Compactly supported Poincaré lemma

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0. Introduction

We begin by introducing some notation and definitions to get our investigation of compactly supported cohomolgy started. We then proceed to compute the compactly supported cohomology groups of simple spaces like $* = \mathbb{R}^0$, $\mathbb{R}^1$ and $\mathbb{R}^2$. From these special cases, we will see how to generalize the computation to $\mathbb{R}^n$. Thereafter, we will prove the general version of the Poincaré Lemma, which states $H_c^\bullet(M \times \mathbb{R}) \cong H_c^{\bullet-1}(M)$. This paper follows the method of proof found in [B&T-1982].

1. Definitions and Notation

Some may find it useful to have the following constructions in mind throughout this paper.

Notation 1.1: Let <u>SmMan</u> be the category with objects all finite dimensional smooth manifolds and with morphisms all smooth maps.

Notation 1.2: Let <u>ISmMan</u> $\subset$ <u>SmMan</u> be the subcategory with objects all smooth manifolds and morphisms all maps $f : M \to N \in$ <u>SmMan</u> so that f is a diffeomorphism on its image and $\text{im}(f) \subseteq N$ is open. (The "I" stands for inclusion).

Notation 1.3: Let <u>PSmMan</u> $\subset$ <u>SmMan</u> be the subcategory with objects all smooth manifolds and morphisms all proper maps $f : M \to N \in$ <u>SmMan</u>. Recall that $f : M \to N$ is proper if for any compact subset $C \subseteq N$, then $f^{-1}(C)$ is also compact.

Definition 1.4: For $M \in$ <u>SmMan</u> define $\Omega_c^\bullet(M) = \{\omega \in \Omega^\bullet(M) \mid \text{supp}(\omega) \text{ is compact }\}$. Recall that $\text{supp}(\omega) = \overline{\{x \in M \mid \omega_x \neq 0\}}$.

We would like to extend $\Omega_c^\bullet$ to a functor. This cannot be done in an obvious (and meaningful) way to all of <u>SmMan</u>. Two extensions are possible using <u>ISmMan</u> or <u>PSmMan</u>.

Definition 1.5: Let $f : M \to N \in$ <u>PSmMan</u>. Then if $\omega \in \Omega_c^\bullet(N)$, the pull-back $f^*\omega$ has compact support in M, since $\text{supp}(f^*\omega) = f^{-1}(\text{supp}(f))$. Then definining $\Omega_c^\bullet :$ <u>PSmMan</u> $\to \mathfrak{C}$ by $\Omega_c^\bullet(f : M \to N) = f^* : \Omega_c^\bullet(N) \to \Omega_c^\bullet(M)$ gives a contravariant functor where $\mathfrak{C}$ can be taken to be <u>Ch</u> $(\mathbb{R}-$<u>Mod</u>$)$, i.e. chain complexes of $\mathbb{R}$-modules (vector spaces) with degree 0 chain maps as morphisms.

Definition 1.6: Let $f : M \to N \in$ <u>ISmMan</u>. Then if $\omega \in \Omega_c^\bullet(M)$, we may define

$$f_*\omega_x = \begin{cases} \left(f^{-1}\right)^* \omega_x & \text{if } x \in \text{im}(f) \\ 0 & \text{otherwise .} \end{cases}$$

This is a compactly supported form on N since $\text{supp}(f_*\omega) = f(\text{supp}(\omega))$ is compact, and it is smooth because $\text{supp}(f_*\omega)$ is a compact subset of $f(M) \subseteq N$ which is open, so extending by 0 is smooth. Then $\Omega_c^\bullet(f : M \to N) = f_* : \Omega_c^\bullet(M) \to \Omega_c^\bullet(N)$ defines a (covariant) functor $\Omega_c^\bullet :$ <u>ISmMan</u> $\to$ <u>Ch</u> $(\mathbb{R}-$<u>Mod</u>$)$. Throughout this paper, we will be concerned with this extension of $\Omega_c^\bullet$.

2. Computation of $H_c^\bullet(*)$ and $H_c^\bullet(\mathbb{R})$

Definition 2.7: For $M \in \underline{\text{ISmMan}}$, define $H_c^\bullet(M) = H^\bullet(\Omega_c^\bullet(M))$. We may consider $H_c^\bullet : \underline{\text{ISmMan}} \to \mathfrak{C}$ as a functor where $\mathfrak{C}$ is either the category of $\mathbb{N}_0$-graded $\mathbb{R}$-modules with degree 0 morphisms, or the category of graded-commutative $\mathbb{R}$-algebras with degree 0 graded algebra homomorphisms.

We would like to compute $H_c^\bullet(M)$ for some $M \in \underline{\text{SmMan}}$. Perhaps the easiest case to start with is when $M = \mathbb{R}^0 = *$. We first observe that

$$\begin{aligned} \Omega_c^\bullet(*) &= 0 \longrightarrow \Omega_c^0(*) \longrightarrow 0 \\ \Omega_c^\bullet(*) &= 0 \longrightarrow \mathbb{R} \longrightarrow 0 \end{aligned}$$

from which it immediately follows that $H_c^0(*) = \mathbb{R}$ and $H_c^k(*) = 0$ for $k \neq 0$.

Now let us compute $H_c^\bullet(\mathbb{R})$. We have

$$\Omega_c^\bullet(\mathbb{R}) = 0 \longrightarrow \Omega_c^0(\mathbb{R}) \xrightarrow{d} \Omega_c^1(\mathbb{R}) \longrightarrow 0.$$

First we see that

$$\begin{aligned} H_c^0(\mathbb{R}) &= \ker\left(d^0\right) \\ &= \{f \in C_c^\infty(\mathbb{R}) \mid \frac{df}{dt} \equiv 0\} \\ &= \{f \in C_c^\infty(\mathbb{R}) \mid \forall t \in \mathbb{R}, f(t) = c\} \\ &= \{0\}. \end{aligned}$$

Next we observe that $\ker d^1 = \Omega_c^1(\mathbb{R}) = \{f(t)\,\mathrm{dt} \mid f \in C_c^\infty(\mathbb{R})\}$ and $\operatorname{im} d^0 = \{\frac{\partial f}{\partial t}\,\mathrm{dt} \mid f \in \Omega_c^0(\mathbb{R})\}$. A standard approach in homological algebra for determining the structure of $\Omega_c^1(\mathbb{R}) / \operatorname{im} d^0$ is to find an $\mathbb{R}$-linear map $A : \Omega_c^1(\mathbb{R}) \twoheadrightarrow \mathbb{R}^k$ where $\mathbb{R}^k$ is a guess of what $H_c^1(\mathbb{R})$ should be, and then show that $\ker A = \operatorname{im} d^0$. The nicest linear map on compactly supported 1-forms around is $\int_{\mathbb{R}} : \Omega_c^1(\mathbb{R}) \to \mathbb{R}$, so let's try that. We first notice that $\int_{\mathbb{R}}$ is a surjective linear map, since there exists a bump function $\rho : \mathbb{R} \to \mathbb{R}$ with compact support such that $\int_{\mathbb{R}} \rho\,\mathrm{dt} = 1$. Let us fix this choice and call $e = \rho\,\mathrm{dt}$. Now we verify that indeed $\ker \int_{\mathbb{R}} = \operatorname{im} d^0$. First let $f \in \Omega_c^0(\mathbb{R})$ with $\operatorname{supp}(f) \subseteq [a,b]$. Then

$$\begin{aligned} \int_{\mathbb{R}} df &= \int_{\mathbb{R}} \frac{\partial f}{\partial t}\,\mathrm{dt} \\ &= \int_{[a,b]} \frac{\partial f}{\partial t}\,\mathrm{dt} \\ &= f(b) - f(a) \\ &= 0. \end{aligned}$$

so $\operatorname{im} d^0 \subseteq \ker \int_{\mathbb{R}}$. Now suppose $g(t)\,\mathrm{dt} \in \ker \int_{\mathbb{R}}$. Then we can consider $G(t) = \int_{-\infty}^t g\,\mathrm{dt}$. The fundamental theorem of calculus tells us that G is differentiable with $\frac{dG}{\mathrm{dt}}(t) = g(t)$. If $G(t)$ is compactly supported, then it will follow that $g(t)\,\mathrm{dt} \in \operatorname{im} d^0$. Since $g(t)\,\mathrm{dt}$ is compactly supported, $\int_{-\infty}^t g\,\mathrm{dt}$ is constant when $t \notin \operatorname{supp}(g)$. For $t < a := \inf\{x \mid x \in \operatorname{supp}(g)\}$, we have $G(t) = 0$ and for $t > b := \sup\{x \mid x \in \operatorname{supp}(g)\}$ we have $G(t) = \int_{\mathbb{R}} g\,\mathrm{dt} = 0$; hence G is compactly supported with $\operatorname{supp}(G) \subseteq [a,b]$. Thus it follows $\ker \int_{\mathbb{R}} = \operatorname{im} d^0$, so $H_c^1(\mathbb{R}) \cong \mathbb{R}$.

The previous calculation shows that two 1-forms are cohomologous if and only if they have the same total integral. It therefore is reasonable to try to single out a representative 1-form for each distinct class. We have in fact already done this with our choice of a bump 1-form e satisfying $\int_{\mathbb{R}} e = 1$. So for $g\,\mathrm{dt} \in \Omega_c^1(\mathbb{R})$, we can take the representative of its cohomology class to be $\left(\int_{\mathbb{R}} g\,\mathrm{dt}\right) e$. That is, $g\,\mathrm{dt} - \left(\int_{\mathbb{R}} g\,\mathrm{dt}\right) \rho\,\mathrm{dt}$ should be a boundary of something. The argument above says that

$$g\,\mathrm{dt} - \left(\int_{\mathbb{R}} g\,\mathrm{dt}\right) \cdot \rho\,\mathrm{dt} = d\left(\int_{-\infty}^{t} g\,\mathrm{dt} - \left(\int_{\mathbb{R}} g\,\mathrm{dt}\right) \cdot \int_{-\infty}^{t} \rho\,\mathrm{dt}\right).$$

We can look at this in terms of chain maps and chain homotopies. We have two chain maps $\mathrm{id}_{\Omega_c^\bullet(\mathbb{R})} : \Omega_c^\bullet(\mathbb{R}) \to \Omega_c^\bullet(\mathbb{R})$ and $r : \Omega_c^\bullet(\mathbb{R}) \to \Omega_c^\bullet(\mathbb{R})$ where $r^0 = 0$ and $r^1(g\,\mathrm{dt}) = \left(\int_{\mathbb{R}} g\,\mathrm{dt}\right) \rho\,\mathrm{dt}$. The claim that $g\,\mathrm{dt}$ and $r(g\,\mathrm{dt})$ determine the same cohomology class is equivalent to saying that id and r induce the same maps on cohomology. To prove this, it would suffice to come up with a chain homotopy $K : \Omega_c^\bullet \to \Omega_c^\bullet$. We, in fact, have already done this! By necessity, define $K^0 = 0$, and $K^1(g\,\mathrm{dt}) = \int_{-\infty}^{t} g\,\mathrm{dt} - \left(\int_{\mathbb{R}} g\,\mathrm{dt}\right) \int_{-\infty}^{t} \rho\,\mathrm{dt}$. The linearity of K^1 follows from linearity of integration. A diagram may help make the situation clear:

$$\begin{tikzcd}
0 \arrow[r] & \Omega_c^0(\mathbb{R}) \arrow[r, "d"] \arrow[dl, "K^0"'] \arrow[d, shift right, "r^0"'] \arrow[d, shift left, "\mathrm{id}"] & \Omega_c^1(\mathbb{R}) \arrow[r] \arrow[dl, "K^1"'] \arrow[d, shift right, "r^1"'] \arrow[d, shift left, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \Omega_c^0(\mathbb{R}) \arrow[r, "d"] & \Omega_c^1(\mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

To check that K is a chain homotopy between id and r, we need to check that $\mathrm{id} - r = \pm(Kd + dK)$. That is, for $f \in \Omega_c^0(\mathbb{R})$, we need $f = \pm K df$ and for $g\,\mathrm{dt} \in \Omega_c^1(\mathbb{R})$ we need $g\,\mathrm{dt} - r(g\,\mathrm{dt}) = d(K(g\,\mathrm{dt}))$. We compute

$$\begin{aligned}
K(df)(t) &= K\left(\frac{\partial f}{\partial t}\right)(t) \\
&= \int_{-\infty}^{t} \frac{\partial f}{\partial t}\,\mathrm{dt} - \left(\int_{\mathbb{R}} \frac{\partial f}{\partial t}\,\mathrm{dt}\right) \cdot \int_{-\infty}^{t} \rho\,\mathrm{dt} \\
&= \int_{-\infty}^{t} \frac{\partial f}{\partial t}\,\mathrm{dt} \\
&= f(t)
\end{aligned}$$

which verifies the first condition. We have already seen that $g\,\mathrm{dt} - r^1(g\,\mathrm{dt}) = d(K(g\,\mathrm{dt}))$, which is the second condition. Hence K is a chain homotopy between id and r.

3. Computation of $H_c^\bullet(\mathbb{R}^n)$

The key to generalizing the previous computation to $\mathbb{R}^n$ is to rethink the map r. Let us consider $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ where we use coordinates $x_1, ..., x_n$ on $\mathbb{R}^n$ and t on $\mathbb{R}$. There are two maps to define in order to generalize r in a useful manner.

First we define a chain map $\pi_* : \Omega_c^\bullet(\mathbb{R}^n \times \mathbb{R}) \to \Omega_c^{\bullet-1}(\mathbb{R}^n)$. To define π_*, observe first that two types of forms generate $\Omega_c^\bullet(\mathbb{R}^n \times \mathbb{R})$: i.) forms without "dt", i.e. $f(x,t)\,\mathrm{dx}_I$ where $f \in \Omega_c^0(\mathbb{R}^n \times \mathbb{R})$, $I = (i_1, ... i_j)$, and $\mathrm{dx}_I = \mathrm{dx}_{i_1}\,\mathrm{dx}_{i_2} \cdots \mathrm{dx}_{i_j}$; ii.) forms with "dt", i.e. $f(x,t)\,\mathrm{dx}_I\,\mathrm{dt}$. We define $\pi_*(f(x,t)\,\mathrm{dx}_I) = 0$ and $\pi_*(f(x,t)\,\mathrm{dx}_I\,dt) = \left(\int_{\mathbb{R}} f(x,t)\,\mathrm{dt}\right) \mathrm{dx}_I$ and extend by linearity. That π_* is a chain map is easily verified.

The next map is $e_* : \Omega_c^{\bullet-1}(\mathbb{R}^n) \to \Omega_c^{\bullet}(\mathbb{R}^n \times \mathbb{R})$ which is induced by our choice of $e = \rho\,\mathrm{dt} \in \Omega_c^1(\mathbb{R})$; define $e_*(\omega) = \omega \wedge e$. Again, it is straightforward to verify that e_* is a chain map.

We now observe that $r = e_* \circ \pi_*$ when we take $n = 0$ in the above constructions. It is also easy to verify that $\pi_* \circ e_* = \mathrm{id}_{\Omega_c^{\bullet}(\mathbb{R}^n)}$. If we can show—just as in the case of $\mathbb{R}$—that $e_* \circ \pi_* \simeq \mathrm{id}$, then it will follow that $H^{\bullet}(\Omega_c^{\bullet}(\mathbb{R}^n \times \mathbb{R})) \cong H^{\bullet}\left(\Omega_c(\mathbb{R}^n)^{\bullet-1}\right) \cong H^{\bullet-1}(\Omega_c^{\bullet}(\mathbb{R}^n))$ so we can compute $H_c^{\bullet}(\mathbb{R}^n)$ inductively. An intuitive idea of what this is saying is that compactly supported top dimensional forms $f(x,t)\,\mathrm{dx}_1 \cdots \mathrm{dx}_n$ can be integrated over $\mathbb{R}^n$ coordinate by coordinate using Fubini's Theorem which then gives us a real number which determines the cohomology class of $f(x,t)\,\mathrm{dx}_1 \cdots \mathrm{dx}_n$.

The picture of the general situation is this:

$$\begin{tikzcd}
0 \arrow[r] & \Omega_c^0(\mathbb{R}^n \times \mathbb{R}) \arrow[r, "d"] \arrow[d, "\pi_*"] & \Omega_c^1(\mathbb{R}^n \times \mathbb{R}) \arrow[r, "d"] \arrow[d, "\pi_*"] & \cdots \arrow[r, "d"] & \Omega_c^{n+1}(\mathbb{R}^n \times \mathbb{R}) \arrow[r] \arrow[d, "\pi_*"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d, "e_*"] & \Omega_c^0(\mathbb{R}^n) \arrow[r, "d"] \arrow[d, "e_*"] & \cdots \arrow[r, "d"] & \Omega_c^n(\mathbb{R}^n) \arrow[r] \arrow[d, "e_*"] & 0 \\
0 \arrow[r] & \Omega_c^0(\mathbb{R}^n \times \mathbb{R}) \arrow[r, "d"] & \Omega_c^1(\mathbb{R}^n \times \mathbb{R}) \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Omega_c^{n+1}(\mathbb{R}^n \times \mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

We are trying to construct a chain homotopy K between $\mathrm{id}_{\Omega_c^{\bullet}(\mathbb{R}^n \times \mathbb{R})}$ and $e_* \circ \pi_*$ which looks like

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & \Omega_c^{k-1}(\mathbb{R}^n \times \mathbb{R}) \arrow[r, "d"] \arrow[d, "\pi_*"] \arrow[ddl, bend right, "K"'] \arrow[dd, bend right, "\mathrm{id}"'] & \Omega_c^{k}(\mathbb{R}^n \times \mathbb{R}) \arrow[r, "d"] \arrow[d, "\pi_*"] \arrow[ddl, bend right, "K"'] \arrow[dd, bend right, "\mathrm{id}"'] & \Omega_c^{k+1}(\mathbb{R}^n \times \mathbb{R}) \arrow[r, "d"] \arrow[d, "\pi_*"] \arrow[ddl, bend right, "K"'] \arrow[dd, bend right, "\mathrm{id}"'] & \cdots \\
\cdots \arrow[r, "d"] & \Omega_c^{k-2}(\mathbb{R}^n) \arrow[r, "d"] \arrow[d, "e_*"] & \Omega_c^{k-1}(\mathbb{R}^n) \arrow[r, "d"] \arrow[d, "e_*"] & \Omega_c^{k}(\mathbb{R}^n) \arrow[r, "d"] \arrow[d, "e_*"] & \cdots \\
\cdots \arrow[r, "d"] & \Omega_c^{k-1}(\mathbb{R}^n \times \mathbb{R}) \arrow[r, "d"] & \Omega_c^{k}(\mathbb{R}^n \times \mathbb{R}) \arrow[r, "d"] & \Omega_c^{k+1}(\mathbb{R}^n \times \mathbb{R}) \arrow[r, "d"] & \cdots
\end{tikzcd}$$

As with π_*, we will define K on type i. and type ii. forms separately, which will then determine K. For type i. forms define $K(g(x,t)\,\mathrm{dx}_I) = 0$, and for type ii. forms define

$$K(g(x,t)\,\mathrm{dx}_I\,\mathrm{dt}) = \left(\int_{-\infty}^{t} g(x,t)\,dt\right)\mathrm{dx}_I - \left(\int_{\mathbb{R}} g(x,t)\,\mathrm{dt}\right)\left(\int_{-\infty}^{t} e\right)\mathrm{dx}_I$$

which bears a striking resemblance with the definition of K in the case for $\mathbb{R}$. K is again linear by linearity of integration, and we will now show that K is a chain homotopy operator (up to sign).

Lemma 3.8: (Poincaré Lemma) We have $H_c^{\bullet}(\mathbb{R}^n \times \mathbb{R}) \cong H_c^{\bullet-1}(\mathbb{R}^n)$.

Corollary 3.9:

$$H_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We prove this lemma by proving $\Omega_c^\bullet(\mathbb{R}^n \times \mathbb{R})$ is chain homotopy equivalent to $\Omega_c^{\bullet-1}(\mathbb{R}^n)$. All that remains is to show $\mathrm{id} - e_*\pi_* = \pm(dK - Kd)$. This is just a complicated calculation, which we include for completeness.

We check this first on k-forms of type i., i.e. $g(x,t)\,\mathrm{dx}_I$. We compute

$$\begin{aligned}(\mathrm{id} - e_*\pi_*)(g\,\mathrm{dx}_I) &= g\,\mathrm{dx}_I - e_*(\pi_*(g\,\mathrm{dx}_I))\\ &= g\,\mathrm{dx}_I - e_*(0)\\ &= g\,\mathrm{dx}_I\end{aligned}$$

and

$$\begin{aligned}(dK - Kd)(g\,\mathrm{dx}_I) &= -K(d(g\,\mathrm{dx}_I))\\ &= -K\left(\left(\sum_i \frac{\partial g}{\partial x_i}\,\mathrm{dx}_i\,\mathrm{dx}_I\right) + \frac{\partial g}{\partial t}\,\mathrm{dt}\,\mathrm{dx}_I\right)\\ &= -K\left((-1)^q\left(\sum_i \frac{\partial g}{\partial x_i}\,\mathrm{dx}_I\,\mathrm{dx}_i\right) + (-1)^q\frac{\partial g}{\partial t}\,\mathrm{dx}_I\,\mathrm{dt}\right)\\ &= (-1)^{k+1}\left(\int_{-\infty}^t \frac{\partial g}{\partial t}\,\mathrm{dt}\right)\mathrm{dx}_I - \left(\int_{\mathbb{R}} \frac{\partial g}{\partial t}\,\mathrm{dt}\right)\left(\int_{-\infty}^t e\right)\mathrm{dx}_I\\ &= (-1)^{k+1}\,g\,\mathrm{dx}_I\,.\end{aligned}$$

Now we check the equality on forms of type ii., i.e. $g(x,t)\,\mathrm{dx}_I\,\mathrm{dt}$:

$$(\mathrm{id} - e_*\pi_*)\,g(x,t)\,\mathrm{dx}_I\,\mathrm{dt} = g(x,t)\,\mathrm{dx}_I\,\mathrm{dt} - \left(\int_{\mathbb{R}} g(x,t)\,\mathrm{dt}\right)\mathrm{dx}_I \wedge e$$

and

$$\begin{aligned}dK(g\,\mathrm{dx}_I\,\mathrm{dt}) &= d\left(\left(\int_{-\infty} g\,\mathrm{dt}\right)\mathrm{dx}_I - \left(\int_{\mathbb{R}} g\,\mathrm{dt}\right)\left(\int_{-\infty}^t e\right)\mathrm{dx}_I\right)\\ &= g\,\mathrm{dt}\,\mathrm{dx}_I + \sum_i\left(\int_{-\infty}^t \frac{\partial g}{\partial t}\,\mathrm{dt}\right)\mathrm{dx}_i\,\mathrm{dx}_I - \left(\int_{\mathbb{R}} g\,\mathrm{dt}\right)e\,\mathrm{dx}_I\\ &\qquad - \sum_i\left(\int_{\mathbb{R}} \frac{\partial g}{\partial x_i}\,\mathrm{dt}\right)\left(\int_{-\infty}^t e\right)\mathrm{dx}_i\,\mathrm{dx}_I\\ Kd(g\,\mathrm{dx}_I\,\mathrm{dt}) &= K\left(\sum_i \frac{\partial g}{\partial x_i}\,\mathrm{dx}_i\,\mathrm{dx}_I\,\mathrm{dt}\right)\\ &= \sum_i\left(\int_{-\infty}^t \frac{\partial g}{\partial x_i}\,\mathrm{dt}\right)\mathrm{dx}_i\,\mathrm{dx}_I - \sum_i\left(\int_{\mathbb{R}} \frac{\partial g}{\partial x_i}\,\mathrm{dt}\right)\left(\int_{-\infty}^t e\right)\mathrm{dx}_i\,\mathrm{dx}_I\end{aligned}$$

In computing $(dK - Kd)(g\,\mathrm{dx}_I\,\mathrm{dt})$, the red and blue terms cancel with each other respectively so we are left with

$$\begin{aligned}(dK - Kd)(g\,\mathrm{dx}_I\,\mathrm{dt}) &= g\,\mathrm{dt}\,\mathrm{dx}_I - \left(\int_{\mathbb{R}} g\,\mathrm{dt}\right)e\,\mathrm{dx}_I\\ &= (-1)^k\left(g\,\mathrm{dx}_I\,\mathrm{dt} - \left(\int_{\mathbb{R}} g\,\mathrm{dt}\right)\mathrm{dx}_I\,e\right)\\ &= (-1)^k(\mathrm{id} - e_*\pi_*)(g\,\mathrm{dx}_I\,\mathrm{dt})\,.\end{aligned}$$

Hence the result follows. ▲

4. Generalization to arbitrary smooth manifolds

Observe that the computation of $H_c(\mathbb{R}^n)$ shows that the functor H_c is not homotopy invariant. It is at least invariant under diffeomorphisms. The generalized Poincaré Lemma can then be seen as a first attempt to figure out the relation between the compactly supported cohomology of homotopic manifolds in a restricted case, i.e. trivial line bundles over a manifold M. This can be extended easily to trivial finite dimensional vector bundles over a manifold M. Computing the compactly supported cohomology of general vector bundles $\pi : E \to M$ will have to wait for another paper. We will establish the following theorem in this section.

Theorem 4.10: We have $H_c^\bullet(M \times \mathbb{R}) \cong H_c^{\bullet-1}(M)$.

Essentially everything done in the previous section will go through with only minor modifications if we can figure out a similar description for forms of "type i. and type ii." Note that our description in the previous section was heavily reliant on working with $\mathbb{R}^n$. Let us define forms of type i. on M to be those given by $f\pi^*\phi$ where $f \in \Omega_c^0(M \times \mathbb{R})$ and $\phi \in \Omega^\bullet(M)$, and forms of type ii. to be those forms given by $f\pi^*\phi\,\mathrm{dt}$. A straightforward partition of unity argument shows that $\Omega_c^\bullet(M \times \mathbb{R})$ is generated as an $\mathbb{R}$ vector space by forms of type i. and type ii.

With this definition of forms of type i. and type ii., we can proceed to define the relevant maps $\pi_* : \Omega_c^\bullet(M \times \mathbb{R}) \to \Omega_c^{\bullet-1}(M)$, $e_* : \Omega_c^{\bullet-1}(M) \to \Omega_c^\bullet(M \times \mathbb{R})$ and K. For a type i. form $f\pi^*\omega$, define $\pi_*(f\pi^*\omega) = 0$, and for a type ii. form $f\pi^*\omega\,\mathrm{dt}$, define $\pi_*(f\pi^*\omega\,\mathrm{dt}) = \left(\int_{\mathbb{R}} f\,\mathrm{dt}\right)\omega$. For any form $\omega \in \Omega_c^{\bullet-1}(M)$ define $e_*(\omega) = \omega \wedge e$ where we recall that $e \in \Omega_c^1(\mathbb{R})$ given by $e = \rho\,\mathrm{dt}$.

Once again, we can verify that π_* and e_* are cochain maps, and that $\pi_* e_* = \mathrm{id}_{\Omega_c^{\bullet-1}(M \times \mathbb{R})}$. So to prove the theorem, it suffices to show $e_*\pi_* \simeq \mathrm{id}$. The chain homotopy K in this case is defined by $K(f\pi^*\omega) = 0$ and

$$K(f\pi^*\omega\,\mathrm{dt}) = \left(\int_{-\infty}^t f\,\mathrm{dt}\right)\omega - \left(\int_{\mathbb{R}} f\,\mathrm{dt}\right)\left(\int_{-\infty}^t e\right)\omega.$$

It is then straightforward (yet complicated) to verify that $\mathrm{id} - \pi_* e_* = \pm(dK - Kd)$, and is entirely analogous to the proof in the case for $M = \mathbb{R}^n$.

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