Σ Theory

Glen M. Wilson

Adviser: Dr. Clifford

The College of New Jersey

Finitely Presented Groups

Definition: A group is finitely presented if its group structure can be described with finitely many generators and finitely many relations.

A group G is finitely generated if there is some finite number of elements $g_1, g_2, \ldots, g_n \in G$ for which all other elements of G can be expressed as a finite product of these elements.

A group G is finitely presented if it is finitely generated, say by $g_1, \ldots, g_n$, and $G \cong F(g_1, \ldots, g_n)/\mathcal{R}$ where $\mathcal{R}$ is finitely generated as a normal subgroup of the free group on $g_1, \ldots, g_n$.

Example: $\mathbb{Z} \times \mathbb{Z} \cong \langle a, b \mid ab = ba \rangle$

Finitely Presented Groups

Questions:

- How can one determine if a given group is finitely presented?
- Is there something computable that tells us if a group is finitely presented or not?

Σ theory provides an answer to these questions for metabelian groups.

Metabelian Groups

Definition: A metabelian group is a group G for which there exists $A \lhd G$ such that A and G/A are Abelian groups. That is, there is a short exact sequence

$$0 \longrightarrow A \longrightarrow G \longrightarrow Q \longrightarrow 1$$

where A and Q are Abelian.

Example:

- The class of metabelian groups contains more than just Abelian groups

$$1 \longrightarrow \mathbb{Z}_3 \longrightarrow S_3 \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

- The alternating groups A_n for $n \geq 5$ are non-Abelian simple groups; hence they are not metabelian groups.

Q-Modules

Definition: A Q-module is an Abelian group A with an action of a group Q on A, i.e.

- $\cdot : Q \times A \to A$
- $1 \cdot a = a$ for all $a \in A$
- $(qq') \cdot a = q \cdot (q' \cdot a)$

which satisfies:

- $q \cdot (a + a') = q \cdot a + q \cdot a'$ for all $a, a' \in A$.

This is equivalent to there being an "actual" $\mathbb{Z}[Q]$-module structure on A where $\mathbb{Z}[Q]$ is the group ring.

Q-MODULE STRUCTURE ON A

$$0 \longrightarrow A \longrightarrow G \longrightarrow Q \longrightarrow 1$$

- G acts on A by conjugation because A is a normal subgroup of G.
- Since A is Abelian, it acts trivially on itself; hence there is a well defined $G/A \cong Q$ action on A.
  We can use this extra structure of Q on A to determine when G is finitely presented.
- Given a Q-module A, we define the semi-direct product $A \rtimes Q$ where $(a, q) * (b, r) = (a + q \cdot b, qr)$.

$$0 \longrightarrow A \xrightarrow{\iota} A \rtimes Q \xrightarrow{\pi} Q \longrightarrow 1$$

Observations

We consider a general extension of Q by A

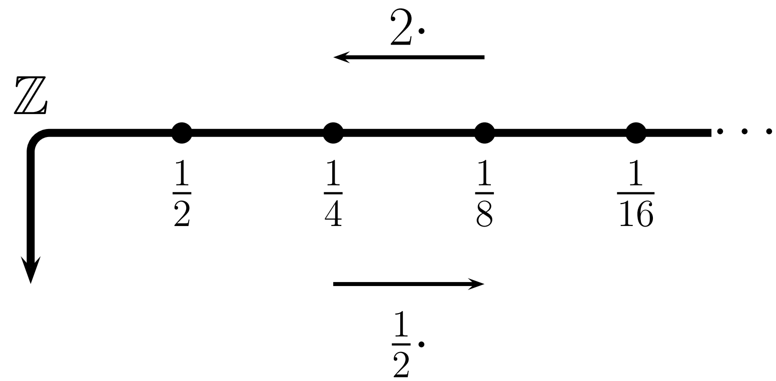
$$0 \longrightarrow A \longrightarrow G \longrightarrow Q \longrightarrow 1.$$

- Q not finitely generated $\Rightarrow$ G not finitely presented
- A not finitely generated Q-module $\Rightarrow$ G not finitely presented
- A and Q finitely generated Abelian groups $\Rightarrow$ G finitely presented

Thus the interesting case is where Q is a finitely generated Abelian group and A is infinitely generated as an Abelian group, but finitely generated as a Q-module.

Example

- $A = \mathbb{Z}[1/2]$, i.e. the dyadic rationals, and take $Q = \langle q \rangle$ the infinite cyclic group.
- Q-module structure on A given by
$$q \cdot x = \frac{1}{2}x.$$
- $\{1\}$ generates $\mathbb{Z}[1/2]$ as a Q-module.



- $\mathbb{Z}[1/2]$ is not a finitely generated Abelian group.
- We don't need all powers of q to finitely generate A with this action; $\{q^n \mid n \geq 0\}$ suffices.

Question: Is $\mathbb{Z}[1/2] \rtimes \mathbb{Z}$ finitely presented?

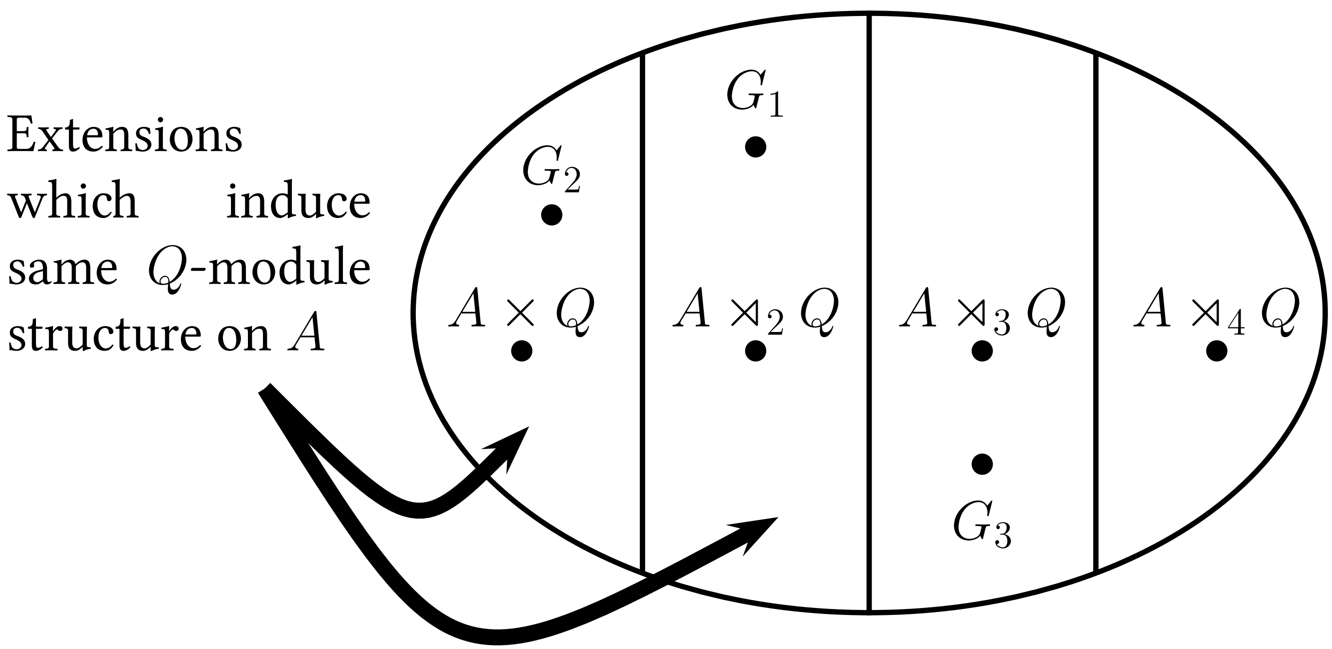
First Main Result of Σ Theory

If G is metabelian with

$$0 \longrightarrow A \longrightarrow G \longrightarrow Q \longrightarrow 1$$

then G is finitely presented if and only if $A \rtimes Q$ is.

Extensions of Q by A



Special Case

In the special case when $Q = \langle q \rangle$, define

$$Q_+ = \{q^n \mid n \geq 0\}, Q_- = \{q^n \mid n \leq 0\}.$$

Then the Σ-invariant may be defined as

$$\Sigma_A = \{Q_\varepsilon \mid A \text{ is finitely generated over } Q_\varepsilon\}$$

In this case, the second main result tells us that

- G is finitely presented if and only if $\Sigma_A \neq \emptyset$,
- i.e. A is finitely generated over Q_+ or Q_-.

Example

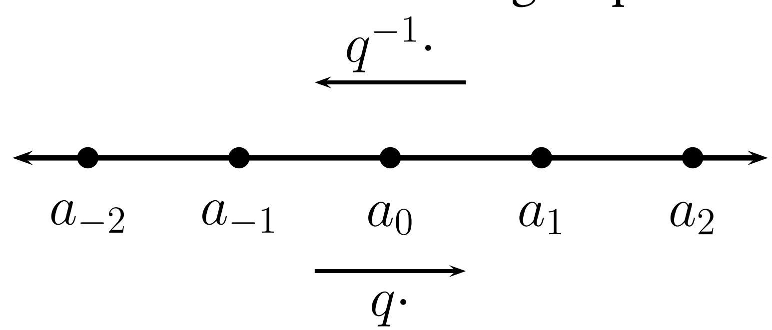
Let $A = \mathbb{Z}[1/2]$, i.e. the dyadic rationals; let $Q = \langle q \rangle$.

- Q-module structures on A are given by $q \cdot x = \pm 2^n x$ where $n \in \mathbb{Z}$. As long as $n \neq 0$, A will be finitely generated over Q.
- Any Q-module structure on A which makes A finitely generated over Q has $\Sigma_A \neq \emptyset$.
- Thus by the main result, every extension G of $\mathbb{Z}$ by $\mathbb{Z}[1/2]$ which induces one of these module structures is finitely presented.
- In the case $q \cdot x = x$, the split extension extension is $\mathbb{Z}[1/2] \rtimes \mathbb{Z} \cong \mathbb{Z}[1/2] \times \mathbb{Z}$ is not even finitely generated.

Example

Let $A = \oplus_{i\in\mathbb{Z}}\mathbb{Z}a_i$ the free Abelian group on the generators $\{a_i \,|\, i \in \mathbb{Z}\}$, and let $Q = \langle q\rangle$. A Q-module structure on A is given by $q \cdot a_i = a_{i+1}$ and extended by linearity.

- A is not finitely generated as an Abelian group.



- A is finitely generated by $\{a_0\}$ as a Q-module.
- A is not finitely generated over $Q_+ = \{q^n \,|\, n \geq 0\}$ or $Q_- = \{q^n \,|\, n \leq 0\}$.
- Therefore no extension which induces this Q-module structure on A is finitely presented. In particular $\mathbb{Z} \wr \mathbb{Z} = \oplus_{i\in\mathbb{Z}}\mathbb{Z}a_i \rtimes \mathbb{Z}$ is not finitely presented.

Interesting Computation

- Let $A = \oplus_{i \in \mathbb{Z}} \mathbb{Z} a_i$ the free Abelian group on the generators $\{a_i \mid i \in \mathbb{Z}\}$, and let $Q = \langle q \rangle$.
- Every extension of Q by A is not finitely generated.
- Every Q-module structure on A which finitely generates A over Q needs "both sides" of Q to do it. That is, A is never finitely generated over Q_+ or Q_-.