural transformation* $\tau : F \to G$ is a family of morphisms $\tau_X : FX \to GX$ which make for any $f \in \mathfrak{C}(X, X')$ the following diagram commute:

$$\begin{tikzcd} FX \arrow[r, "\tau_X"] \arrow[d, "Ff"] & GX \arrow[d, "Gf"] \\ FX' \arrow[r, "\tau_{X'}"] & GX' \end{tikzcd}$$

Definition 1.11: A *natural equivalence* is a natural transformation in which τ_X is an invertible morphism in $\mathfrak{D}$ for all $X \in \mathfrak{C}$. We write $\tau : F \xrightarrow{\sim} G$ to denote a natural equivalence.

Definition 1.12: An *isomorphism* of categories is a full, faithful functor which is also a bijection on objects. A weaker notion is that of equivalence.

Definition 1.13: An *equivalence* of categories is a functor $F : \mathfrak{C} \to \mathfrak{D}$ for which there is a functor $G : \mathfrak{D} \to \mathfrak{C}$ and natural equivalences $F \circ G \xrightarrow{\sim} \mathrm{id}_{\mathfrak{D}}$ and $G \circ F \xrightarrow{\sim} \mathrm{id}_{\mathfrak{C}}$.

Definition 1.14: A *concrete category* is a category $\mathfrak{C}$ equipped with a faithful functor $U : \mathfrak{C} \to \underline{\mathrm{Set}}$. As we can may identify a morphism $f \in \mathcal{M}\mathfrak{C}$ with Uf, we may think of each object of $\mathfrak{C}$ as having an underlying set and each morphism f as a set function Uf.

Example 1.15: Not all categories are complete. The homotopy category of pointed topological spaces $\underline{\mathrm{hTop}}_*$ is not concrete. This was proven by Freyd in [F-1970].

Definition 1.16: Given a category $\mathfrak{C}$, the *opposite category* of $\mathfrak{C}$, denoted $\mathfrak{C}^{op}$ is the category with $\mathcal{O}\mathfrak{C}^{op} = \mathcal{O}\mathfrak{C}$ and $\mathfrak{C}^{op}(X, X') = \mathfrak{C}(X', X)$.

Another way to see this is to define for a morphism $f \in \mathcal{M}\mathfrak{C}$ the opposite morphism of f to be a symbolic arrow $f^{op} : \mathrm{cod}(f) \to \mathrm{dom}(f)$. Then $\mathcal{M}\mathfrak{C}^{op} = \{f^{op} \mid f \in \mathcal{M}\mathfrak{C}\}$. The opposite morphisms are defined so that for a composable pair f, g, $(f \circ g)^{op} = g^{op} \circ f^{op}$. In essence, all we are doing is drawing all of the arrows in $\mathfrak{C}$ backwards.

Definition 1.17: A contravariant functor $F : \mathfrak{C} \to \mathfrak{D}$ is a functor $F : \mathfrak{C}^{op} \to \mathfrak{D}$. We make the convention that the symbol F of a contravariant functor always represents the functor $F : \mathfrak{C}^{op} \to \mathfrak{D}$.

Definition 1.18: Given a functor $F : \mathfrak{C} \to \mathfrak{D}$, the opposite functor $F^{op} : \mathfrak{C}^{op} \to \mathfrak{D}^{op}$ is defined by $F^{op}(f^{op} : X' \to X) = (Ff)^{op} : FX' \to FX$.

Definition 1.19: An indexing category is a small category I, where $\mathcal{M}I = \{\mathrm{id}_x \mid x \in \mathcal{O}I\}$.

Definition 1.20: For a small category I, we define $\mathfrak{C}^I$ to be the category with objects all functors $F : I \to \mathfrak{C}$ and morphisms all natural transformations between functors $F \in \mathcal{O}\mathfrak{C}^I$.

Notation 1.21: In our commutative diagrams, dotted arrows represent unique induced maps or unique maps whose existence is in question, while dashed arrows stand for induced maps or maps whose existence is in question without any conditions on uniqueness. All diagrams are to be assumed commutative unless otherwise noted.

2. Basic Properties of Adjoint Functors

Definition 2.22: Let $F : \mathfrak{C} \to \mathfrak{D}$ and $G : \mathfrak{D} \to \mathfrak{C}$ be functors. We say that F is left adjoint to G (or equivalently G is right adjoint to F) if there exists a natural equivalence $\eta : \mathfrak{D}(F-,-) \xrightarrow{\sim} \mathfrak{C}(-,G-)$ between the functors $\mathfrak{D}(F-,-), \mathfrak{C}(-,G-) : \mathfrak{C}^{op} \times \mathfrak{D} \to \underline{\mathrm{Set}}$. In this case, we write $\eta : F \dashv G$.

Given $\eta : F \dashv G$, we now list the most important properties of such a natural equivalence.

1. As η is a natural equivalence, given $\alpha : X' \to X$ and $\beta : Y \to Y'$ the commutativity of the diagram

$$\begin{tikzcd}
\mathfrak{D}(FX,Y) \arrow[r, "\eta=\eta_{X,Y}"] \arrow[d, "F\alpha^*"'] & \mathfrak{C}(X,GY) \arrow[d, "\alpha^*"] \\
\mathfrak{D}(FX',Y) \arrow[r, "\eta=\eta_{X',Y}"] \arrow[d, "\beta_*"'] & \mathfrak{C}(X',GY) \arrow[d, "G\beta_*"] \\
\mathfrak{D}(FX',Y') \arrow[r, "\eta=\eta_{X',Y'}"] & \mathfrak{C}(X',GY')
\end{tikzcd}$$

   captures the fact that η is a natural transformation. For $\phi \in \mathfrak{D}(FX,Y)$, the equation $\eta(\beta \circ \phi \circ F\alpha) = G\beta \circ \eta(\phi) \circ \alpha$ obtained by walking around the perimeter of the above diagram is equivalent to the naturality of η. By setting one of β or α as the identity, we recover the commutativity of the top and bottom squares respectively.

   We thus have the fundamental equation

$$\eta(\beta \circ \phi \circ F\alpha) = G\beta \circ \eta(\phi) \circ \alpha$$

   which implies

$$\eta(\phi \circ F\alpha) = \eta(\phi) \circ \alpha$$
$$\eta(\beta \circ \phi) = G\beta \circ \eta(\phi).$$

   We can interpret these equations as a distributivity property of η.

2. Taking $Y = FX$ and $\phi = \mathrm{id}_{FX}$, we define the unit of the adjunction to be the natural transformation $\varepsilon : \mathrm{id}_{\mathfrak{C}} \to GF$ with $\varepsilon_X := \eta(\mathrm{id}_{FX}) : X \to GFX$.

   Now taking $X = GY$, we define the counit of the adjunction to be the natural transformation $\delta : FG \to \mathrm{id}_{\mathfrak{D}}$ defined by $\delta_Y := \eta^{-1}(\mathrm{id}_{GY})$.

   To prove that these are natural transformations, we must show that the following diagram is commutative, i.e. $\varepsilon_X \circ \alpha = GF\alpha \circ \varepsilon_{X'}$.

$$\begin{tikzcd}
X' \arrow[r, "\varepsilon_{X'}"] \arrow[d, "\alpha"'] & GFX' \arrow[d, "GF\alpha"] \\
X \arrow[r, "\varepsilon_X"] & GFX
\end{tikzcd}$$

One may use the distributivity equations above, or consult the following diagram which is commutative by the naturality of η to determine that $\eta(F\alpha) = \varepsilon_X \circ \alpha$.

$$\begin{tikzcd}
\mathfrak{D}(FX, FX) \arrow[r, "\eta"] \arrow[d, "F\alpha^*"'] & \mathfrak{C}(X, GFX) \arrow[d, "\alpha^*"] \\
\mathfrak{D}(FX', FX) \arrow[r, "\eta"] & \mathfrak{C}(X', GFX) \\
\mathfrak{D}(FX', FX') \arrow[r, "\eta"] \arrow[u, "\alpha_*"] & \mathfrak{C}(X', GFX') \arrow[u, "GF\alpha_*"']
\end{tikzcd}$$

Alternatively, we compute

$$\begin{tikzcd}
\eta(\mathrm{id} \circ F\alpha) \arrow[r, equal] \arrow[d, equal] & \eta(\mathrm{id}) \circ \alpha \arrow[r, equal] & \varepsilon_X \circ \alpha \\
\eta(F\alpha \circ \mathrm{id}) \arrow[r, equal] & GF\alpha \circ \eta(\mathrm{id}) \arrow[r, equal] & GF\alpha \circ \varepsilon'_X
\end{tikzcd}$$

Entirely similar computations show that the counit δ is a natural transformation as well.

3. The following two equations hold as well:

$$\delta F \circ F\varepsilon = \mathrm{id}$$

$$G\delta \circ \varepsilon G = \mathrm{id}$$

To prove the validity of these two equations, we compute using the fundamental equation

$$\eta(\delta_{FX} \circ F\varepsilon_X) = \eta(\delta_{FX}) \circ \varepsilon_X = \varepsilon_X = \eta(\mathrm{id}_{FX})$$

which implies $\delta_{FX} \circ F\varepsilon_X = \mathrm{id}_{FX}$ as η is a bijection. A similar computation establishes the other equation.

4. The following two equations explicitly describe the natural equivalence η in terms of the unit, counit and functors F and G.

$$\eta(\psi) = G\psi \circ \varepsilon_X, \text{ with } \psi : FX \to Y;$$

$$\eta^{-1}(\zeta) = \delta_Y \circ F\zeta, \text{ with } \zeta : X \to GY.$$

One can prove the first equation with the following diagram, chasing id_{FX} around:

$$\begin{tikzcd}
\mathfrak{D}(FX, FX) \arrow[r, "\eta"] \arrow[d, "\psi_*"'] & \mathfrak{C}(X, GFX) \arrow[d, "G\psi_*"] \\
\mathfrak{D}(FX, Y) \arrow[r, "\eta"] & \mathfrak{C}(X, GY)
\end{tikzcd}$$

The above objects derived from the adjunction η are indeed enough to reconstruct it. We formulate this in the following proposition.

Proposition 2.23: If $\varepsilon : \mathrm{id} \to GF$ and $\delta : FG \to \mathrm{id}$ are natural transformations and if the equation $\delta F \circ F\varepsilon = \mathrm{id}$ and $G\delta \circ \varepsilon G = \mathrm{id}$ hold, then $\eta : F \dashv G$, defined by $\eta(\phi) = G\phi \circ \varepsilon_X$, is a natural equivalence which shows F is left adjoint to G. Furthermore, ε and δ are the unit and counit of the adjunction η respectively.

Conversely, if $\eta : F \dashv G$ is a natural equivalence, then $\varepsilon_X := \eta(\mathrm{id}_{FX})$ and $\delta_Y := \eta^{-1}(\mathrm{id}_{GY})$ define natural transformations which satisfy the above equations.

Proof. The first part of the proposition is the only thing that remains to be proven. We thus show that η is natural by verifying that the equation $\eta(\beta \circ \phi \circ F\alpha) = G\beta \circ \eta(\phi) \circ \alpha$ holds:

$$\begin{aligned}
\eta(\beta \circ \phi \circ F\alpha) = {} & G(\beta \circ \phi \circ F\alpha) \circ \varepsilon_{X'} \\
& G\beta \circ G\phi \circ GF\alpha \circ \varepsilon_{X'} \\
& G\beta \circ G\phi \circ \varepsilon_X \circ \alpha \text{ (by naturality of } \varepsilon) \\
& G\beta \circ \eta(\phi) \circ \alpha.
\end{aligned}$$

We now define $\xi : \mathfrak{C}(-, G-) \to \mathfrak{D}(F-, -)$ by $\xi(\psi) := \delta_Y \circ F\psi$, for $\psi : X \to GY$. We show that ξ is inverse to η, thus proving that η is a natural equivalence.

$$\begin{aligned}
\xi(\eta(\phi)) &= \delta_Y \circ F \circ \eta(\phi) \\
&= \delta_Y \circ F(G\phi \circ \varepsilon_X) \\
&= \delta_Y \circ FG\phi \circ F\varepsilon_X \\
&= \phi \circ \delta_{FX} \circ F\varepsilon_X, \text{ (by naturality of } \delta) \\
&= \phi.
\end{aligned}$$

One similarly shows that $\eta\xi = \mathrm{id}$, from which we conclude $\xi = \eta^{-1}$ and η is a natural equivalence. Thus $\eta : F \dashv G$. Define $\varepsilon'_X := \eta(\mathrm{id}_{FX})$ and $\delta'_Y := \eta^{-1}(\mathrm{id}_{GY})$. We compute

$$\begin{aligned}
\varepsilon'_X &= \eta(\mathrm{id}_{FX}) \\
&= G(\mathrm{id}_{FX}) \circ \varepsilon_X \\
&= \mathrm{id}_{GFX} \circ \varepsilon_X \\
&= \varepsilon_X
\end{aligned}$$

and

$$\begin{aligned}
\delta'_Y &= \eta^{-1}(\mathrm{id}_{GY}) \\
&= \delta_Y \circ F(\mathrm{id}_{GY}) \\
&= \delta_Y \circ \mathrm{id}_{FGY} \\
&= \delta_Y
\end{aligned}$$

as desired. ■

Proposition 2.24: If $\eta : F \dashv G$ and $\eta' : F \dashv G'$, then there exists a natural equivalence between G and G'. We remark that for all $Y \in \mathfrak{D}$, we have $GY \cong G'Y$. Alternatively, G determines F up to natural equivalence.

Proof. We prove the first claim. We have the natural equivalence

$$\begin{tikzcd}
\mathfrak{C}(-, GY) \arrow[r, "\eta^{-1}"] \arrow[rr, bend left, "\sim"] & \mathfrak{D}(F-, Y) \arrow[r, "\eta'"] & \mathfrak{C}(-, G'Y).
\end{tikzcd}$$

We thus define $\theta_Y := \eta' \circ \eta^{-1}(\mathrm{id}_{GY}) : GY \to G'Y$, i.e. $\theta = G'\delta \circ \varepsilon' G$. It follows from the above derived equations that $\overline{\theta} := G\delta' \circ \varepsilon G'$ is the inverse to θ. Thus θ induces an equivalence between G and G' as desired. ■

Proposition 2.25: Let $F : \mathfrak{C} \to \mathfrak{D}$, $F' : \mathfrak{D} \to \mathfrak{E}$ be functors, and suppose there exist G and G' such that $\eta : F \dashv G$ and $\eta' : F' \dashv G'$. Then $\eta_{-,G'-} \circ \eta'_{F-,-} : F'F \dashv GG'$.

$$\begin{tikzcd}
\mathfrak{C} \arrow[r, "F"'] \arrow[r, phantom, shift left=3, "\top"] & \mathfrak{D} \arrow[l, bend right=50, "G"'] \arrow[r, "F'"'] \arrow[r, phantom, shift left=3, "\top"] & \mathfrak{E} \arrow[l, bend right=50, "G'"']
\end{tikzcd}$$

Proof. Clear. ■

3. (Co-)Universal Constructions

We now have the necessary terminology and properties to define and study universal constructions in mathematics. The following definitions follow essentially from [H-1970].

Definition 3.26: Let $\mathfrak{C}, \mathfrak{D}$ be categories. A universal construction with respect to the functor $G : \mathfrak{D} \to \mathfrak{C}$ is a left adjoint to G with the unit of the adjunction.

Definition 3.27: Let $\mathfrak{C}, \mathfrak{D}$ be categories. A couniversal construction with respect to the functor $F : \mathfrak{C} \to \mathfrak{D}$ is a right adjoint to F with the counit of the adjunction.

In the literature, the term universal construction is often abused and used for both universal and couniversal constructions. We use the terminology precisely as defined above. In addition, we also use the term (co-)universal property to mean the property that a (co-)universal construction determines in the categories $\mathfrak{C}$ and $\mathfrak{D}$. We will discuss this terminology more in section 4.

Example 3.28: Let $\mathfrak{C}$ be a category; let I be an indexing category and consider the diagonal functor $P : \mathfrak{C} \to \mathfrak{C}^I$, i.e. $P(X)(\cdot) = X$ and $P(X)(\cdot \to \cdot) = \mathrm{id}_X$. A right adjoint R to P along with the counit δ of the adjunction $\eta : P \dashv R$ defines the product in the category $\mathfrak{C}$ over the indexing category I. We often denote R by $\prod$. This is a *couniversal* construction, yet it is called the *product* (and not the coproduct).

Example 3.29: Let $\mathfrak{C}$ be a category; let I be an indexing set (i.e. a small category with only the identity morphisms) and consider the diagonal functor $P : \mathfrak{C} \to \mathfrak{C}^I$. A left adjoint L to P along with the unit ε of the adjunction $\eta : L \dashv P$ defines the coproduct in the category $\mathfrak{C}$ over the indexing set I. This is a *universal* construction.

Proposition 3.30: The description of the product as an adjunction is equivalent to the following usual universal property: the object $R\{X_i\}$ with morphisms $\pi_i : R\{X_i\} \to X_i$ is the product of $\{X_i\}$ in $\mathfrak{C}$ if for any other object $Y \in \mathcal{O}\mathfrak{C}$ with morphisms $\phi_i : Y \to X_i$ there exists a unique morphism $\phi' : Y \to R\{X_i\}$ such that $\pi_i \circ \phi' = \phi_i$ for all $i \in I$.

Proof. Given $\eta : P \dashv R$, and $\{X_i\} \in \mathfrak{C}^I$, we show that $R\{X_i\}$ along with δ satisfies the usual universal property. That is, we assume we are given a diagram

$$\begin{tikzcd} R\{X_i\} \arrow[r, "\delta"] & \{X_i\} \\ Y \arrow[u, dotted, "\phi'"] \arrow[ur, "\phi"'] & \end{tikzcd}$$

in $\mathfrak{C}^I$ with Y and ϕ arbitrary, and seek a unique morphism ϕ' which makes the diagram commute. As $\eta : \mathfrak{C}^I(PY, \{X_i\}) \xrightarrow{\sim} \mathfrak{C}(Y, R\{X_i\})$, we take $\phi' := \eta(\phi) : Y \to R\{X_i\}$ which satisfies $\delta \circ P\phi' = \eta^{-1}(\phi') = \eta^{-1}(\eta(\phi)) = \phi$, i.e. the above diagram can be completed in a unique way so that it commutes.

We now prove the other direction. We assume that for any $\{X_i\} \in \mathcal{O}\mathfrak{C}^I$ there is a morphism $\delta : PR\{X_i\} \to \{X_i\}$ such that the following holds: for any $Y \in \mathfrak{C}$ with a morphism $\phi : PY \to \{X_i\}$ there exists a unique morphism $\phi' : Y \to R\{X_i\}$ such that $\phi = \delta \circ \phi'$.

We begin by showing that $R : \mathfrak{C}^I \to \mathfrak{C}$ is a functor. We thus suppose we have a diagram

$$\begin{tikzcd}
PR\{Z_i\} \arrow[r, "\mathfrak{z}"] & \{Z_i\} \\
PR\{Y_i\} \arrow[r, "\mathfrak{y}"] & \{Y_i\} \arrow[u, "\beta"'] \\
PR\{X_i\} \arrow[r, "\mathfrak{x}"] & \{X_i\} \arrow[u, "\alpha"']
\end{tikzcd}$$

and seek to define for a morphism $\alpha : \{X_i\} \to \{Y_i\}$ an induced morphism $R\alpha : R\{X_i\} \to R\{Y_i\}$ and show that $R(\beta \circ \alpha) = R\beta \circ R\alpha$. We define $R\alpha := \alpha'$. We see that the above diagram induces a commutative diagram

$$\begin{tikzcd}
PR\{Z_i\} \arrow[rr, "\mathfrak{z}"] & & \{Z_i\} \\
PR\{Y_i\} \arrow[u, dotted, "PR\beta\mathfrak{y}"'] \arrow[urr, "\beta\mathfrak{y}"] \arrow[rr, "\mathfrak{y}"] & & \{Y_i\} \arrow[u, "\beta"'] \\
PR\{X_i\} \arrow[u, dotted, "PR\alpha\mathfrak{x}"'] \arrow[uu, dashed, bend left=20, "PR\beta\mathfrak{y} \circ PR\alpha\mathfrak{x}"] \arrow[uu, dotted, bend left=60, "PR(\beta\alpha\mathfrak{x})"] \arrow[uurr, "\beta\alpha\mathfrak{x}"'] \arrow[urr, "\alpha\mathfrak{x}"'] \arrow[rr, "\mathfrak{x}"] & & \{X_i\}. \arrow[u, "\alpha"']
\end{tikzcd}$$

We remark that the equality $PR\beta\mathfrak{y} \circ PR\alpha\mathfrak{x} = PR(\beta\alpha\mathfrak{x})$ follows by the uniqueness of the induced maps from our assumption. We thus have concluded that $R : \mathfrak{C}^I \to \mathfrak{C}$ is a functor.

We now show the morphism $\delta : PR\{X_i\} \to \{X_i\}$ defines a natural transformation $\delta : PR \to \mathrm{id}_{\mathfrak{C}^I}$. Let $\alpha : \{X_i\} \to \{Y_i\} \in \mathfrak{C}^I$. Then by our assumption, we obtain the commutative diagram

$$\begin{tikzcd}
PR\{X_i\} \arrow[r, "\delta_{\{X_i\}}"] \arrow[d, dotted] \arrow[dr, "\alpha\delta_{\{X_i\}}"] & \{X_i\} \arrow[d, "\alpha"] \\
PR\{Y_i\} \arrow[r, "\delta_{\{Y_i\}}"] & \{Y_i\}
\end{tikzcd}$$

which proves that δ is a natural equivalence. Thus we may define a natural transformation $\xi : \mathfrak{C}(Y, R\{X_i\}) \to \mathfrak{C}^I(PY, \{X_i\})$ by $\xi\psi := \delta \circ P\psi$. To complete the proof, we now show ξ is a natural equivalence. Define $\eta : \mathfrak{C}^I(PY, \{X_i\}) \to \mathfrak{C}(Y, R\{X_i\})$ by setting $\eta(\phi) := \phi'$, i.e. the induced map $\phi' : Y \to R\{X_i\}$ seen in our assumption. We verify that $\eta\xi = \mathrm{id} = \xi\eta$. Because

$$\begin{tikzcd}
PR\{X_i\} \arrow[r, "\delta"] & \{X_i\} \\
PY \arrow[u, "P\phi"'] \arrow[u, dotted, bend left=60, "P\eta(\delta(P\phi))"] \arrow[ur, "\delta\circ P\phi"'] &
\end{tikzcd}$$

we have

$$\begin{aligned}
\eta(\xi\phi) &= \eta(\delta \circ P\phi) \\
&= \phi.
\end{aligned}$$

Similarly, we consider $\psi : PY \to \{X_i\}$ and obtain

$$\begin{aligned}
\xi(\eta(\psi)) &= \delta \circ P\eta(\psi) \\
&= \psi
\end{aligned}$$

from the commutative diagram

$$\begin{tikzcd}
PR\{X_i\} \arrow[r, "\delta"] & \{X_i\} \\
PY \arrow[u, "P\eta(\psi)"] \arrow[ur, "\psi"] &
\end{tikzcd}$$

We thus conclude that $\xi = \eta^{-1}$, and hence $\eta : P \dashv R$ as desired. ■ Many familiar categories admit small products and coproducts, and the constructions are what one expects. The following example provides a motivationg example to think of what products and coproducts are doing.

Example 3.31: Consider a partially ordered set $(X, \leq)$ which we interpret as a category with $\mathcal{O}X = X$ and $\mathcal{M}X = \{x \to y \,|\, x \leq y\}$. The product in such a category (if it exists) is easily seen to be the infimum with the help of the following diagram:

$$\begin{tikzcd}
Y \arrow[rr] \arrow[dr, dotted] & & r_i \\
& \prod\{r_i\} \arrow[ur] \arrow[dr] & \\
Y \arrow[rr] \arrow[ur, dotted] & & r_j
\end{tikzcd}$$

Interpreting the above diagram yields the property: for any $Y \in X$ such that $Y \leq r_i$ for all $i \in I$, we then have $Y \leq \prod \{r_i\}$, which is the defining condition for the infimum. Similarly, one sees that the coproduct in $(X, \leq)$ is the supremum.

Example 3.32: Let $\mathfrak{C}$ be a concrete category. Consider the faithful functor $U : \mathfrak{C} \to \underline{\text{Set}}$—typically called the underlying functor—which comes with the assumption that $\mathfrak{C}$ is concrete. The universal construction with respect to U defines the universal property of free objects. That is, a left adjoint $L \dashv U$ along with the unit ε of the adjunction.

One similarly defines cofree objects as the couniversal construction with respect to U.

Example 3.33:[*Limits and Colimits*] The limit is similar in construction to the product. Let I be a small category (not necessarily an indexing category). The functor of interest is the diagonal functor $P : \mathfrak{C} \to \mathfrak{C}^I$. The couniversal construction with respect to P then yields the limit, while the universal construction yields the colimit. We typically write the limit with respect to a functor $F \in \mathfrak{C}^I$ as $\lim F$, while the colimit is denoted by either $\operatorname{colim} F$. So we have $\operatorname{colim} \dashv P$ and $P \dashv \lim$.

In the literature, one often encounters the terms "direct limit" and "inverse limit". These are special instances of colimits and limits respectively. These use the special indexing category $I = \mathbb{N}$ with morphisms $n \to m$ iff $n \leq m$. We can also consider the opposite category $\mathbb{N}^{op}$ which has morphisms $n \to m$ iff $n \geq m$.

If $F \in \mathbb{C}^{\mathbb{N}}$, then we call $\operatorname{colim} F$ the *direct limit* and write it as $\varinjlim F$. Alternatively, using the notation of a *directed system* $(F(n), f_n)$, we write $\operatorname{colim} F = \varinjlim F(n)$. Note directed systems "go to the right":

$$F(1) \xrightarrow{f_1} F(2) \xrightarrow{f_2} F(3) \xrightarrow{f_3} \cdots$$

If $F \in \mathbb{C}^{\mathbb{N}^{op}}$, then we call $\lim F$ the *inverse limit* and write it as $\varprojlim F$. Alternatively, using the notation of an *inverse system* $(F(n), f_n)$, we write $\lim F = \varprojlim F(n)$. Note that inverse systems "go to the left":

$$\cdots \longrightarrow F(3) \xrightarrow{f_3} F(2) \xrightarrow{f_2} F(1)$$

The concept of direct limit and inverse limit can be generalized slightly to other indexing categories I and still retain some nice properties. If the indexing category I is *filtered,* then a functor $F : I \to \mathfrak{C}$ is called a directed system. If the indexing category J is such that J^{op} is filtered, we say J is cofiltered and $G : J \to \mathfrak{C}$ is an inverse system. Again, in this case, we write $\operatorname{colim} F = \varinjlim F$ and $\lim G = \varprojlim G$.

Example 3.34: A classical example of a limit which is not a product is an inverse limit in $\underline{\text{Rng}}$, For some functor $\mathfrak{X} : I \to \underline{\text{Rng}}$, a description of the product of the $\mathfrak{X}_i$ is given by

$$\prod \mathfrak{X}_i = \{\overline{x} : \mathbb{N} \to \cup_{\mathbb{N}} \mathfrak{X}_i \,|\, x_n := \overline{x}(n) \in \mathfrak{X}_n\}$$

with the usual component-wise operations. Then the limit of a functor $\mathfrak{X} : I \to \underline{\text{Rng}}$ is given by

$$\lim \mathfrak{X} = \left\{ \overline{x} \in \prod \mathfrak{X}_i \,|\, \mathfrak{X}_{ij}(x_i) = x_j, \forall i \geq j \right\}$$

where $\mathfrak{X}_{ij} = \mathfrak{X}(f_{ij})$.

Specifically, we consider the functor $\mathfrak{R}^p : I \to \underline{\text{Rng}}$ with $\mathfrak{R}^p(i) := \mathbb{Z}_{p^i}$ and $\mathfrak{R}^p_{ij} : \mathbb{Z}_{p^i} \twoheadrightarrow \mathbb{Z}_{p^j}$ the canonical surjection. The object $\varprojlim \mathfrak{R}^p = \lim \mathfrak{R}^p$ is called the p-adic integers. The elements of $\lim \mathfrak{R}^p$ are often called coherent sequences.

Example 3.35:[*Equalizers and coequalizers*] Consider the category $I = \bullet \rightrightarrows \bullet$. For an object $F \in \mathfrak{C}^I$, we call $\lim F$ the equalizer, and $\operatorname{colim} F$ the coequalizer. Equalizers and coequalizers are of fundamental importance in studying limits and colimits because of the following proposition.

Proposition 3.36: Let $\mathfrak{C}$ be a category for which all products and equalizers exist. Then $\mathfrak{C}$ is complete. Dually, if $\mathfrak{C}$ has all coproducts and all coequalizers, then $\mathfrak{C}$ is cocomplete.

Proof.

$\blacksquare$

Observe that in the previous examples, the limit was a subobject of the product. This is true in general. By duality, a similar relationship is shared between colimits and coproducts. The definition of monomorphism (and by duality, epimorphism) will be developed in the course of the proof. Of course, [ML-1971] is the standard reference.

Proposition 3.37: Let $\mathfrak{C}$ be a category which has small products. For a small category I, define its underlying indexing category I', i.e. $\mathcal{O}I' = \mathcal{O}I$ and $\mathcal{M}I' = \{\mathrm{id}_x \,|\, x \in \mathcal{O}I'\}$. Suppose there is a right adjoint to $P^I_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C}^I$. Then for $\mathfrak{X} \in \mathfrak{C}^I$ and $U\mathfrak{X} : I' \to \mathfrak{C}$ is the induced functor, there exists a monomorphism $\lim \mathfrak{X} \rightarrowtail \prod U\mathfrak{X}$. By duality, there is an epimorphism $\coprod U\mathfrak{X} \to \operatorname{colim} \mathfrak{X}$.

Proof. Define $U : \mathfrak{C}^I \to \mathfrak{C}^{I'}$ by $U\,(\mathfrak{X} : I \to \mathfrak{C}) : I' \to \mathfrak{C}$, by $(U\mathfrak{X})i = \mathfrak{X}(i)$ on objects of $i \in \mathcal{O}I$. For $\tau : \mathfrak{X} \to \mathfrak{Y}$ define $U(\tau) : U\mathfrak{X} \to U\mathfrak{Y}$ by $U(\tau)_i = \tau_i$ for all $i \in I$. This is evidently a functor.

It is clear that the inside triangle of the following diagram commutes.

$$\begin{tikzcd}
& \mathfrak{C} \arrow[dl, "P_{\mathfrak{C}}^{UI}"'] \arrow[dr, "P_{\mathfrak{C}}^{I}"] & \\
\mathfrak{C}^{UI} \arrow[ur, bend left, "\prod"] & & \mathfrak{C}^{I} \arrow[ll, "U"] \arrow[ul, bend right, "\lim"']
\end{tikzcd}$$

The counit of the adjunction defining the limit gives us a map $\delta_{\mathfrak{X}} : P \lim \mathfrak{X} \to \mathfrak{X} \in \mathfrak{C}^I$. The underlying maps give us $U\delta_{\mathfrak{X}} \in \mathfrak{C}^{I'}$. Through the adjunction for the product $\eta : \mathfrak{C}^{I'}(P \lim \mathfrak{X}, U\mathfrak{X}) \xrightarrow{\sim} \mathfrak{C}(\lim \mathfrak{X}, \prod U\mathfrak{X})$ consider $\overline{\delta} = \eta(\delta_{\mathfrak{X}})$. By the universality of the product, the following diagram commutes

$$\begin{tikzcd}
P\Pi U\mathfrak{X} \arrow[r, "\delta^{I'}_{U\mathfrak{X}}"] & U\mathfrak{X} \\
P \lim \mathfrak{X} \arrow[u, dotted, "P\overline{\delta}"] \arrow[ur, "U\delta^I_{\mathfrak{X}}"] &
\end{tikzcd}$$

To show that $\overline{\delta} : \lim \mathfrak{X} \to \prod U\mathfrak{X}$ is a monomorphism, we need to show that for any Y and any $f_1, f_2 : Y \to \lim \mathfrak{X}$ such that $\overline{\delta} f_1 = \overline{\delta} f_2$, it is then the case that $f_1 = f_2$.

Through the adjunction $\mathfrak{C}(Y, \lim \mathfrak{X}) \xrightarrow{\sim} \mathfrak{C}^I(PY, \mathfrak{X})$, there correspond unique $\overline{f_i} : PY \to \mathfrak{X}$ such that $\delta_{\mathfrak{X}} \circ Pf_i = \overline{f_i}$ for $i = 1, 2$. That is the following diagram commutes for each i

$$\begin{tikzcd}
P \lim \mathfrak{X} \arrow[r, "\delta_{\mathfrak{X}}"] & \mathfrak{X} \\
PY \arrow[u, "Pf_i"] \arrow[ur, "\overline{f_i}"] &
\end{tikzcd}$$

Splicing the two diagrams above together, while also applying U to the one directly above, we obtain

$$\begin{tikzcd}
P\Pi U\mathfrak{X} \arrow[r, "\delta^{I'}_{U\mathfrak{X}}"] & U\mathfrak{X} \\
P \lim \mathfrak{X} \arrow[u, dotted, "P\overline{\delta}"] \arrow[ur, "U\delta^I_{\mathfrak{X}}"] & \\
PY \arrow[u, shift left, "Pf_1"] \arrow[u, shift right, "Pf_2"'] \arrow[uur, shift left, "U\overline{f_2}"] \arrow[uur, shift right, "U\overline{f_1}"'] &
\end{tikzcd}$$

Note that the individual maps that $U\overline{f_i}$ is comprised of do not change; the fact that they satisfy certain commutativity properties with the diagram $\mathfrak{X}$ is all that is forgotten.

Observe now that as $\overline{\delta} f_1 = \overline{\delta} f_2 \in \mathfrak{C}(Y, \prod U\mathfrak{X}) \xrightarrow{\sim} \mathfrak{C}^{I'}(PY, U\mathfrak{X})$, the maps $U\overline{f_1} = U\overline{f_2}$. It follows that $\overline{f_1} = \overline{f_2}$, and therefore $f_1 = f_2$ as desired. ■

We remind the reader that the notions of monomorphism and epimorphism aren't always what one expects. The typical example is that in Haus and Man a map $f : X \to Y$ is an epimorphism if and only if $\operatorname{im} f$ is a dense subset of Y by the Coincidence Theorem [S-1992]. In particular, $\iota : \mathbb{Q} \hookrightarrow \mathbb{R}$ is an epimorphism!

Example 3.38: Pull-backs and push-outs are a specific case of limits and colimits that we will use later on. Consider $I = \bullet \leftarrow \bullet \rightarrow \bullet$. The universal construction, i.e. colimit or left adjoint, with respect to the constant functor $P : \mathfrak{C} \to \mathfrak{C}^I$ defines the push-out in $\mathfrak{C}$. One frequently used example of a push-out is that of an amalgamated product in $\underline{\mathrm{Gp}}$. Given

$$\begin{tikzcd} N \arrow[r, "f_1"] \arrow[d, "f_2"'] & G \\ H & \end{tikzcd}$$

the push out of this diagram is given by $G *_N H = G * H / \langle f_1(n)(f_2(n))^{-1} \rangle_{G*H}$ with the canonical maps $\varepsilon_1 : G \to G *_N H$ and $\varepsilon_2 : H \to G *_N H$. We once again see how the colimit is a quotient object of the coproduct.

Now consider $J = \bullet \rightarrow \bullet \leftarrow \bullet$. The couniversal construction with respect to the constant functor $P : \mathfrak{C} \to \mathfrak{C}^J$ defines the pull-back.

Example 3.39: Our last example may be considered to be the motivating example for the definition of adjoint functors. It is in fact one of the very first things Kan mentions in his pioneering paper [K-1958, p. 294] on the subject. The example is the adjointness of $- \otimes B \dashv \hom(B, -)$.

Consider the category $\underline{\mathrm{Mod}}_\Lambda$ of right Λ-modules where Λ is a ring with identity (not necessarily commutative). Then for any $B \in \underline{\mathrm{Mod}}$, we may consider the functors $- \otimes_\Lambda B : \underline{\mathrm{Mod}}_\Lambda \to \underline{\mathrm{Ab}}$ and $\hom(B, -) : \underline{\mathrm{Ab}} \to \underline{\mathrm{Mod}}_\Lambda$. We then see that

$$\eta : \hom_{\mathbb{Z}}(A \otimes_\Lambda B, C) \xrightarrow{\sim} \hom_\Lambda(A, \hom_{\mathbb{Z}}(B, C))$$

where $\overline{f} := \eta(f : B \otimes A \to C)$ is defined by $\overline{f}(a)(b) = f(b \otimes a)$ and extended by linearity. To see the equivalence, we define an inverse to η by $\overline{g} := \eta^{-1}(g : A \to \hom_{\mathbb{Z}}(B, C))$ where $\overline{g}(b \otimes a) = g(a)(b)$ and $\overline{g}$ is extended to all of $B \otimes A$ by linearity.

4. Universal (Co-)Universal Constructions

In what follows, we sweep any set-theoretic difficulties under the rug. The reader who objects should just consider the case where $\mathfrak{i} = (-)^I$, i.e. the morphism category construction.

Definition 4.40: Consider a (meta)functor $\mathfrak{i} : \underline{\mathrm{Cat}} \to \underline{\mathrm{Cat}}$ which preserves adjoints, i.e. if $F \dashv G$, then $\mathfrak{i}F \dashv \mathfrak{i}G$; consider a natural transformation $P^{\mathfrak{i}}_{-} : \mathrm{id}_{\underline{\mathrm{Cat}}} \to \mathfrak{i}$. For any category $\mathfrak{C}$, we define a left adjoint to $P^{\mathfrak{i}}_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{i}\mathfrak{C}$ to be a universal universal construction, and a right adjoint to $P^{\mathfrak{i}}_{\mathfrak{C}}$ to be a universal couniversal construction. We typically will abbreviate these as UUCs and UCCs.

One reason why the adjoint preserving property is included in this definition is because of the following useful proposition.

Proposition 4.41: Consider $F : \mathfrak{C} \to \mathfrak{D} \in \underline{\mathrm{Cat}}$. If $L \dashv F$ and a given UCC $(\mathfrak{i}, P^{\mathfrak{i}}_{-}$ exists in both $\mathfrak{C}$ and $\mathfrak{D}$, then the UCC commutes in $\mathfrak{D}$. That is, if $P^{\mathfrak{i}}_{\mathfrak{C}} \dashv \mathcal{R}^{\mathfrak{i}}_{\mathfrak{C}}$ and $P^{\mathfrak{i}}_{\mathfrak{D}} \dashv \mathcal{R}^{\mathfrak{i}}_{\mathfrak{D}}$, then $\mathcal{R}^{\mathfrak{i}}_{\mathfrak{D}} \circ \mathfrak{i}F \cong F \circ \mathcal{R}^{\mathfrak{i}}_{\mathfrak{C}}$. A similar result holds for all manner of permutations of left and right adjoints.

Proof. For the general result, the following diagram is helpful.

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\mathfrak{C} \arrow[r, "F"] \arrow[d, "P^{\mathfrak{i}}_{\mathfrak{C}}"'] & \mathfrak{D} \arrow[l, bend right, "R"'] \arrow[l, bend left, "L"] \arrow[d, "P^{\mathfrak{i}}_{\mathfrak{D}}"] \\
\mathfrak{i}\mathfrak{C} \arrow[r, "\mathfrak{i}F"] \arrow[u, bend left, "\mathcal{R}^{\mathfrak{i}}_{\mathfrak{C}}"] \arrow[u, bend right, "\mathcal{L}^{\mathfrak{i}}_{\mathfrak{C}}"'] & \mathfrak{i}\mathfrak{D} \arrow[l, bend right, "\mathfrak{i}L"'] \arrow[l, bend left, "\mathfrak{i}R"] \arrow[u, bend left, "\mathcal{L}^{\mathfrak{i}}_{\mathfrak{D}}"] \arrow[u, bend right, "\mathcal{R}^{\mathfrak{i}}_{\mathfrak{D}}"']
\end{tikzcd}$$

By proposition 25, we have $\mathfrak{i}L \circ P^{\mathfrak{i}}_{\mathfrak{D}} \dashv \mathcal{R}^{\mathfrak{i}}_{\mathfrak{D}} \circ \mathfrak{i}F$ and $P^{\mathfrak{i}}_{\mathfrak{C}} \circ L \dashv F \circ \mathcal{R}^{\mathfrak{i}}_{\mathfrak{C}}$. We see that $P^{\mathfrak{i}}_{\mathfrak{C}} \circ L = \mathfrak{i}L \circ P^{\mathfrak{i}}_{\mathfrak{D}}$ by the naturality of $P^{\mathfrak{i}}_{-} : \mathrm{id}_{\underline{\mathrm{Cat}}} \to \mathfrak{i}$, and therefore, by proposition 24, there is a natural equivalence $\mathcal{R}^{\mathfrak{i}}_{\mathfrak{D}} \circ \mathfrak{i}F \cong F \circ \mathcal{R}^{\mathfrak{i}}_{\mathfrak{C}}$. The proof of the remaining parts follow analogously.

$L \dashv F$	UUCs commute in $\mathfrak{C}$, i.e. $\mathcal{L}^{\mathfrak{i}}_{\mathfrak{C}} \circ \mathfrak{i}L \cong L \circ \mathcal{L}^{\mathfrak{i}}_{\mathfrak{D}}$	$L \dashv F$	UCCs commute in $\mathfrak{D}$, i.e. $\mathcal{R}^{\mathfrak{i}}_{\mathfrak{D}} \circ \mathfrak{i}F \cong F \circ \mathcal{R}^{\mathfrak{i}}_{\mathfrak{C}}$
$F \dashv R$	UUCs commute in $\mathfrak{D}$, i.e. $\mathcal{L}^{\mathfrak{i}}_{\mathfrak{D}} \circ \mathfrak{i}F \cong F \circ \mathcal{L}^{\mathfrak{i}}_{\mathfrak{C}}$	$F \dashv R$	UCCs commute in $\mathfrak{C}$, i.e. $\mathcal{R}^{\mathfrak{i}}_{\mathfrak{C}} \circ \mathfrak{i}R \cong R \circ \mathcal{R}^{\mathfrak{i}}_{\mathfrak{D}}$

∎

One can relax the existence of $\mathcal{R}^{\mathfrak{i}}_{\mathfrak{D}}$ to get that for $\mathfrak{X} \in \mathfrak{i}\mathfrak{C}$, the object $F \circ \mathcal{R}^{\mathfrak{i}}_{\mathfrak{C}}(\mathfrak{X})$ satisfies the relevant couniversal property in $\mathfrak{D}$. This is of importance in the later sections.

Proposition 4.42: Consider $F : \mathfrak{C} \to \mathfrak{D}$ and suppose $L \dashv F$. If the UCC $(\mathfrak{i}, P^{\mathfrak{i}}_{-})$ exists in $\mathfrak{C}$, i.e. there is $P^{\mathfrak{i}}_{\mathfrak{C}} \dashv \mathcal{R}^{\mathfrak{i}}_{\mathfrak{C}}$, then for $\mathfrak{X} \in \mathfrak{i}\mathfrak{C}$ the object $F \circ \mathcal{R}^{\mathfrak{i}}_{\mathfrak{C}}(\mathfrak{X})$ with $\mathfrak{i}F(\delta_{\mathfrak{X}})$ satisfies the couniversal property determined by $P^{\mathfrak{i}}_{\mathfrak{D}}$ for $\mathfrak{i}F(\mathfrak{X})$. That is, for any $Y \in \mathfrak{D}$ and any map $\phi : P^{\mathfrak{i}}_{\mathfrak{D}}Y \to \mathfrak{i}F\mathfrak{X}$, there is a unique $\overline{\phi}$ making the following diagram commute

$$\begin{tikzcd}
PF\mathcal{R}\mathfrak{X} \arrow[r, "\mathfrak{i}F\delta_{\mathfrak{X}}"] & \mathfrak{i}F\mathfrak{X} \\
PY \arrow[u, "\overline{\phi}"] \arrow[ur, "\phi"'] &
\end{tikzcd}.$$

Proof. The proof is straight forward when one considers the relationship between the above diagram and

$$\begin{tikzcd} P\mathcal{R}\mathfrak{X} \arrow[r, "\delta_{\mathfrak{X}}"] & \mathfrak{X} \\ PLY \arrow[u, dotted, "\overline{\phi}'"] \arrow[ur, "\phi' := \eta^{-1}_{PY,\mathfrak{X}}(\phi)"'] & \end{tikzcd}$$

where $\eta_{PY,\mathfrak{X}} : \mathfrak{i}\mathfrak{C}(\mathfrak{i}LPY, \mathfrak{X}) \xrightarrow{\sim} \mathfrak{i}\mathfrak{D}(PY, \mathfrak{i}F\mathfrak{X})$. ■

We now give a few examples of UUCs and UCCs.

Example 4.43: The UUCs and UCCs that motivated the general definition are limits and colimits. They both arise from a small category I and considering the (meta)functor $-^I : \underline{\mathrm{Cat}} \to \underline{\mathrm{Cat}}$ and the diagonal natural transformation $P_- : \mathrm{id}_{\underline{\mathrm{Cat}}} \to -^I$. Specifically, we have

$$\begin{tikzcd} \underline{\mathrm{Cat}} \arrow[d, "-^I"'] & \mathfrak{C} \arrow[r, "F"] & \mathfrak{D} \arrow[d, mapsto] \\ \underline{\mathrm{Cat}} & \mathfrak{C}^I \arrow[r, "F^I"] & \mathfrak{D}^I \end{tikzcd}$$

where F^I works component-wise on maps of a morphism $\phi : \mathfrak{X} \to \mathfrak{Y} \in \mathfrak{C}^I$. Such a ϕ is composed of morphisms $\phi_i : \mathfrak{X}_i \to \mathfrak{Y}_i$ for each $i \in I$. Then $F^I(\phi) : F^I(\mathfrak{X}) \to F^I(\mathfrak{Y})$ is given by applying F to each ϕ_i. It is easy to verify that $-^I$ preserves adjointness.

It is likewise simple to verify that $P_- : \mathrm{id}_{\underline{\mathrm{Cat}}} \to -^I$ given by the diagonal functor on each $\mathfrak{C}$ is a natural equivalence. That is, for any $F : \mathfrak{C} \to \mathfrak{D}$, the following diagram commutes

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "F"] \arrow[d, "P_{\mathfrak{C}}"'] & \mathfrak{D} \arrow[d, "P_{\mathfrak{D}}"] \\ \mathfrak{C}^I \arrow[r, "F^I"] & \mathfrak{D}^I. \end{tikzcd}$$

Example 4.44: The constant (meta)functors $C_{\mathfrak{A}}$ for some $\mathfrak{A} \in \underline{\mathrm{Cat}}$ provide the simplest possible $\mathfrak{i}$ for constructing new UUCs and UCCs. Coming up with a $P_- : \mathrm{id}_{\underline{\mathrm{Cat}}} \to C_{\mathfrak{A}}$ seems to pose a greater challenge. One example that uses this constant (meta)functor is the construction of initial and terminal objects. These are constructed by taking $\mathfrak{i} = C_{\underline{1}}$ where $\underline{1} = \bullet$ is the category with one object and one morphism, and $P_{\mathfrak{C}} : \mathfrak{C} \to \underline{1}$ defined by $P_-(X \to Y) = \mathrm{id}_{\bullet}$. It is easy to verify that this satisfies all of the conditions of a UUC and UCC.

Other examples of UUCs and UCCs have not been easy to find. The statement of Freyd's Adjoint Functor Theorem provides a basis to conjecture that (co-)limits, initial and terminal objects are all of the possible UUCs and UCCs, in some suitable sense.

5. I have a functor, but does it have an adjoint!?

The previous section offers some techniques to show adjoints don't exist. Suppose we are considering $F : \mathfrak{C} \to \mathfrak{D}$ and wish to know if it has a left or right adjoint. One method would be to see if a UUC (resp. UCC) exists for $\mathfrak{C}$ and determine whether F preserves the UUC (resp. UCC) or not. Another property to check is the (co-)solution set condition.

Definition 5.45: Let $F : \mathfrak{C} \to \mathfrak{D}$ be a functor and consider $Y \in \mathfrak{D}$. A solution set for Y is a set $\{X_i \in \mathfrak{C} \,|\, i \in I\}$ and $\{f_i : Y \to FX_i\} \,|\, i \in I\}$ where I is a set (yes, a set!) if: for any $X \in \mathfrak{C}$ and any $\phi : Y \to FX$ there exists an i and $\overline{\phi} : X_i \to X$ such that the following diagram commutes

$$\begin{tikzcd} FX_i \arrow[d, "F\overline{\phi}"'] & Y \arrow[l, "f_i"'] \arrow[dl, "\phi"] \\ FX & . \end{tikzcd}$$

Remark 5.46: It is clear that if $L \dashv F$ then $\{FLY\}$ with $\{\varepsilon_Y : Y \to FLY\}$ is a solution set for Y. Thus for a left adjoint to F to exist, the functor F must satisfy the solution set condition.

Definition 5.47: Let $F : \mathfrak{C} \to \mathfrak{D}$ be a functor and consider $Y \in \mathfrak{D}$. A cosolution set[1] for Y is a set $\{X_i \in \mathfrak{C} \,|\, i \in I\}$ and $\{f_i : FX_i \to Y\}$ where I is a set if: for any $X \in \mathfrak{C}$ and any $\phi : FX \to Y$ there exists an i and $\overline{\phi} : X \to X_i$ such that the following diagram commutes

$$\begin{tikzcd} FX_i \arrow[r, "f_i"] & Y \\ FX \arrow[u, "F\overline{\phi}"] \arrow[ur, "\phi"'] & . \end{tikzcd}$$

Remark 5.48: It is clear tht if $F \dashv R$ then $\{FRY\}$ with $\{\delta_Y : FRY \to Y\}$ is a cosolution set for Y. Thus for a right adjoint to F to exist, the functor F must satisfy the cosolution set condition.

The following theorem due to Freyd provides a partial converse to the above remarks. The conditions are slightly idealized, however, and it will not be of much use to us. We state the theorem without proof; see [ML-1971, § V.6] or [M-1967, § V.3] for a proof.

Theorem 5.49: (Freyd's Adjoint Functor Theorem) Suppose $\mathfrak{C}$ is a small-complete category (that is, all limits for I a small category exist) which has $\mathfrak{C}(X,Y)$ a set for all objects X, Y. Then a functor $F : \mathfrak{C} \to \mathfrak{D}$ has a left adjoint if and only if F preserves all small limits and there is a solution set for all $Y \in \mathfrak{D}$.

[1] This terminology is not necessarily standard.

Question 5.50: Can we get a statement like this without restrictions on the category and just on the functor?

Example 5.51: In an introductory course on real analysis, one learns that $\mathbb{Q}$ is not complete, and $\mathbb{R}$ is the completion of $\mathbb{Q}$. From this, we conclude that the constant functor $P : \mathbb{Q} \to \mathbb{Q}^I$ does not have a left or right adjoint when I is infinite. The category $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ of extended real numbers is complete and cocomplete. We may now ask if the canonical inclusion $\iota : \mathbb{Q} \to \overline{\mathbb{R}}$ has a left or right adjoint.

Observe that ∞ is the terminal object of $\overline{\mathbb{R}}$ and $-\infty$ is the initial object of $\overline{\mathbb{R}}$. If a left (resp. right) adjoint of ι were to exist, it would have to send the initial (resp. terminal) object of $\overline{\mathbb{R}}$ to the initial (resp. terminal) object of $\mathbb{Q}$. As $\mathbb{Q}$ does not have initial or terminal objects, no such adjoints can exist.

Even if we add to $\mathbb{Q}$ an initial and terminal object, there still are no adjoints to the inclusion $\iota : \overline{\mathbb{Q}} \to \overline{\mathbb{R}}$. Let us write $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\pm\infty\}$. Let us suppose there exists a right adjoint $\iota \dashv R$ and we will derive a contradiction. Consider an irrational number $x \in \overline{\mathbb{R}}$. Since R is right adjoint to the inclusion, we must have $\overline{\mathbb{Q}}(Rx, Rx) \cong \overline{\mathbb{R}}(\iota Rx, x)$ which implies $Rx \leq x$. Now consider a sequence of rational numbers x_n such that $\lim x_n = x$ and for all n we have $x_n < x$. Since $Rx < x$, there exists some N so that for all $n > N$ we have $Rx < x_n < x$. In other words, $\overline{\mathbb{Q}}(x_n, Rx) = \emptyset$. Yet the adjunction $\iota \dashv R$ implies

$$\emptyset = \overline{\mathbb{Q}}(x_n, Rx) \cong \overline{\mathbb{R}}(\iota x_n, x) \neq \emptyset$$

which is a contradiction. Thus there can be no adjunction $\iota \dashv R$. An entirely similar argument shows no left adjoint to ι can exist.

6. Adjoints to Contravariant Functors

We can interpret all of the results established above for contravariant functors $F : \mathfrak{C} \to \mathfrak{D}$, i.e. a functor $F : \mathfrak{C}^{op} \to \mathfrak{D}$. For sake of clarity, we write out what an adjunction $F \dashv G$ looks like for a contravariant functor F.

Remark 6.52: Consider $F : \mathfrak{C} \to \mathfrak{D}$ a contravariant functor. An adjunction $F \dashv G$, where $G : \mathfrak{D} \to \mathfrak{C}^{op}$, is given by a natural equivalence

$$\eta_{X,Y} : \mathfrak{D}(FX, Y) \xrightarrow{\sim} \mathfrak{C}^{op}(X, GY) = \mathfrak{C}(GY, X).$$

Example 6.53: A nice, commonly encountered example of such an adjunction appears in the study of vector spaces. Let $\underline{\mathrm{Vect}}_k$ be the category of all vector spaces over a field k. Define $D = \hom(-, k) : \underline{\mathrm{Vect}}_k \to \underline{\mathrm{Vect}}_k$; this is a contravariant functor. This recast in terms of *op*-categories and functors is $D : \underline{\mathrm{Vect}}_k^{op} \to \underline{\mathrm{Vect}}_k$. We claim $D^{op} \dashv D$, and also that $D \not\dashv D^{op}$.

It is simple enough to verify that the following transformation is natural which establishes $D^{op} \dashv D$.

$$\begin{aligned} \eta_{V,W} : \underline{\mathrm{Vect}}_k^{op}(D^{op}(V), W) &\xrightarrow{\sim} \underline{\mathrm{Vect}}_k(V, D(W)) \\ F^{op} : \mathrm{Hom}(V, k) \to W &\mapsto ((\eta_{V,W} F)(v))(w) = (F(w))(v). \end{aligned}$$

This adjunction tells us the useful fact that D sends UCCs in $\underline{\mathrm{Vect}}_k^{op}$, i.e. UUCs in $\underline{\mathrm{Vect}}_k$, to UCCs in $\underline{\mathrm{Vect}}_k$. In particular, $\mathrm{hom}(\coprod V_i, k) \cong \prod \mathrm{hom}(V_i, k)$.

We now show that there is no adjunction $D \dashv D^{op}$. If $D \dashv D^{op}$ were the case, D would have to send UUCs in $\underline{\mathrm{Vect}}_k^{op}$, i.e. UCCs in $\underline{\mathrm{Vect}}_k$, to UUCs in $\underline{\mathrm{Vect}}_k$. We particularly investigate what D does on coproducts in $\underline{\mathrm{Vect}}_k$.

Consider $I = \mathbb{N}$ a countable indexing category, and the diagram $\mathfrak{X} : I \to \underline{\mathrm{Vect}}_k$ given by $\mathfrak{X}(i) = k$ for all $i \in I$. Then $\prod \mathfrak{X} = \prod_{\mathbb{N}} k$ is in fact the free k vector space on an uncountable number of generators. This is easy to establish using a cardinality argument and the fact that Zorn's lemma implies $\prod \mathfrak{X}$ does have a basis. It therefore follows that $D(\prod \mathfrak{X}) = \mathrm{hom}(\prod \mathfrak{X}, k)$ is the product of an uncountable number of copies of k. This is clearly not isomorphic to $\coprod D^I \mathfrak{X}$, the sum of a countable number of copies of k. Therefore there is no adjunction $D \dashv D^{op}$.

This example can be used as justification for considering the existence of an adjoint between categories as saying the categories are equivalent. The dual functor is an isomorphism for finite dimensional vector spaces, but then fails for infinite dimensional vector spaces. However, there is still an adjoint relationship $D^{op} \dashv D$.

If we wish to consider those functors which have adjoints as a new kind of equivalence of categories, it is then of great importance to understand UUCs and UCCs as they are preserved under adjoints, and thus posess some of the intrinsic structure of a category.

7. Functors in Algebra

We focus mainly on homological algebra. Pehaps other algebraic constructions will be added in the future.

List of things to cover:

1. Abelian categories. Move kernel here.
2. Exactness and adjoints [Weibel] pp 51–58.
3. Derived functors of the inverse limit, [Weibel] pp. 80–86.

8. Functors in Topology

The question that motivates this section is, Does the de-Rham cohomology functor, in any of its forms, have an adjoint? One might anticipate that it does, as it sends coproducts to products—a hallmark characteristic of what it means for there to be an adjoint! To proceed, we verify that Freyd's adjunct functor theorem does not apply.

First off, SmMan is not cocomplete. We will show that the pushout of

$$\begin{tikzcd} \{0\} \arrow[r, hook] \arrow[d, hook] & \mathbb{R} \\ \mathbb{R} & \end{tikzcd}$$

does not exist in SmMan.

The reader is cautioned from using intuition about colimits in Top when thinking about colimits in SmMan and Man. We have the following example that shows we need to be careful!

Example 8.54: Consider the diagram

$$\begin{tikzcd} \mathbb{R} \setminus \{0\} \arrow[r, hook] \arrow[d, hook] & \mathbb{R} \\ \mathbb{R} & \end{tikzcd}$$

It is clear that the colimit (push-out) of this diagram in Top is the line with two origins. However, in Man and SmMan the colimit is just $\mathbb{R}$ itself. This follows as any map $f : \mathbb{R} \to X$ is uniquely determined by the values of f on $\mathbb{R} \setminus \{0\}$ when X is Hausdorff. Consider $X \in$ Man and maps $\phi_1, \phi_2 : \mathbb{R} \to X$ which make the diagram commute below, i.e. $\phi_1|_{\mathbb{R}\setminus\{0\}} = \phi_2|_{\mathbb{R}\setminus\{0\}}$. Then by the Coincidence Theorem, $\phi_1 = \phi_2$ and hence taking $\overline{\phi} = \phi_1 = \phi_2$ shows that $\mathbb{R}$ is the colimit in Man, which is not homeomorphic to the line with two origins.

$$\begin{tikzcd} \mathbb{R} \setminus \{0\} \arrow[r, hook] \arrow[d, hook] & \mathbb{R} \arrow[d, "\mathrm{id}"'] \arrow[ddr, bend left, "\phi_2"] & \\ \mathbb{R} \arrow[r, "\mathrm{id}"] \arrow[drr, bend right, "\phi_1"'] & \mathbb{R} \arrow[dr, dotted, "\overline{\phi}"] & \\ & & X \end{tikzcd}$$

In order to establish that SmMan is not cocomplete, we will utilize the contravariant functor $C^\infty(-, \mathbb{R}) :$ SmMan $\to$ $\mathrm{Alg}_{\mathbb{R}}$. The main property it posesses is that it sends those colimits which exist to limits. As limits are better understood in $\mathrm{Alg}_{\mathbb{R}}$, we can compute the limit of $C^\infty(\mathbb{R} \leftarrow \{0\} \to \mathbb{R})$ and see if that $\mathbb{R}$-algebra is in the image of $C^\infty(-, \mathbb{R})$. In order to do this efficiently, we utilize the following result about ideals in rings of the form $C^\infty(M, \mathbb{R})$ where $M \in$ SmMan and defer the proof to the end of the paper.

Proposition 8.55: Let $M \in$ SmMan. For a point $p \in M$, define the I_p to be the ideal $I_p = \{f \mid f(p) = 0\}$ in $C^\infty(M, \mathbb{R})$. This ideal is maximal, and the product ideals $I_p^n = \langle g_1 \cdots g_n \,|\, g_i \in I_p \rangle$ have the following description

$$\begin{aligned} I_p^n &= \left\{ f\colon \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(p) = 0 \,\forall \alpha, |\alpha| \le k-1 \right\} \\ &= \{ f\colon f \,\text{has}\, k-1 \,\text{order contact with}\, 0 \} \\ &= \{ f\colon \text{the Taylor series of}\, f \text{has no terms of degree} \le k-1 \} \end{aligned}$$

where $\alpha \in \mathbb{N}_0^n$ and $|\alpha| = \sum_{i=1}^n \alpha_i$ is the sum of the entries. Also $x^\alpha = \prod x_i^{\alpha_i}$ for $x \in \mathbb{R}^n$.

The above shows that the ideals I_p^n contain local information. Define two functions $f, g \in C^\infty(M, \mathbb{R})$ to be locally equivalent at p if there exists an open set $U \subseteq M$ with $p \in U$ such that $f|_U = g|_U$; we write $f \sim_p g$ and denote equivalence classes as $[f]_p$ which are called germs. The collection of all germs forms an $\mathbb{R}$-algebra under the canonical operations which we denote by $C_p^\infty(M, \mathbb{R})$. Define $\mathfrak{m}_p = \{ [f]_p \,|\, f(p) = 0 \}$. Consider the canonical projection $\phi : C^\infty(M, \mathbb{R}) \to C_p^\infty(M, \mathbb{R})$, then $I_p^n = \phi^{-1}(\mathfrak{m}_p^n)$.

Remark 8.56: If one restricts ones attention to the category $\mathcal{A}$Man, the above proposition can be gotten much cheaper! It is perhaps instructive for the reader to work through this case to see how the machinery works.

Proposition 8.57: The diagram $\mathbb{R} \leftarrow \{0\} \to \mathbb{R}$ does not have a colimit in SmMan.

Proof. Suppose that the colimit does exist and call it M, and denote the maps of the unit as ε_1 and ε_2 as seen in the following diagram

$$\begin{tikzcd} \mathbb{R} \setminus \{0\} \arrow[r, hook] \arrow[d, hook] & \mathbb{R} \arrow[d, "\varepsilon_2"] \\ \mathbb{R} \arrow[r, "\varepsilon_1"] & M. \end{tikzcd}$$

As $C^\infty(-, \mathbb{R})$ sends colimits to limits, we have

$$C^\infty(M, \mathbb{R}) \cong \{ (f_1, f_2) \,|\, f_i \in C^\infty(\mathbb{R}, \mathbb{R}), f_1(0) = f_2(0) \} \,.$$

Denote $x := \varepsilon_1(0) = \varepsilon_2(0) \in M$. We then investigate the local properties of M at x by studying the ideals I_x^n / I_x^{n+1}. We demonstrate that $\dim_\mathbb{R} I_x^n / I_x^{n+1} = 2$ for all n which is impossible for a smooth manifold.

We first establish the dimension computation of I_p^n / I_p^{n+1} in $C^\infty(\mathbb{R}, \mathbb{R})$ with $p \in \mathbb{R}$. If $f \in I_p^n$, then $f - f^{(n)}(p)(x-p)^n \in I_p^{n+1}$. Hence $I_p^n / I_p^{n+1} = \langle (x-p)^n \rangle_\mathbb{R}$.

We now compute $\dim_\mathbb{R} I_x^n / I_x^{n+1}$ in $C^\infty(M, \mathbb{R})$. We make the convention that $J_p^n := I_p^n$ in $C^\infty(\mathbb{R}, \mathbb{R})$. Observe that $I_x^n = \langle (f_{11} \cdots f_{1n}, f_{21} \cdots f_{2n}) \,|\, f_{ij} \in J_0 \rangle$. Therefore, the quotient $I_x^n / I_x^{n+1} = \langle (x^n, 0), (0, x^n) \rangle_\mathbb{R}$ from which we conclude $\dim_\mathbb{R} I_x^n / I_x^{n+1} = 2$.

For a general manifold X, the computation of $\dim_\mathbb{R} I_p^n / I_p^{n+1}$ is local, so it suffices to carry out the computation at a point $p \in X$ through the local coordinate charts in $\mathbb{R}^k$, which we carry out now. Consider $f \in I_p \subseteq C^\infty(\mathbb{R}^k, \mathbb{R})$. Then $f - \sum \frac{\partial f}{\partial x_i}(p) \cdot (x_i - p_i) \in I_p^2$, and it is clear that $(x_i - p_i) - (x_j - p_j) \notin I_p^2$ from which we conclude $I_p/I_p^2 = \langle x_1, ..., x_k \rangle_{\mathbb{R}}$.

We now verify that if $X \in \underline{\text{SmMan}}$ and $\dim_{\mathbb{R}} I_p/_p^2 = 2$ (i.e. X is locally 2-dimensional at p) that $\dim_{\mathbb{R}} I_p^2/I_p^3 = 3$ from which we conclude that $M \notin \underline{\text{SmMan}}$. Since the computation of $\dim_{\mathbb{R}} I_p^2/I_p^3$ doesn't depend on if it is done in the manifold X or in $\mathbb{R}^2$, we cary out the computation for $\mathbb{R}^2$. We see that for $f \in I_p^2$,

$$f - \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(p) \cdot (x - p_1)^2 - \frac{1}{2}\frac{\partial^2 f}{\partial y^2} \cdot (y - p_2)^2 - \frac{\partial^2 f}{\partial xy} \cdot (x - p_1)(x - p_2) \in I_p^3.$$

It is therefore evident that $I_p^2/I_p^3 = \langle x^2, y^2, xy \rangle_{\mathbb{R}}$; hence the result. ■

Therefore we cannot use Freyd's adjoint functor theorem to prove the existence or nonexistence of an adjoint to H^*_{dR}. There are a few ways to see that it does not have a left or right adjoint, however: we can show the (co-)solution set conditions do not hold, or compute H^*_{dR} of limits and colimits and see if they are not sent to the corresponding colimits and limits. We carry out both methods to see that no $H^i_{dR} : \underline{\text{SmMan}} \to \underline{\text{Vect}_{\mathbb{R}}}$ has an adjoint, and also that $H^*_{dR} : \underline{\text{SmMan}} \to \underline{\text{Alg}_{\mathbb{R}}}$ does not have an adjoint.

Recall that the Knneth formula tells us that $H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$. Thus the case when $X = Y = \mathbb{S}^1$ tells us the cohomology of the torus, which is clearly not $H^*(\mathbb{S}^1) \times H^*(\mathbb{S}^1)$. Therefore there is no right adjoint to H^* as a functor to $\underline{\text{Alg}_{\mathbb{R}}}$. One can construct similar examples to see that no H^i has a right adjoint either.

To show that H^* does not have a left adjoint, we need to investigate colimits that are more complicated than just coproducts. The Mayer-Vietoris sequence on two open sets tells us how to compute the de Rham cohomology of nice push-out diagrams. The long exact sequence can in fact tell us how far away H^i of a push-out is from being the pull-back. The Mayer-Vietoris sequence tells us that if $\{U, V\}$ is an open cover of a smooth manifold M, then

$$\begin{tikzcd} U \cap V \arrow[r, shift left, "\iota_U"] \arrow[r, shift right, "\iota_V"'] & U \coprod V \arrow[r, "\pi"] & M \end{tikzcd}$$

induces a sequence

$$\Omega^*(M) \xrightarrow{\pi^*} \Omega^*(U) \prod \Omega^*(V) \xrightarrow{\iota_U^* - \iota_V^*} \Omega^*(U \cap V)$$

and then a long exact sequence of cohomology

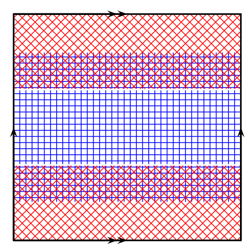
$$\begin{tikzcd} 0 \arrow[r] & H^0(M) \arrow[r] & H^0(U) \times H^0(V) \arrow[r] & H^0(U \cap V) \arrow[dll, "\theta"] \\ & H^1(M) \arrow[r] & H^1(U) \times H^1(V) \arrow[r] & H^1(U \cap V) \cdots \end{tikzcd}$$

Let us write $M = \operatorname{colim}(U \leftarrow U \cap V \to V)$. If H^i were to send this colimit to a limit, the following sequence must be exact

$$0 \longrightarrow H^i(M) \longrightarrow H^i(U) \times H^i(V) \longrightarrow H^i(U \cap V)$$

that is, $H^i(M) \cong \{(f,g) \in H^i(U) \times H^i(V) \mid f|_{U\cap V} = g|_{U \cap V}\}$ which is clearly the description of $\lim\big(H^i(U) \to H^i(U \cap V) \leftarrow H^i(V)\big)$.

We see by the Mayer-Vietoris sequence, that the exactness will fail in general for $i > 0$, and what controls it is the connecting map θ. A concrete example of this arises by considering the following two set cover of the torus



To show that H^0 has no left adjoint, we show that the solution set condition fails. In our definition for the solution set condition, take $Y = \mathbb{R}$ and consider a general set $\{X_i\}$ of smooth manifolds with maps $\{f_i : \mathbb{R} \to H^0(X_i)\}$. Each space X_i decomposes as a disjoint union of its connected components, which we write $X_i = \coprod_{\mathcal{J}} X_{ij}$ where $\mathcal{J}$ is some set. Then $H^0(X_i) = \prod_{\mathcal{J}} \mathbb{R}e_{ij}$. We now take X to be a point, or any smooth manifold with only one connected component, so that $H^0(X) = \mathbb{R}$. If $f_i(e_1) = \sum_{l=1}^{k} r_{j_l} e_{ij_l}$ for $r_{j_l} \neq 0$ and $k \geq 1$, take $\phi(e_1) = (2 \cdot \prod r_{j_l})e_1$. Thus for any map $\overline{\phi} : X \to X_i$, the space X must be sent entirely into one connected component X_{it}. Thus $H^0\overline{\phi}(e_{it}) = e_1$ and $H^0\overline{\phi}(e_{ij}) = 0$ for all other j. Hence

$$\begin{aligned} H^0\overline{\phi}(f_i(e_1)) &= \begin{cases} r_t e_1 & \text{or} \\ 0 \end{cases} \\ &\neq 2 \cdot \prod r_{j_l} e_1. \end{aligned}$$

Now if $f_i(e_1) = 0$, simply take $\phi(e_1) \neq 0$. Thus it is clear that the solution set condition does not hold, hence the result.

Remark 8.58: Behind the proof that not all colimits exist in SmMan is a very general technique that relates to our discussion about the completion of $\mathbb{Q}$. If we want to show that a subset $A \subseteq \mathbb{Q}$ does not have (co-)product in $\mathbb{Q}$, it is much easier to construct the completion $\overline{\mathbb{R}}$ of $\mathbb{Q}$ and see what the (co-)product of $\iota(A)$ is there. Since $\iota : \mathbb{Q} \hookrightarrow \mathbb{R}$ preserves all of those (co-)products which do exist in $\mathbb{Q}$, we can see what the $\prod \iota(A)$ is, and see if it is in $\iota\mathbb{Q}$ or not. We really did just this to show that not all colimits exist in SmMan, just with more complicated machinery and objects. We knew that $\underline{\mathrm{Alg}_{\mathbb{R}}}$ had all limits, and knew that $C^\infty(-,\mathbb{R})$ sent those colimits which do exist to limits. We then investigated the ring $C^\infty(M,\mathbb{R})$ to see if it was in the image of $C^\infty(-,\mathbb{R})$ and saw that it wasn't; hence the colimit did not exist in $\underline{\mathrm{SmMan}}$.

One can ask the same question for limits in $\underline{\mathrm{SmMan}}$. Instead of using $C^\infty(-,\mathbb{R})$, one can use the concept of a Hausdorff Frlicher space to carry out the method. The investigation of Frlicher spaces is left to another paper. See *http://ncatlab.org* for more details on the construction of Frlicher spaces.

9. Ideals in $C^\infty(M,\mathbb{R})$

We now prove proposition 55.

Proposition 9.59: Let $M \in \underline{\mathrm{SmMan}}$. For a point $p \in M$, define the I_p to be the ideal $I_p = \{f \mid f(p) = 0\}$ in $C^\infty(M,\mathbb{R})$. This ideal is maximal, and the product ideals $I_p^n = \langle g_1 \cdots g_n \mid g_i \in I_p\rangle$ have the following description

$$\begin{aligned} I_p^n &= \left\{ f\colon \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(p) = 0 \,\forall \alpha, |\alpha| \le k-1 \right\} \\ &= \{f\colon f \,\text{has}\, k-1 \,\text{order contact with}\, 0\} \\ &= \{f\colon \text{the Taylor series of}\, f \text{has no terms of degree} \le k-1\} \end{aligned}$$

where $\alpha \in \mathbb{N}_0^n$ and $|\alpha| = \sum_{i=1}^n \alpha_i$ is the sum of the entries. Also $x^\alpha = \prod x_i^{\alpha_i}$ for $x \in \mathbb{R}^n$.

In particular, we show:

1. The ideal I_p^2 has the above description in $C^\infty(\mathbb{R}^k,\mathbb{R})$;
2. By induction, the ideals I_p^n admit the above description in $C^\infty(\mathbb{R}^k,\mathbb{R})$;
3. Using partitions of unity, we obtain the general result in $C^\infty(M,\mathbb{R})$.

Proof.

1. We first prove that if $f \in I_p$ and $\frac{\partial f}{\partial x_i}(p) = 0$ for all $1 \le i \le k$, then $f = \sum g_{1i} g_{2i} \in I_p^2$. Define $\phi(t) = f(p + t(x-p))$ for a general $x \in \mathbb{R}^k$. Then

$$\begin{aligned} f(x) = \phi(1) - \phi(0) &= \int_0^1 \frac{d\phi}{dt}(t)dt \\ &= \int_0^1 \sum_{i=1}^k (x_i - p_i) \cdot \frac{\partial f}{\partial x_i}(p + t(x-p))dt \\ &= \sum (x_i - p_i) \cdot \int_0^1 \frac{\partial f}{\partial x_i}(p + t(x-p))dt. \end{aligned}$$

It is clear that defining $g_{1i}(x) = (x_i - p_i)$ and $g_{2i}(x) = \int_0^1 \frac{\partial f}{\partial x_i}(p+t(x-p))dt$ yields the desired decomposition as $g_{2i}(p) = \int_0^1 \frac{\partial f}{\partial x_i}(p)dt = \int_0^1 0dt = 0$.

Showing that $f \in I_p^2$ has all first order derivatives vanishing at p is a straightforward application of the product rule.

2. The general case follows by induction. Suppose f satisfies $\frac{\partial^{|\alpha|}}{\partial x^\alpha}(p) = 0$ for all $|\alpha| < n$. We then show that

$$f(x) = \sum_{|\alpha|=n-1} (x-p)^\alpha \int\limits_{[0,1]^{n-1}} \left(t_1^{n-2} t_2^{n-3} \cdots t_{n-2} \frac{\partial^{n-1} f}{\partial x^\alpha}(p + t_1 \cdots t_{n-1}(x-p)) \right) dV$$

which is then in I_p^n.

Assume the result holds for n and we show that it is true for $n+1$. Under the assumption that f satisfies $\frac{\partial^{|\alpha|}}{\partial x^\alpha}(p) = 0$ for all $|\alpha| < n+1$ we have

$$f(x) = \sum_{|\alpha|=n-1} (x-p)^\alpha \int\limits_{[0,1]^{n-1}} \left(t_1^{n-2} t_2^{n-3} \cdots t_{n-2} \frac{\partial^{n-1} f}{\partial x^\alpha}(p + t_1 \cdots t_{n-1}(x-p)) \right) dV$$

It is then clear that

$$\frac{\partial}{\partial x_i} \left(\int\limits_{[0,1]^{n-1}} \left(t_1^{n-2} t_2^{n-3} \cdots t_{n-2} \frac{\partial^{n-1} f}{\partial x^\alpha}(p + t_1 \cdots t_{n-1}(x-p)) \right) dV \right)(p)$$

$$= \int\limits_{[0,1]^{n-1}} 0dV = 0.$$

Hence we may apply the result obtained for $n = 2$ to each integral expression to get the desired representation of f.

3. The general results follows fairly easily from the above result using partitions of unity to spread the local result to the whole manifold. Specifically, let $U \subseteq M$ be an open set about p which is homeomorphic to $\mathbb{R}^k$ through a local chart $\psi : U \to \mathbb{R}^k$. It is easy to see that if $f : M \to \mathbb{R} \in I_p^n$, then f satisfies the partial derivative condition.

We now suppose $f : M \to \mathbb{R}$ satisfies $\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(p) = 0$ for all α such that $|\alpha| < n$. In this case, $f \circ \psi^{-1}$ satisfies the derivative conditions at $\psi(p)$ so we may use the result proved above to obtain $f\psi^{-1} = \sum h_{i1} \cdots h_{in}$ where each $h_{ij}(\psi(p)) = 0$, and thus, $f|_U = \sum h_{i1} \circ (h_{i1} \circ \psi) \cdots (h_{in} \circ \psi)$. We denote $h_{ij} \circ \psi = g_{ij}$.

To extend each g_{ij} to all of M, we use a partition of unity $\{\phi_U, \phi_V\}$ subordinate to $\{U, V\}$ where $V = M \setminus W$ where $W \subset U$ is closed with

$p \in W$. Then we compute

$$\begin{aligned} f &= (\phi_U + \phi_V)^n \cdot f \\ &= \sum_{\ell=0}^{n} \binom{n}{\ell} \phi_U^{n-\ell} \phi_V^{\ell} f. \end{aligned}$$

We can make sense out of this by observing ϕ_V has support in V, and thus by construction $\phi_V \in I_p$. Thus $\phi_V^n \cdot f \in I_p^n$. All other terms $\phi_V^{n-\ell} \phi_V^{\ell} f$ with $\ell \neq 0$ have support in $U \cap V$ where the description $f = \sum g_{ij}$ holds. Thus

$$\begin{aligned} \phi_V^{n-\ell} \phi_U^{\ell} f &= \phi_V^{n-\ell} \phi_U^{\ell} \left(\sum g_{i1} \cdots g_{in} \right) \\ &= \sum \phi_U^{n-\ell} \phi_V^{\ell} g_{i1} \cdots g_{in} \\ &= \sum (\phi_U g_{i1})(\phi_U g_{i2}) \cdots (\phi_U g_{i(n-\ell-1)})(\phi_U g_{i(n-\ell) \cdots g_{in}}) \phi_U^{\ell} \end{aligned}$$

and it is evident since ϕ_U has support in U that each $(\phi_U g_{ij})$ is a well defined smooth function on M with $(\phi_U g_{ij})(p) = 0$; hence the result.

■

References

[F-1970] Freyd, P.: Homotopy is not concrete, *Reprints in Theory and Applications of Categories*, No. 6, 1–10 (2004).

[A-2004] Adámek, A.; Herrlich, H.; Strecker, G.: *Abstract and Concrete Categories; The Joy of Cats*, on-line edition 2004.

[H-1970] Hilton, P.; Stammbach, U.: *A Course in Homological Algebra*, New York (1970).

[K-1958] Kan, Daniel: Adjoint Functors, *Transactions of the American Mathematics Society.* **87**, 294–329 (1958).

[ML-1971] Mac Lane, S.: *Categories for the Working Mathematician* New York (1971).

[M-1967] Mitchell, B.: *Theory of Categories*, New York (1967).

[S-1992] Sieradski, A.: *An Introduction to Topology and Homotopy*, Boston (1992).

[MO1-2010] *Ideals in the ring of smooth endomorphisms of the real line*

http://mathoverflow.net/questions/21614/ideals-in-the-ring-of-smooth-endomorphisms-of-the-real-line

[MO2-2010] *Colimits in the Category of Smooth Manifolds*

http://mathoverflow.net/questions/19116/colimits-in-the-category-of-smooth-manifolds