Finitely Presented Metabelian Groups

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0. Introduction

In group theory, it is a fundamental question to determine what information about a group is necessary and sufficient to make a group finitely presented. In this paper, we develop the theory of Σ-invariants due to Robert Bieri and Ralph Strebel [Bieri] which provides an answer to this question for the class of metabelian groups.

We begin our study of Σ-invariants with a digression on group extensions and basic definitions used throughout the paper. From there we define the Σ-invariants and establish their basic properties. We then devote a section to examples of computing the invariant Σ_A. The computation of Σ_A is difficult in general; much work in the field is on developing methods to compute these invariants. In the last section, we sketch the proof of the main theorem.

1. Group Extensions

Definition 1.0.1: Let N and Q be groups. An *extension of Q by N* is a short exact sequence[1]

$$1 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} Q \longrightarrow 1.$$

We remark that the information which determines an extension is the group G along with the two maps ι and π. The reader is cautioned from thinking that an extension is determined solely by the group G.

Definition 1.0.2: A morphism of extensions is a map of the chain complexes. That is, Given (G, ι, π) and (G', ι', π') extensions of Q by N, a morphism $\phi : (G, \iota, \pi) \to (G', \iota', \pi')$ is given by a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r, "\iota"] \arrow[d, "\phi_N"] & G \arrow[r, "\pi"] \arrow[d, "\phi_G"] & Q \arrow[r] \arrow[d, "\phi_Q"] & 1 \\
1 \arrow[r] & N \arrow[r, "\iota'"] & G' \arrow[r, "\pi'"] & Q \arrow[r] & 1
\end{tikzcd}$$

Definition 1.0.3: Two extensions are considered equivalent if there is a morphism $\phi : G \to G'$ so that $\phi_N = \mathrm{id}_N$ and $\phi_Q = \mathrm{id}_Q$. From the 5-lemma, this implies ϕ_G is an isomorphism in the category of groups.

Definition 1.0.4: For groups N and Q, define the set of extensions of Q by N to be $\mathfrak{E}(Q, N)$. Define $E(Q, N) := \mathfrak{E}(Q, N) / \sim$ where $\sim$ is the equivalence of extensions.

If N is an Abelian group, an extension of Q by N posesses more structure. Specifically, Q acts on N via conjugation through G. For $\overline{q} \in Q$, where $q \in G$ is such that $\pi(q) = \overline{q}$ and for any $a \in N$, define $\overline{q} \cdot a := qaq^{-1}$. (Note that we make the standard identification of N

[1] Caution: Some authors define this to be an extension of N by Q. We justify this terminology as it agrees nicely with the cohomology of Q with coefficients in N. See proposition 12.

with its image in G.) In this situation, we typically denote N by A and write it additively. It is routine to verify that this action is well defined. Furthermore, the Q-action on N satisfies $\forall q \in Q, a, a' \in N, q \cdot (a + a') = q \cdot a + q \cdot a'$, i.e. N is a Q-module. Given a group extension of Q by N, we call this Q-module structure on N the induced Q-module structure on N.

Definition 1.0.5: Let Q be a group. A Q-module is an Abelian group A equipped with a Q-action that satisfies:

$$q \cdot (a + a') = q \cdot a + q \cdot a'.$$

In other words, a Q-module structure on A is a group homomorphism $\phi : Q \to \mathrm{End}\,(A)$.

Alternatively, a Q-module structure on A is equivalent to A being a $\mathbb{Z}Q$-module where $\mathbb{Z}Q$ is the group ring of Q.

Definition 1.0.6: Given a Q-module A with $\phi : Q \to \mathrm{End}(A)$, we define $\mathcal{E}_\phi\,(Q, A)$ to be the subset of $E\,(Q, A)$ which contains all extensions that induce the given Q-module structure on A.

Definition 1.0.7: We define the support of $\lambda = \sum_{q \in Q} n_q q \in \mathbb{Z}Q$ to be $\mathrm{supp}\,(\lambda) = \{q \in Q : n_q \neq 0\}$.

The definition of supp is motivated by interpreting $\lambda \in \mathbb{Z}Q$ as a function $\lambda : Q \to \mathbb{Z}$. When $\lambda = \sum_{q \in Q} n_q q$, then $\lambda\,(q) := n_q$. Then one can say that the group ring $\mathbb{Z}Q$ is the collection of all functions $\lambda : Q \to \mathbb{Z}$ with finite support. We will often use this interpretation of group ring elements.

Definition 1.0.8: Consider a Q-module A. Define the semi-direct product of the Q-module A to be $A \rtimes Q$ which is the group on the set $A \times Q$ with operation given by $(a, q) * (a', q') = (a + q \cdot a', qq')$. Note that if the Q-module structure is trivial, i.e. $q \cdot a = a$ for all $q \in Q$, $a \in A$, the semi-direct product is just the direct product $Q \times A$.

Definition 1.0.9: We define the split extension of the Q-module A to be

$$A \overset{\iota}{\rightarrowtail} A \rtimes Q \overset{\pi}{\twoheadrightarrow} Q$$

where $\iota\,(a) = (a, 1)$ and $\pi\,(a, q) = q$.

Example 1.0.10: Investigating the extension classes in $E\,(\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2)$ illustrates that it is possible for there to be more than just the split extension in $\mathcal{E}_\phi(\mathbb{Z}_2, \mathbb{Z}_2^2)$. Observe that

$$1 \longrightarrow \langle i^2 \rangle \longrightarrow Q_8 \longrightarrow Q_8 / \langle i^2 \rangle \longrightarrow 1$$

shows $Q_8 \in \mathcal{E}_\phi(\mathbb{Z}_2, \mathbb{Z}_2^2)$ where $\phi(q) = \mathrm{id}_{\mathbb{Z}_2}$. The split extension in $\mathcal{E}_\phi(\mathbb{Z}_2, \mathbb{Z}_2^2)$ is the direct product $\mathbb{Z}_2^3$. Thus as Q_8 is not Abelian, the above extension of Q_8 is not the split extension.

Example 1.0.11: In $\mathfrak{E}\,(\mathbb{Z}, \mathbb{Z}_3)$ the extensions $(\mathbb{Z}, \iota, \pi_1)$, $(\mathbb{Z}, \iota, \pi_2)$ given by $\iota\,(n) = 3n$, $\pi_1\,(n) = n \mod 3$, $\pi_2\,(n) = 2n \mod 3$ are not equivalent. They are, however, isomorphic. This illustrates that an extension is not determined by the group alone, and thus one must always specify the maps ι and π when considering an extension.

Proposition 1.0.12: Consider a Q-module A, where Q is not necessarily Abelian. Then there is a bijection $\mathcal{E}\,(Q, A) \cong H^2\,(Q, A)$. Through this bijection, we determine a group structure on $\mathcal{E}\,(Q, A)$.

Definition 1.0.13: A metabelian group is a group G for which there exists a short exact sequence $A \rightarrowtail G \twoheadrightarrow Q$ where A and Q are Abelian groups.

We now may ask, "What conditions on a metabelian group are necessary and sufficient for it to be finitely presented?" It was thought before the introduction of Σ-invariants that an answer might depend upon which extension class the extension determined in $\mathcal{E}(Q, A)$. However, the main result of this paper shows that the finite presentability of a metabelian group $A \rightarrowtail G \twoheadrightarrow Q$ is determined by the Q-module A alone. That is, the split extension $A \rtimes Q$ is finitely presented if and only if every extension group in $\mathcal{E}(Q, A)$ is finitely presented. This result is furthermore sharp with regard to $E(Q, A)$. That is, there are examples of Abelian groups Q and A for which there are finitely presented and not finitely presented groups in $E(Q, A)$.

Convention: From here on, Q is a finitely generated Abelian group with finite free rank $\operatorname{rk} Q = n$, i.e. $Q \cong \mathbb{Z}^n \oplus T(Q)$ where $T(Q)$ is the torsion subgroup, and A is finitely generated as a Q-module unless otherwise noted.

2. Introduction to Σ_A

Definition 2.0.14: Let R be a (not necessarily commutative) ring with 1 and let Γ be a totally ordered Abelian group. A valuation v on R with values in Γ is any map $v : R \to \Gamma_\infty$ which satisfies:

i.) $v(xy) = v(x) + v(y)$;

ii.) $v(x+y) \geq \inf\{v(x), v(y)\}$;

iii.) $v(0) = \infty$, $v(1) = 0$.

Remark 2.0.15: The condition $v(1) = 0$ is necessary to dispose of the trivial case $v(K) = \infty$, which does not correspond with the other ideas of valuations and valuation rings. For more on valuation rings and valuations, see [Bourbaki].

Definition 2.0.16: An additive character is a group homomorphism $v : Q \to (\mathbb{R}, +)$.

Additive characters easily induce valuations $v_* : \mathbb{Z}Q \to \mathbb{R}_\infty$ in the traditional sense discussed in the previous definition. We formulate this in the following proposition.

Proposition 2.0.17: For $v \in \operatorname{Hom}(Q, \mathbb{R})$ an additive character, we extend v to a valuation $v_* : \mathbb{Z}Q \to \mathbb{R}_\infty$ by defining:

i.) $v_*(\lambda) = \min\{v(q) : q \in \operatorname{supp}(\lambda)\}$;

ii.) $v_*(0) = \infty$.

Proof. We need to show that for $\lambda, \mu \in \mathbb{Z}Q$ the equations

i.) $v_*(\lambda\mu) = v_*(\lambda) + v_*(\mu)$ and

ii.) $v_*(\lambda + \mu) \geq \inf\{v_*(\lambda), v_*(\mu)\}$

are satisfied.

We write $\lambda = \sum_{q \in \operatorname{supp}(\lambda)} \lambda(q)q$, likewise for μ and we compute

$$\lambda\mu = \sum_{\substack{q \in \operatorname{supp}(\lambda) \\ r \in \operatorname{supp}(\mu)}} (\lambda(q)\mu(r))\, qr.$$

We now see

$$v_* (\lambda\mu) = \min \{v (q + r) : q \in \text{supp} (\lambda) , \ r \in \text{supp} (\mu)\}$$

$$v_* (\lambda) + v_* (\mu) = \min \{v (q) : q \in \text{supp} (\lambda)\} + \min \{v (r) : r \in \text{supp} (\mu)\}$$

and it is clear that $v (qr) = v (q) + v (r)$ will be minimal when $v (q)$ and $v (r)$ are minimal; hence the first equation holds.

From the easily verified inclusion $\text{supp} (\lambda + \mu) \subseteq \text{supp} (\lambda) \cup \text{supp} (\mu)$, the second equation follows. $\bowtie$

Our main tool (the Σ-invariant) comes from studying $\text{Hom} (Q, \mathbb{R})$. Let $Q = \mathbb{Z}^n \oplus T (Q)$ be a decomposition of Q into the direct sum of the torsion subgroup and the complementary space which is isomorphic to $\mathbb{Z}^n$. We define $e_i := (\delta_{i1}, \delta_{i2}, ..., \delta_{in}) \oplus 0$ where δ_{ij} is the Kronecker delta. We now consider the homomorphism $\theta : Q \twoheadrightarrow \mathbb{Z}^n$ defined by $\theta (e_i) := e_i$.

Let $u = (u_1, ..., u_n) \in \mathbb{R}$. We then can define a valuation, v_u in the natural way, by setting $v_u (e_i) = u_i$. In other words, $v_u (q) = \langle u, \theta (q)\rangle$.

The map θ then induces an $\mathbb{R}$-vector-space isomorphism:

Definition 2.0.18: $\theta^* : \mathbb{R}^n \to \text{Hom} (Q, \mathbb{R})$ which is defined by $\theta^* (u) := v_u$.

This construction also holds for any map $\theta : Q \twoheadrightarrow \mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ with finite kernel.

Definition 2.0.19: We are now in the position to define a topology on $\text{Hom} (Q, \mathbb{R})$ via the isomorphism θ^* for our specific θ. However, the induced topology is really independent from the choice of θ as $\text{Hom} (Q, \mathbb{R})$ is a topological vector space, and there is only one topological vector space of dimension n, namely $\mathbb{R}^n$.

We will mainly be interested in the proper submonoids of Q defined for a valuation $v \in \text{Hom} (Q, \mathbb{R})$ as $Q_v := \{q \in Q \mid v (q) \geq 0\}$. Suppose that $v \in \text{Hom} (Q, \mathbb{R})$ and $k \in \mathbb{R}^+$. Then v and $k \cdot v$ are both additive characters which additionally satisfy $Q_v = Q_{k \cdot v}$. We thus introduce an equivalence relation on $\text{Hom} (Q, \mathbb{R}) \setminus \{0\}$ as we are really interested only in the submonoids of Q determined by additive characters.

Definition 2.0.20: For $v, w \in \text{Hom} (Q, \mathbb{R}) \setminus \{0\}$, we say $v \sim w$ iff there exists $k \in \mathbb{R}^+$ such that $v = k \cdot w$. This defines an equivalence relation and we can thus construct the topological quotient space of this relation; we annote it as $S (Q) := (\text{Hom} (Q, \mathbb{R}) \setminus \{0\}) / \sim$.

Elements of $S (Q)$ will be written as $\overline{v}$ for $v \in \text{Hom} (Q, \mathbb{R}) \setminus \{0\}$.

Proposition 2.0.21: The space $S (Q)$ is homeomorphic to $\mathbb{S}^{n-1}$.

A monoid Q_v determines a *monoid ring* $\mathbb{Z}Q_v$ in the same way that a group ring is constructed. That is, $\mathbb{Z}Q_v = \{\lambda : Q_v \to \mathbb{Z} \mid |\text{supp}(\lambda)| < \infty\}$ is the free Abelian group with a basis given by Q_v with multiplication defined by extending the operation in Q_v over the addition by distributivity, i.e. $(\sum_q n_q q)(\sum_r m_r r) = \sum_{q,r} n_q m_r qr$.

Definition 2.0.22: Let $q \in Q$. Then define $H_q := \{\overline{v} \mid v (q) > 0\}$.

The subsets H_q of $S (Q)$ correspond to the hemispheres in S^{n-1}. This can be seen easily via the map θ as hemispheres in S^{n-1} are defined for a direction $u \in S^{n-1}$ as $H_u := \left\{x \in S^{n-1} \mid \langle u, x\rangle > 0\right\}$. We thus see the important fact that these sets are open.

We will find it useful to generalize this notation to the group ring $\mathbb{Z}Q$.

Definition 2.0.23: For $\lambda \in \mathbb{Z}Q$ define $H_\lambda := \{\overline{v} \mid v_* (\lambda) > 0\}$.

Proposition 2.0.24: $H_\lambda = \bigcap_{q \in \text{supp}(\lambda)} H_q$, hence H_λ is also open.

Proof. This is clear by definition of $v_*(\lambda) = \min\{v(q) \mid q \in \operatorname{supp}(\lambda)\}$. $\bowtie$

We may now define our main tool Σ_A.

Definition 2.0.25: Let A be a finitely generated Q-module. Then

$$\Sigma(Q, A) := \Sigma_A := \{\overline{v} \in S(Q) \mid A \text{ is finitely generated over } \mathbb{Z}Q_v\}.$$

2.1. Properties of Σ_A

Proposition 2.1.26: $\Sigma_A = \bigcup_{\lambda \in C(A)} H_\lambda$ where $C(A) = \{\lambda \in \mathbb{Z}Q \mid \forall a \in A,\ \lambda a = a\}$ is the centralizer. Furthermore, Σ_A is an open subset of $S(Q)$.

Proof. Note that it suffices to prove that $v \in \Sigma_A$ if and only if there exists $\lambda \in C(A)$ for which $v_*(\lambda) > 0$. Also observe the case where $\operatorname{rk} Q = 0$ holds trivially as $S(Q) = \emptyset$.

Now assume $\operatorname{rk} Q = n \geq 1$. Consider $\overline{v} \in \Sigma(Q, A)$; we construct an element $\lambda \in C(A)$ which satisfies $v_*(\lambda) > 0$. By assumption, there exists $\mathcal{A} = \{a_1, ..., a_k\} \subseteq A$ which generates A over $\mathbb{Z}Q_v$. Pick $q \in Q$ such that $v(q) > 0$. Such a q exists as v is a non-trivial additive character on Q. Then for all $a_i \in \mathcal{A}$, there is an expression $q^{-1} \cdot a_i = \sum_{j=1}^k \lambda_{ij} a_j$, or equivalently $\sum_{j=1}^k (\delta_{ij} - q \cdot \lambda_{ij}) a_j = 0$.

We thus can arrange these expressions into matrix notation as

$$(\delta_{ij} - q \cdot \lambda_{ij})_{i,j=1}^k \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix} = 0,$$

and we set $A := (\delta_{ij} - q \cdot \lambda_{ij})_{i,j=1}^k$ and $\overline{a} = {}^T(a_1, a_2, ..., a_k)$. Multiplying by A^{adj}—the adjoint matrix of A—we obtain

$$\det A \cdot \overline{a} = A^{adj} A \cdot \overline{a} = 0,$$

hence $\det A$ annihilates every a_i. Furthermore, by the commutativity of $\mathbb{Z}Q_v$ it is easily seen that $\det A$ annihilates all of A. One can inductively compute $\det A$ to be of the form $\det A = 1 - q \cdot \mu$ where $\mu \in \mathbb{Z}Q_v$. As $\mu \in \mathbb{Z}Q_v$, $\mu \neq q^{-1}$, $\det A$ is a non trivial element in the annihilator ideal. Thus $\lambda := q \cdot \mu$ is a nontrivial element of $C(A)$. We now finally compute that

$$v_*(q\mu) = v(q) + v_*(\mu) > 0$$

as $v(q) > 0$ and $\mu = \sum n_r \cdot r$ with $r \in Q_v$ and $v_*(\mu) \geq \inf\{v(r) \mid r \in \operatorname{supp}(\mu)\} \geq 0$ by definition of Q_v.

Now suppose that $v : Q \to \mathbb{R}$ such that there exists $\lambda \in C(A)$ with $v_*(\lambda) > 0$. Then as $\mathbb{R}$ is an Archimedean group, there exists $m > 0$ for any nonzero $\mu \in \mathbb{Z}Q$ such that $mv_*(\lambda) > -v_*(\mu)$. Thus $v_*(\mu\lambda^m) = v_*(\mu) + mv_*(\lambda) > 0$ and $\mu\lambda^m \in \mathbb{Z}Q_v$. Suppose $a \in A$. then $\mu a = \mu\lambda^m a \in \mathbb{Z}Q_v a$. Therefore $\mathbb{Z}Qa = \mathbb{Z}Q_v a$. Given a finite generating set $\mathcal{A} \subseteq A$, we conclude that $\mathbb{Z}Q\mathcal{A} = \mathbb{Z}Q_v\mathcal{A}$, hence $\mathcal{A}$ is a finite generating set of A over $\mathbb{Z}Q_v$. Hence $v \in \Sigma_A$ and the proposition follows. $\bowtie$

Proposition 2.1.27:

1. $\Sigma(Q, A) = \Sigma_{\mathbb{Z}Q/\operatorname{Ann}(A)}$, i.e. $\Sigma_A = \Sigma(Q, \mathbb{Z}Q/\operatorname{Ann}(A))$

2. Let $A' \overset{\iota}{\rightarrowtail} A \overset{\pi}{\twoheadrightarrow} A''$ be an exact sequence of $\mathbb{Z}Q$-modules. Then

$$\Sigma(Q, A) = \Sigma(Q, A') \cap \Sigma(Q, A'').$$

Proof.

1. It suffices to show that $C(A) = C(\mathbb{Z}Q/\operatorname{Ann}(A))$ by proposition 26. In the proof, we will use the identity $C(M) = \operatorname{Ann}(M) + 1$ for all Q-modules M.

   Let $\lambda \in C(A)$. Then $\lambda - 1 \in \operatorname{Ann}(A)$, hence for all $\mu \in \mathbb{Z}Q$ we have $(\lambda - 1)\mu \in \operatorname{Ann}(A)$. Therefore $\overline{(\lambda - 1)\mu} = \overline{\lambda - 1}\overline{\mu} = 0$ in $\mathbb{Z}Q/\operatorname{Ann}(A)$, from which we conclude $\lambda - 1 \in \operatorname{Ann}(\mathbb{Z}Q/\operatorname{Ann}(A)) = C(\mathbb{Z}Q/\operatorname{Ann}(A)) - 1$ and hence, $\lambda \in C(\mathbb{Z}Q/\operatorname{Ann}(A))$.

   Now suppose $\lambda \in C(\mathbb{Z}Q/\operatorname{Ann}(A))$. Then $\lambda\overline{1} = \overline{1}$ from which we conclude $\overline{\lambda - 1} \in \operatorname{Ann}(A) = C(A) - 1$; hence $\lambda \in C(A)$ as desired.

2. Let $v \in \Sigma_A$. By definition there exists a finite subset $\mathcal{A}$ of A which generates A over $\mathbb{Z}Q_v$. As A projects onto A'' via π, the set $\pi\mathcal{A} = \{\pi(a) \mid a \in \mathcal{A}\}$ clearly generates A'' over $\mathbb{Z}Q_v$, hence $v \in \Sigma_{A''}$.

   We observe that $C(A) \subseteq C(A')$. Therefore by the above theorem, it follows directly that $\Sigma_A \subseteq \Sigma_{A'}$. This paragraph and the previous one have established that $\Sigma_A \subseteq \Sigma_{A'} \cap \Sigma_{A''}$.

   Now let $v \in \Sigma_{A'} \cap \Sigma_{A''}$. There then exist finite sets $\mathcal{A}' \subseteq A'$ and $\mathcal{A}'' \subseteq A''$ which generate A' and A'' respectively over $\mathbb{Z}Q_v$. Let $s : A'' \to A$ be a function which satisfies $\mathrm{id}_A = \pi \circ s$. Then the set $\mathcal{A} = \{\iota(a') \mid a' \in \mathcal{A}'\} \cup \{s(a'') \mid a'' \in \mathcal{A}''\}$ can be easily shown to generate A over $\mathbb{Z}Q_v$, hence $v \in \Sigma_A$ and the result follows.

$\bowtie$

2.2. Computations of Σ_A

Example 2.2.28: Consider the case when $Q = \mathbb{Z} = (q)$, $A = \oplus_{i\in\mathbb{Z}}\mathbb{Z}a^i$, and $q \cdot \sum n_i a^i = \sum n_i a^{i+1}$.

$$0 \longrightarrow \oplus_{i\in\mathbb{Z}}\mathbb{Z}a_i \longrightarrow \oplus_{i\in\mathbb{Z}}\mathbb{Z}a_i \rtimes \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 1$$

This is specifically the case of the wreath product $G = \mathbb{Z} \wr \mathbb{Z}$. The wreath product itself is a very useful and interesting construction in group theory. In fact, the category of groups with the wreath product has the structure of a non-commutative monoid!

We now show that $\Sigma_A = \emptyset$. We begin by fixing $\theta : Q \to \mathbb{R}$ to be $\theta\left(q^i\right) = i$. Observe that $S(Q) = \{v_1, v_{-1}\}$ where $v_1\left(q^i\right) = i$ and $v_{-1}\left(q^i\right) = -i$.

Suppose $\mathcal{A} = \{\alpha_1, ..., \alpha_k\}$ is a finite set of A. We show that $\mathcal{A}$ is not a generating set of A over $\mathbb{Z}Q_1$ and $\mathbb{Z}Q_{-1}$ (we abuse notation here). For each i, we have $\alpha_i = \sum_j \alpha_{ij}a^j$ where only finitely many $\alpha_{ij} \neq 0$. The set $\{j \mid \alpha_{ij} \neq 0\}$ is finite and thus posesses a maximum and minimum; denote them as M and m respectively. For any $\lambda \in \mathbb{Z}Q_1$, we have $\lambda \cdot \alpha_i \in \oplus_{l\geq m}\mathbb{Z}a^l$; hence $a^{m-1} \notin \langle \alpha_1, ..., \alpha_k \rangle_{\mathbb{Z}Q_1}$. For $\lambda \in \mathbb{Z}Q_{-1}$ we similarly have $\lambda \cdot \alpha_i \in \oplus_{l\leq M}\mathbb{Z}a_i$; thus $\mathcal{A}$ is not a generating set over $\mathbb{Z}Q_{-1}$.

More generally, for *any* $\mathbb{Z}Q$-module structure on A which makes A a finitely generated $\mathbb{Z}Q$-module, we have $\Sigma_A = \emptyset$.

Let $Q = (q) \cong \mathbb{Z}$ and $A = \oplus_{i\in\mathbb{Z}}\mathbb{Z}a_i$, i.e. the free $\mathbb{Z}$-module on $\{a_i \mid i \in \mathbb{Z}\}$. Suppose there is a Q-module structure on A which makes A a finitely generated Q-module, and furthermore, suppose A is also a tame Q-module. That is, A is finitely generated over $Q_1 = \{q^n \mid n \geq 0\}$ or $Q_{-1} = \{q^{-n} \mid n \geq 0\}$. Without loss of generality, assume that A is finitely generated over Q_1. For if it were finitely generated over Q_{-1}, we could just consider the action of $Q = (r)$ on A by $r \cdot a = q^{-1} \cdot a$. This new action then finitely generates A over Q_1.

Without loss of generality, we may assume that $\mathcal{A} = \left\{a_{i_j} \mid 1 \leq j \leq k\right\}$ is a finite generating set of A over Q_1. For if $\{\alpha_1, ...\alpha_t\}$ generates A finitely over Q_1, then so does $\cup_1^t \operatorname{supp}(\alpha_i)$.

Then $Q_1\mathcal{A} = \left\{q^n a_{i_j} \mid n \geq 0, 1 \leq j \leq k\right\}$ generates A as an Abelian group. This set is not necessarily linearly independent over $\mathbb{Z}$. We now construct a $\mathbb{Z}$-linearly independent subset of $Q_1\mathcal{A}$ which we denote as $\mathfrak{Q}$. Begin with $\mathfrak{Q}^1 := \mathcal{A}$. Given $\mathfrak{Q}^{n-1}$, we first construct $V_{n,0} = \emptyset$ and define

$$V_{n,j} = \begin{cases} V_{n,j-1} \cup \left\{q^n a_{i_j}\right\} & \text{if } \mathfrak{Q}^{n-1} \cup V_{n,j-1} \cup \left\{q^n a_{i_j}\right\} \text{ is linearly independent over } \mathbb{Z} \\ V_{n,j} = V_{n,j-1} & \text{otherwise.} \end{cases}$$

then $\mathfrak{Q}^n = \mathfrak{Q}^{n-1} \cup V_{n,k}$. It is not the case that $\langle \mathfrak{Q} \rangle_{\mathbb{Z}} = A$. However, $A/\langle \mathfrak{Q} \rangle$ is a torsion Abelian group. This follows as $q^n a_{i_j} \notin \mathfrak{Q}$ implies all consecutive powers $q^{n+\ell}a_{i_j} \notin \mathfrak{Q}$. We see this as for $q^{n_{i_j}} \notin \mathfrak{Q}$ implies there is a non-trivial equation relating an integer multiple of $q^n a_{i_j}$ to the elements in $\mathfrak{Q}^{n-1} \cup V_{n,j-1}$. Multiplying this equation by q and perhaps some integer yields a non-trivial equation realating some integer multiple of $q^n a_{i_j}$ to the elements in $\mathfrak{Q}^n \cup V_{n+1,j-1}$.

There must exist some ℓ for which $\{q^n a_{i_\ell} \mid n \geq 0\} \subseteq \mathfrak{Q}$. To see this, we work over $\mathbb{Q}$. The set $Q_1\mathcal{A}$ generates $\oplus\mathbb{Q}a_i$. Furthermore, our construction of $\mathfrak{Q}$ yields a basis for $\oplus\mathbb{Q}a_i$. As $\oplus\mathbb{Q}a_i$ is infinite dimensional, the set $\mathfrak{Q}$ must be infinite.

From this realization, there are at most $k-1$ independent equations obtained between the elements of $Q_1\mathcal{A}$. Now for all $1 \leq j \leq n$ there are integers m_j such that $m_j q^{-1} a_{i_j}$ is a $\mathbb{Z}$-linear combination in terms of $\mathfrak{Q}$. Multiplying each one of these expressions by q and possibly some integer gives us k linear relations among the elements of $\mathfrak{Q}$. Define $j' = \inf\left\{r \mid q^r a_{i_j} \notin \mathfrak{Q}\right\}$. For example

$$mq^{-1}a_{i_t} = \sum_{r=0}^{j'}\sum_j \mu_{rj} q^r a_{i_j} \tag{1}$$

$$ma_{i_t} = \sum_{r=0}^{j'}\sum_j \mu_{rj} q^{r+1} a_{i_j}. \tag{2}$$

There are at most $k-1$ terms in the right-hand side of this equation which are not in $\mathfrak{Q}$. These are those terms $\mu_{rj'}q^{j'+1}a_{i_j}$. After multiplying equation 2 by some integer, we can reduce every term $\mu_{rj'}q^{j'+1}a_{i_j}$ into terms of $\mathfrak{Q}$. Note that in 2, the "degree 0" terms, i.e. terms of the form $cq^0 a_{i_j}$, $c \in \mathbb{Z}$, come from reducing the terms $\mu_{rj'}q^{j'+1}a_{i_j}$. However, as there are at most $k-1$ equations used in each equation to reduce it into terms of $\mathfrak{Q}$,, at least one of the obtained equations has to be non-trivial. This follows as one of ma_{i_j} is not in the Abelian group generated by the $k-1$ degree 0 parts of the relations used. There is therefore a non-trivial linear relationship among the elements of $\mathfrak{Q}$, which is a contradiction.

Example 2.2.29: Consider the case where $Q = \mathbb{Z} = (q)$, $A = \mathbb{Z}[1/2]$ and $q \cdot x = 2x$ (note that $q^{-1} \cdot x = \frac{1}{2}x$). With the description of $S(Q)$ as in the previous example, we will show $\Sigma_A = \{v_{-1}\}$. It is clear that $\{1\}$ generates A over $\mathbb{Z}Q_{-1}$ as $q^{-n} \cdot 1 = \frac{1}{2^n}$, and $\langle 2^n \mid n \leq 0 \rangle_{\mathbb{Z}} = A$. An argument similar to the one in the previous example shows that no finite set $\mathcal{A} \subseteq A$ generates A over $\mathbb{Z}Q_1$.

In general, any Q-module structure on A is tame (see definition 39). First off, $q \cdot - : A \to A$ must be an automorphism of A. This automorphism is $\mathbb{Z}$-linear, but by induction on n, we see that $q\cdot$ is $\mathbb{Z}[1/2]$-linear from the equation $2(q \cdot 2^{-n-1}) = q \cdot 2^{-n}$. Hence $q\cdot \in \mathrm{GL}_1(\mathbb{Z}[1/2])$, i.e. $q \cdot 1 = \pm 2^{-k}$ for some $k \in \mathbb{Z}$. An entirely analogous computation to that in the preceeding paragraph shows that either $\Sigma_A = \{v_1\}$ or $\Sigma_A = \{v_{-1}\}$. Tameness will play an important role in our proof of the main result.

From this computation, we note that Σ_A is never all of $S(Q)$. This is due entirely to the fact that A is not a finitely generated Abelian group; see theorem 36 for more details.

Example 2.2.30: Consider the case where $Q = \mathbb{Z} = (q)$, $A = \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$, and the $\mathbb{Z}Q$-module structure on A is given by

$$q \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

It is then not too difficult to see that $\Sigma_A = \{v_1\}$ with the help of the computations made in the previous example and by the linearity of the action.

This choice of Q and A exhibit an interesting property that the previous two examples do not. This property is that there exist $\mathbb{Z}Q$-module structures on A for which A is tame and for which A is not tame. We will see the importance of this when we get to the main theorem of this paper.

With Q and A as above, the $\mathbb{Z}Q$-module structure on A given by

$$q \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

makes A a non-tame $\mathbb{Z}Q$-module. That is, $\Sigma_A = \emptyset$.

3. Geometric Lemma

The common tool in the proofs to come is the geometric lemma which we prove in this section.

Definition 3.0.31: Let X be a set. Define $\mathfrak{f}X := \{A \subseteq X : |A| < \infty\}$.

Throughout this section, $\mathcal{F} \in \mathfrak{f}\mathfrak{f}\mathbb{R}^n$ and $L \in \mathcal{F}$. We also define the open ball of radius ρ centered at x by $B(\rho; x) := \{y \in \mathbb{R}^n \mid |y - x| < \rho\}$. If $x = 0$, we simply write B_ρ.

Definition 3.0.32: An element $x \in \mathbb{R}^n$ can be taken from B_ρ by $\mathcal{F}$ if there exists $L \in \mathcal{F}$ such that $x + L \subseteq B_\rho$ or if $x \in B_\rho$. If there exists $L \in \mathcal{F}$ such that $x + L \subseteq B_\rho$, we say x can be properly taken from B_ρ by $\mathcal{F}$ with respect to L.

Definition 3.0.33: The set of points $x \in \mathbb{R}^n$ which can be taken from B_ρ by $\mathcal{F}$ will be denoted by $T(\mathcal{F}, \rho)$.

As a preliminary observation, we see that a point $x \in \mathbb{R}^n$ can be properly taken from B_ρ by $\mathcal{F}$ with respect to L if and only if $x \in \bigcap_{y \in L} B(-y; \rho)$. Thus we have

$$T(\mathcal{F}, \rho) = \bigcup_{L \in \mathcal{F}} \left(\bigcap_{y \in L} B(-y; \rho) \right) \cup B_\rho.$$

Lemma 3.0.34: Suppose $\mathcal{F} \in \mathfrak{f}\mathfrak{f}\mathbb{R}^n$ such that for any $x \in \mathbb{R}^n \setminus \{0\}$ there exits $L \in \mathcal{F}$ such that $\langle x, y \rangle > 0$ for all $y \in L$. Then there exists $\rho_0 > 0$ and a function $\varepsilon : \{\rho \mid \rho > \rho_0\} \to \mathbb{R}^+$ such that for all $\rho > \rho_0$ we have $B_{\rho+\varepsilon(\rho)} \subseteq T(\mathcal{F}, \rho)$. Furthermore, ε is an increasing function.

The casual reader can easily skip the details of the proof. The essential geometric fact we are utilizing (in imprecise language) is that spheres flatten out as their radius gets larger.

Proof. Define $f : S^{n-1} \to \mathbb{R}$ by $f(u) := \max_{L \in \mathcal{F}} \min_{y \in L} \{\langle u, y \rangle \mid y \in L \in \mathcal{F}\}$. One can show that f is continuous utilizing the continuity of $\langle -, - \rangle$ without too much difficulty. By the compactness of S^{n-1} there exists $v \in S^{n-1}$ such that $f(v) = \inf\left\{f(u) \mid u \in S^{n-1}\right\} > 0$. Set $C := f(v)$. Also define $D := \max_{L \in \mathcal{F}} \max_{y \in L} \{|y| \mid y \in L \in \mathcal{F}\} > 0$.

Set $\rho_0 = \frac{D^2}{2C}$ and $\varepsilon(\rho) = C - \left(\frac{D^2}{2\rho}\right)$. By our choice of C, for all $x \in \mathbb{R}^n \setminus \{0\}$ there exists $L_x \in \mathcal{F}$ such that $\min_{y \in L_x} \left\{\langle \frac{-x}{|x|}, y \rangle \mid y \in L_x\right\} \geq C$. We see this as

$$\max_{L \in \mathcal{F}} \min_{y \in L} \left\{\langle \frac{-x}{|x|}, y \rangle\right\} = f\left(\frac{-x}{|x|}\right) \geq C.$$

Thus for $|x| > \rho_0$ and $y \in L_x$ we have

$$\begin{aligned} |x+y|^2 &= |x|^2 + 2\langle x/|x|, y \rangle |x| + |y|^2 \\ &\leq |x|^2 - 2C|x| + D^2 \\ &\leq |x|^2 \end{aligned}$$

where the last inequality is justified by

$$\begin{aligned} -2C|x| + D^2 &< -2C\rho_0 + D^2 \\ &= -D^2 + D^2 = 0. \end{aligned}$$

Moreover,

$$\begin{aligned} |x+y| - |x| &= \frac{|x+y|^2 - |x|^2}{|x+y| + |x|} \\ &\leq \frac{-2C|x| + D^2}{2|x|} \\ &= -\varepsilon(|x|) \end{aligned}$$

hence $|x+y| \leq |x| - \varepsilon(|x|)$. Thus for all x such that $\rho_0 < \rho \leq |x| < \rho + \varepsilon(\rho)$ we have $|x+y| \leq |x| - \varepsilon(|x|) < \rho + \varepsilon(\rho) - \varepsilon(|x|) < \rho$; that is, $x + y \in B_\rho$ for all $y \in L_x$, hence $x \in T(\mathcal{F}, \rho)$ and $B_{\rho+\varepsilon(\rho)} \subseteq T(\mathcal{F}, \rho)$. $\bowtie$

Example 3.0.35: To make this concrete, we look at the example when $n = 2$ and $\mathcal{F} = \{\{(1,0)\}, \{(0,1)\}, \{(-1,0)\}, \{(0,-1)\}\}$. We would like to demonstrate what restricts the size of τ in the equation $B_\rho \subseteq B_\tau \subseteq T(\mathcal{F}, \rho)$. It is clear that the points $(x, y) \in \partial T(\mathcal{F}, \rho)$ which are closest to B_ρ (for $\rho > 1/\sqrt{2}$) are those points which satisfy $x = y$ or $x = -y$, and the points which are furthest away are those of the form $(x, 0)$ or $(0, y)$. We concentrate our attention to the first quadrant as the others follow by symmetry. We compute the point $(x, x) \in \partial T(\mathcal{F}, \rho)$ to be given by $\left(\frac{1+\sqrt{-1+2\rho^2}}{2}, \frac{1+\sqrt{-1+2\rho^2}}{2}\right)$ which has length $\frac{1}{\sqrt{2}}\left(1+\sqrt{-1+2\rho^2}\right)$. For $\rho > 1/\sqrt{2}$ we have $\rho < \frac{1}{\sqrt{2}}\left(1+\sqrt{-1+2\rho^2}\right) < \frac{1}{\sqrt{2}} + \rho$. Thus the biggest ball we can fit into $T(\mathcal{F}, \rho)$ is going to be determined by the value of $\frac{1}{\sqrt{2}}\left(1+\sqrt{-1+2\rho^2}\right)$ which is an increasing function for $\rho > 1/\sqrt{2}$. Thus the "flattening" is seen as ρ gets larger and the distance between the cusps at the points $(x, \pm x)$ and the ball B_ρ increases. We note that this distance is bounded by $1/\sqrt{2}$.

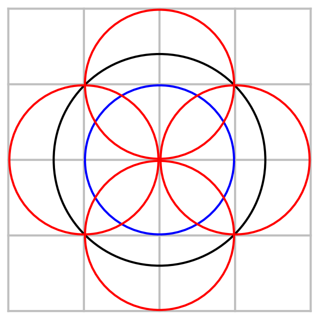

Figure 1: The blue circle is the boundary of the ball B_1 which we are taking points from. The black circle is the boundary of the largest ball which fits in $T(\mathcal{F}, 1)$, namely $B_{2/\sqrt{2}}$.

4. Characterization of Q-modules A with $\Sigma_A = S(Q)$

We choose to prove the following theorem for a few reasons: first off, it gives a useful computation of Σ_A and builds our intuition about what the Σ-invariant is describing; chiefly, the proof illustrates the way we will be applying the Geometric Lemma (lemma 34) for our major result to come. We have indeed built up all the necessary concepts needed for this section, so we go right to the statement of the theorem and its proof.

Theorem 4.0.36: Let Q be a finitely generated Abelian group of free rank n; let A be a finitely generated Q-module. Then $\Sigma_A = S(Q)$ if and only if A is a finitely generated Abelian group.

Proof. We begin by fixing a homomorphism $\theta : Q \twoheadrightarrow \mathbb{Z}^n \hookrightarrow \mathbb{R}^n$, or for sake of convenience, the explicit one described above. We now define $X_\rho := \theta^{-1}(B_\rho)$. The sets X_ρ satisfy the two following properties:

i.) for $\rho < \tau$, $X_\rho \subseteq X_\tau$;

ii.) $\bigcup_\rho X_\rho = Q$.

Suppose now that $\Sigma_A = S(Q)$. Then by Propositions 21, 24 and 26, we have $S^{n-1} = S(Q) = \bigcup_{\lambda \in C(A)} H_\lambda$, hence $\{H_\lambda \mid \lambda \in C(A)\}$ is an open cover of the compact topological space S^{n-1}. There thus exists a finite subcover, and hence a finite subset $\Lambda \subseteq C(A)$ such that $\{H_\lambda \mid \lambda \in \Lambda\}$ covers S^{n-1}.

Define $\mathcal{F} := \{\{\theta(q) \mid q \in \operatorname{supp}(\lambda)\} \mid \lambda \in \Lambda\} \in \mathfrak{ff}\mathbb{R}^n$. As valuations $v \in S(Q)$ can be interpreted as inner products, i.e. $\exists u \in S^{n-1}$ such that $v(q) = \langle u, \theta(q) \rangle$, it follows by the definition of Σ_A and our assumption that $\mathcal{F}$ satisfies the hypothesis of the Geometric Lemma. Thus there exists a $\rho_0 > 0$ and an increasing function $\varepsilon : \{\rho \mid \rho > \rho_0\} \to \mathbb{R}^+$ such that $B_{\rho+\varepsilon(\rho)}$ can be taken from B_ρ.

Interpreting this result in Q, we obtain for $q \in X_{\rho+\varepsilon(\rho)}$ an element $\lambda \in \Lambda$ such that $\theta(q) + \theta(r) \in B_\rho$ for all $r \in \operatorname{supp}(\lambda)$. As $\theta(q) + \theta(r) = \theta(qr) \in B_\rho$, we have $qr \in X_\rho$. Thus $q\lambda = \sum_{r \in \operatorname{supp}(\lambda)} n_r (qr) \in \mathbb{Z}X_\rho$.

Now consider $a \in A$ and $q \in X_{\rho+\varepsilon(\rho)}$. Then $q \cdot a = q \cdot (\lambda \cdot a) = q\lambda \cdot a \in X_\rho a$. However, as $q \cdot a \in X_{\rho+\varepsilon(\rho)}$ is a general element, we conclude that $\mathbb{Z}X_{\rho+\varepsilon(\rho)} a \subseteq \mathbb{Z}X_\rho a$ for all $a \in A$. The inclusion $\mathbb{Z}X_\rho a \subseteq \mathbb{Z}X_{\rho+\varepsilon(\rho)} a$ is clear by property i.) of the sets X_ρ. We have thus shown $\mathbb{Z}X_{\rho+\varepsilon(\rho)} a = \mathbb{Z}X_\rho a$ for all $a \in A$. Furthermore, for any $c \in [\rho, \rho + \varepsilon(\rho)]$, we have $\mathbb{Z}X_c a = \mathbb{Z}X_\rho a$.

Fix some $\rho' > \rho_0$ and constant c which satisfies $0 < c < \varepsilon(\rho')$. Thus by induction, one can establish $\mathbb{Z}X_{\rho'} a = \mathbb{Z}X_{\rho'+nc} a$ for all $n \in \mathbb{N}$. It thus follows by property ii.) of the sets X_ρ that $\mathbb{Z}X_\rho a = \mathbb{Z}\left(\bigcup_{t > \rho'} X_t\right) a = \mathbb{Z}Qa$.

Suppose now that $\mathcal{A}$ is a finite basis for A as a Q-module. Then $A = \mathbb{Z}Q\mathcal{A} = \mathbb{Z}X_{\rho'}\mathcal{A}$. However, $X_{\rho'}\mathcal{A} = \{q \cdot a \mid q \in X_{\rho'}, a \in \mathcal{A}\}$ is a finite set which generates A as a $\mathbb{Z}$-module. That is, A is a finitely generated Abelian group.

The implication that A being a finitely generated Abelian group implies that $\Sigma_A = S(Q)$ is immediate by the initial definition of Σ_A. Thus the theorem has been proven. ⋈

5. Tame Q-Modules

In the following sections, all Q-modules A will be right Q-modules.

Definition 5.0.37: Given a Q-module A, we can define an associated left Q-module A^* via the multiplication $q \cdot a := a \cdot q^{-1}$.

Proposition 5.0.38: $\Sigma_{A^*} = -\Sigma_A$

Proof. Suppose $v \in \Sigma_{A^*}$ and let $\mathcal{A} = \{a_1, \ldots a_k\}$ be a basis for A^* over Q_v. Thus for any $a \in A^*$ there exist $\lambda_i \in \mathbb{Z}Q_v$ such that

$$\begin{aligned} a &= \sum_{i=1}^{k} a_i \lambda_i \\ &= \sum_{i=1}^{k} \sum_{j=1}^{l_i} \lambda_{ij} (a_i q_{ij}) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{l_i} \lambda_{ij} \left(q_{ij}^{-1} a_i\right). \end{aligned}$$

As $q_{ij}^{-1} \in Q_{-v}$ for all i, j, we conclude that $\mathcal{A}$ generates A as a Q_{-v}-modules; hence $\Sigma_{A^*} \subseteq -\Sigma_A$. The other inclusion follows analogously, hence the proposition. $\bowtie$

Definition 5.0.39: A Q-module A is tame if it is finitely generated over $\mathbb{Z}Q$ and $\Sigma_A \cup \Sigma_{A^*} = S(Q)$, or equivalently $\Sigma_A \cup -\Sigma_A = S(Q)$.

Proposition 5.0.40:

1. If the Q-module A is tame, then so are its all submodules of A.
2. Homomorphic images of tame modules are tame.
3. Finite direct products of a tame module A are tame.

Proof.

1. Let $A' \subseteq A$, or in other words, $A' \rightarrowtail A$. By Proposition 27, part 2, we have $\Sigma_A \subseteq \Sigma_{A'}$. Thus $S(Q) = \Sigma_A \cup -\Sigma_A \subseteq \Sigma_{A'} \cup -\Sigma_{A'}$, hence A' is tame.
2. Follows as in part 1. of this proof by Proposition 27, part 2.
3. As $C(A) \subseteq C(A^n)$, we have $\Sigma_A \subseteq \Sigma_{A^n}$; hence the proposition follows by the description of Σ in Proposition 26.

$\bowtie$

Proposition 5.0.41: Let $\phi : \tilde{Q} \to Q$ be a homomorphism with $|\operatorname{coker} \phi| < \infty$ and let A be a Q-module. Let $\tilde{A}$ be the $\tilde{Q}$-module obtained by restricting scalars to $\phi(\tilde{Q})$. That is, for $\tilde{q} \in \tilde{Q}$ and $a \in A$, define $\tilde{q} \cdot a := \phi(\tilde{q}) \cdot a$. Any such ϕ induces a map $\phi^* : S(Q) \to S(\tilde{Q})$ given by $\phi^*(v) := v \circ \phi$. We then have:

1. $\phi^*(\Sigma(Q, A)^c) = \Sigma(\tilde{Q}, \tilde{A})^c$:
   a.) $\phi^*(\Sigma(Q, A)^c) \subseteq \Sigma(\tilde{Q}, \tilde{A})^c$,
   b.) $\Sigma(\tilde{Q}, \tilde{A})^c \subseteq \operatorname{im} \phi^*$,
   c.) $(\phi^*)^{-1}\left(\Sigma(\tilde{Q}, \tilde{A})^c\right) \subseteq \Sigma(Q, A)^c$;
2. A is tame if and only if $\tilde{A}$ is tame.

Proof.

1. a.) Observe that $\tilde{A}$ is finitely generated over $\tilde{Q}$ if and only if A is finitely generated over $\phi(\tilde{Q})$. Suppose $v \in \Sigma(Q, A)^c$. Then A is not finitely generated over Q_v. Since $\phi(\tilde{Q}_{v\circ\phi}) = \phi(\tilde{Q}) \cap Q_v \subseteq Q_v$, $\tilde{A}$ is not finitely generated over $\tilde{Q}_{v\circ\phi}$.
   b.) Let $\tilde{v} \in \Sigma(\tilde{Q}, \tilde{A})^c$. Define $\overline{v} : \phi(\tilde{Q}) \to \mathbb{R}$ via $\overline{v} \circ \phi = \tilde{v}$. The map $\overline{v}$ is well defined, for it is well defined at 0. That is, choose any $q \in \ker(\phi)$. We then have $q \cdot a = \phi(q) \cdot a = 1 \cdot a = a$ and thus $q, q^{-1} \in C(\tilde{A})$. Since $\tilde{v} \in \Sigma(\tilde{Q}, \tilde{A})^c$, we have $\tilde{v}(C(\tilde{A})) \leq 0$. We thus conclude $\tilde{v}(q) = 0$, and hence, $\overline{v}$ is well defined. As divisibility is equivalent to injectivity in $\underline{\text{Ab}}$, we can extend $\overline{v}$ to a homomorphism v defined on all of Q as seen in the diagram below. Hence $v \circ \phi = \tilde{v}$ as desired.

$$\begin{tikzcd}
& & \mathbb{R} & \\
Q \arrow[urr, dashed, "v"] & & \phi(\tilde{Q}) \arrow[ll, tail, "\mathrm{id}|_{\phi(\tilde{Q})}"] \arrow[u, "\overline{v}"'] & 1 \arrow[l]
\end{tikzcd}$$

Plainly speaking, there are two scenarios to consider:

i. for $x \in Q \setminus \phi(\tilde{Q})$ there exists $\tilde{q} \in \tilde{Q}$ and $n \in \mathbb{Z} \setminus \{0\}$ such that $n \cdot x = \phi(\tilde{q})$. Then define $v(x) := \frac{\tilde{v}(\tilde{q})}{n}$ which is possible as $\mathbb{R}$ is divisible.

ii. $x \in Q \setminus \phi(\tilde{Q})$ does not satisfy the hypothesis of the previous case. Then x is necessarily of finite order, in which case define $v(x) := 0$;

c.) We now prove the contrapositive of c.), namely, if $v \in \Sigma(Q, A)$, then $\phi^*(v) \in \Sigma(\tilde{Q}, \tilde{A})$. We thus assume that there is a finite set $\mathcal{A}$ which generates A over Q_v, and it remains to show that A is finitely generated over $Q_v \cap \phi(\tilde{Q})$.

By our assumption that coker ϕ is finite, we can choose a representative system $\overline{q_1}, ..., \overline{q_m}$ of coker $\phi = Q/\phi(\tilde{Q})$. Furthermore, we may choose $q_1, ..., q_m$ such that $v(q_i) \leq 0$. For if the free rank of Q is 0, then $v(q) = 0$ for all $q \in Q$. If the free rank of Q is nonzero, there is then an element of infinite order in $\operatorname{im}\phi \cap Q$ which satisfies $v(q) < 0$. Hence if $v(q_i) > 0$, we can replace it with some $q^m q_i$ as $v(q^m q_i) = mv(q) + v(q_i) \leq 0$ holds for sufficiently large $m \in \mathbb{N}$ by the Archimedian property. Thus our assumption about the representative system $\overline{q_1}, ..., \overline{q_m}$ is justified.

We now consider some $a \in A$ given by $a = \sum \lambda_i a_i$ where $\lambda_j = \sum \lambda_{ij} r_{ij}$ where $r_{ij} \in Q_v$, $\lambda_{ij} \in \mathbb{Z}$. Then as $r_{ij} = q_{\ell_{ij}} \phi(s_{ij})$ for all ij, we have

$$\begin{aligned}
a &= \sum \lambda_i a_i \\
&= \sum \left(\sum \lambda_{ij} r_{ij}\right) a_i \\
&= \sum \left(\sum \lambda_{ij} (r_{ij} a_i)\right) \\
&= \sum \left(\sum \lambda_{ij} \phi(s_{ij}) q_{\ell_{ij}} a_i\right)
\end{aligned}$$

and $0 < v(\ell_{ij}) = v\left(q_{\ell_{ij}}\right) + v(\phi(s_{ij}))$. From assumption that $v\left(q_{\ell_{ij}}\right) \leq 0$ we conclude $0 \leq -v\left(q_{\ell_{ij}}\right) < v(\phi(\tilde{s_{ij}}))$. Therefore $\{q_i \cdot a_j\}$ forms a basis over $Q_v \cap \phi(\tilde{Q})$ and is finite.

2. Observe A is tame if and only if $\Sigma(Q, A)^c \subseteq -\Sigma(Q, A)$.

Suppose A is tame. Then

$$\Sigma(\tilde{Q}, \tilde{A})^c = \phi^*(\Sigma(Q, A)^c) \subseteq \phi^*(-\Sigma(Q, A)) \subseteq -\Sigma(\tilde{Q}, \tilde{A})$$

yields $\Sigma(\tilde{Q}, \tilde{A})^c \subseteq -\Sigma(\tilde{Q}, \tilde{A})$ as desired. The rightmost inequality holds as part 1 of the proof works for both left and right modules and we have $-\Sigma(Q, A) = \Sigma(Q, A^*)$.

Now suppose $\tilde{A}$ is tame. Then just as above, we have

$$\Sigma(Q,A)^c = (\phi^*)^{-1}\left(\Sigma(\tilde{Q},\tilde{A})^c\right) \subseteq (\phi^*)^{-1}\left(-\Sigma(\tilde{Q},\tilde{A})\right) \subseteq -\Sigma(Q,A)$$

with which the tameness of A is proven.

$\bowtie$

6. Finitely Presented Metabelian Groups

Theorem 6.0.42: An extension of Q by A is finitely presented if and only if A is a tame Q-module.

The first thing we note from this theorem is that it says the finite presentability of a metabelian group depends only on what the Q-module is, and not on the element it determines in $\mathcal{E}(Q,A) \cong H^2(Q,A)$.

To prove this theorem, we establish a few lemmas first.

Lemma 6.0.43: Let Q_1 be a complementary subgroup of $T(Q)$, i.e. so that $Q_1 \cong \mathbb{Z}^n$, and let $G_1 := \pi^{-1}(Q_1)$. Observe that there is a Q_1 module structure on A via the map $Q_1 \rightarrowtail Q$ by restriction of scalars. Then

i. A is tame over Q if and only if A is tame over Q_1;

ii. G is finitely presented if and only if G_1 is finitely presented.

Proof.

i. Observe that the canonical injection $\iota : Q_1 \rightarrowtail Q$ satisfies the conditions of Proposition 41. Thus our first claim is proven.

ii. Note that $G/G_1 \cong Q/Q_1 \cong T(Q)$ which is a finite group. That is, G_1 is a subgroup of finite index in G. It is a standard result that a subgroup of finite index is finitely presented if and only if the group is finitely presented, and thus, the reduction step follows. A proof can be found in [Lyndon, p 103]

$\bowtie$

6.1. $\Sigma_A \cup -\Sigma_A = S(Q)$ for finitely presented metabelian groups

In this subsection, we show that a finitely presented metabelian group impose a tame Q-module structure on A. The proof of this relies on topological properties of the Cayley complex of G.

Definition 6.1.44: Let $G = \langle \mathfrak{X} \mid \mathfrak{R} \rangle$ be a presentation of the group G. Denote the Cayley complex by $\tilde{\Gamma} = \tilde{\Gamma}(\mathfrak{X},\mathfrak{R})$. The Cayley complex we are interested in has 0-cells given by G, 1-cells given by $G \times \mathfrak{X}$ and 2-cells given by $G \times \mathfrak{R}$. It is convenient to introduce the notation of inverse 1-cells $\left(gx, x^{-1}\right)$ which is the inverse path of (g,x). That is, we consider the disjoint union of cells indexed by these sets, then take the quotient space given by the following rules of gluing:

1. A 1-cell $(g,x) \in G \times \mathfrak{X}$ begins at g and ends at gx; in other words, the boundary of a 1-cell (g,x) is given by $\partial(g,x) = gx - x$.

2. A 2-cell $(g,r) \in G \times \mathfrak{R}$ with $r = y_1 y_2 \cdots y_s$, $y_i \in \mathfrak{X} \cup \mathfrak{X}^{-1}$ has boundary

$$\partial (g,r) = (g, y_1)(gy_1, y_2) \cdots (gy_1 y_2 \cdots y_{s-1}, y_s).$$

We note that "redundant faces" are not deleted, and the cells are unoriented. See [Lyndon, III.4] for more details on this construction.

Proposition 6.1.45: We establish some basic facts the Cayley complex $\tilde{\Gamma}$ of a group $G = \langle \mathfrak{X} \mid \mathfrak{R} \rangle$. We denote by $\tilde{\Gamma}^r$ the r-skeleton of $\tilde{\Gamma}$.

1. The Cayley complex $\tilde{\Gamma}$ is connected and simply connected.
2. G acts on $\tilde{\Gamma}$ by left multiplication. For $g \in G$ the explicit description of the action is that for $h \in \tilde{\Gamma}^0$, $g \cdot h = gh$; for $(h,x) \in \tilde{\Gamma}^1$, $g \cdot (h,x) = (gh,x)$; and for $(h,r) \in \tilde{\Gamma}^2$, $g \cdot (h,r) = (gh,r)$.
3. $\tilde{\Gamma}$ is the universal cover for $G \setminus \tilde{\Gamma} := \tilde{\Gamma}/\sim$, i.e. the quotient space of $\tilde{\Gamma}$ by G. The space $G \setminus \tilde{\Gamma}$ has fundamental group isomorphic to G.
4. If $N \lhd G$, we see by covering space theory that $N \setminus \tilde{\Gamma}$ has fundamental group N, and the action of G on $\tilde{\Gamma}$ induces an action of G/N on $N \setminus \tilde{\Gamma}$.

Proof. These are routine properties of the Cayley complex which can be found in [Lyndon, III.2,III.3,III.4]. ⋈

Theorem 6.1.46: If G is a finitely presented metabelian group, then A is a tame Q-module.

Proof. We assume that G is finitely presented and that $Q \cong \mathbb{Z}^n$. We construct a finite presentation which is well suited for our investigation. Pick the generators $\mathfrak{X} = \mathcal{T} \cup \mathcal{M}$ where $\mathcal{T} = \{t_1, ..., t_n\}$ are such that $\{q_i := \pi(t_i) \mid 1 \le i \le n\}$ is a basis for Q. Now pick a finite set $\mathcal{M}$ which generates A as a normal subgroup of G and is disjoint from $\mathcal{T}$. Note that in this setting, $\mathcal{M}$ generating A as a normal subgroup in G is equivalent to $\mathcal{M}$ being a generating set of A as a Q-module. The set $\mathfrak{X}$ is then necessarily a generating set of G, and there exists a finite set $\mathfrak{R}$ of relations for this generating set which gives a presentation of $G = \langle \mathfrak{X} \mid \mathfrak{R} \rangle$.

We thus consider the Cayley complex of G with respect to the presentation $G = \langle \mathfrak{X} \mid \mathfrak{R} \rangle$ which we denote by $\tilde{\Gamma} := \tilde{\Gamma}(\mathfrak{X}; \mathfrak{R})$. Define $\Gamma := A \setminus \tilde{\Gamma}$. The complex Γ has fundamental group $\pi_1(\Gamma) = A$. The fundamental group $\pi_1(\gamma)$ is equipped with a Q-module structure given by the action of Q on Γ mentioned in proposition 45, and agrees with the action of Q on A. The action may be explicitly described as follows: let $q = \pi(t_1^{s_1} \cdots t_n^{s_n})$ and $\gamma \in \pi_1(\Gamma)$; consider the path

$$\mu = (1, t_1)(q_1, t_1) \cdots (q_1^{s_1}, t_2) \cdots \cdots \left(q_1^{s_1} \cdots q_n^{s_n - 1}, t_n\right);$$

then $q \cdot \gamma = \mu (q\gamma) \mu^{-1}$. That is, in the fundamental groupoid of Γ, which we denote $\Pi(\Gamma)$, the action of Q on $\pi_1(\Gamma)$ is given by conjugating $\pi_1(\Gamma, q)$ by q. In terms of A, the Q-action on $\pi_1(\Gamma)$ coincides with the Q-module structure on A.

We now seek to better understand the 1-complex of Γ. Observe that Γ^1 decomposes as $\Gamma^1 = \Omega \cup \Delta$ where $\Omega := (Q, Q \times \mathcal{M})$ and $\Delta := (Q, Q \times \mathcal{T})$. The picture is that Δ is an n-dimensional grid and Ω is the union of a bouquet[2] of $\operatorname{card}(\mathcal{M})$ circles at each vertex.

[2]i.e. one point union

With these details behind us, we can finally get to the main idea of the proof. Consider a non-trivial additive character $v : Q \to \mathbb{R}$, and fix $\theta : Q \twoheadrightarrow \mathbb{Z}^n \hookrightarrow \mathbb{R}^n$. It is perhaps most intuitively clear if we take $\theta(q_i)) = e_i$. Then associate (with an abuse of notation $v = (v_1, ..., v_n)) \ v(q) = \langle v, \theta(q) \rangle$. Define Γ_v to be the subcomplex generated by the vertices Q_v, i.e. an r-cell is in Γ_v if and only if its boundary lies entirely in Γ_v. Likewise define $\Delta_v = \Gamma_v \cap \Delta$ and $\Omega_v = \Gamma_v \cap \Omega$.

> **Where we are headed**: Our main goal is to apply Van Kampen's theorem for any non-trivial additive character v and suitably chosen $q \in Q_v$ to Γ with subspaces Γ_v and $q\Gamma_{-v}$. The intuitive idea then is that the only way to write the Abelian group A as an amalgamated product is if it is in a trivial way. From this, we will get a Q-module surjection of $\pi_1(\Gamma_{\epsilon v})$ onto A for some $\epsilon = \pm 1$. Applying the Hurewicz transformation gives us a surjection of $H_1(\Gamma_{\epsilon v})$ which is easier to work with. For any v, we can show that $H_1(\Gamma_v)$ is a finitely generated Q_v-module without too much difficulty. It therefore follows that A is either a finitely generated Q_v module or a finitely generated Q_{-v}-module. Hence for any $v \in S(Q)$, either $v \in \Sigma_A$ or $-v \in \Sigma_A$.

We now fill in the details of the proof.

Lemma 6.1.47: Consider a non-trivial additive character v and suppose θ is chosen as above. For any $q \in Q$ that satisfies $v(q) \geq 2\sqrt{2}\|v\|$, the space $\Gamma_v \cap q\Gamma_{-v}$ is path connected. A stronger result in [Bieri, § 4.7], but we content ourselves with this version.

Proof. We are concerned with $\Delta_v \cap q\Delta_{-v}$ in this proof as connectivity is just a question about the 1-skeleton and the loops at a point are irrelevant. Throughout the proof, we identify a point $p = \prod q_i^{p_i}$ with $(p_1, ..., p_n) \in \mathbb{Z}^n$.

As $v(p) = \langle v, p \rangle$, we can extend v to all of $\mathbb{R}^n$ by $v(x) = \langle v, x \rangle$. Then the intersection $\Gamma_v \cap q\Gamma_{-v}$ lies within the region $L = \{x \in \mathbb{R}^n \mid 0 \leq v(x) \leq v(q)\}$. In other words, we are looking at the piece of $\mathbb{R}^n$ between the hyperplanes $P_1 = \{x \mid \langle v, x \rangle = \sum v_i x_i = 0\}$ and $P_1 + \theta(q)$ which are evidently parallel to one another. One can then verify that the distance between these two hyperplanes is $v(q)/\|v\| \geq 2\sqrt{2}$. Because of this, any point $p \in Q_v \cap qQ_{-v}$ has at least $n^2 - n$ "neighboring" points also in $Q_v \cap qQ_{-v}$.

Consider $p = (p_1, ..., p_n) \in Q_v \cap qQ_{-v}$. By a neighboring point, we mean those points $p \pm e_i \pm e_j$ and $p \pm e_i$ for all $1 \leq i < j \leq n$ and signs. Suppose $p + e_i, p - e_i \notin Q_v \cap qQ_{-v}$ as $p \in Q_v \cap qQ_{-v}$ the points $p \pm e_i$ lie in different connected components of $\mathbb{R}^n \setminus L$. Thus $2 = \|p + e_i - (p - e_i)\| > 2\sqrt{2}$ which is absurd. Hence at least one of $p \pm e_i \in Q_v \cap qQ_{-v}$. A similar realization shows us that if $p \pm e_i \pm e_j \notin Q_v \cap qQ_{-v}$, then $-(p \pm e_i \pm e_j) \in Q_v \cap qQ_{-v}$.

Let $p \in Q_v \cap qQ_{-v}$ and $1 \leq k \leq n$; define $\Delta_{p,k}$ to be the 1-subcomplex of $\Delta_v \cap q\Delta_{-v}$ determined by the 0-cells $\{q_1^{p_1} \cdots q_k^r \cdots q_n^{p_n} \mid r \in \mathbb{Z}\} \cap \Delta_v \cap q\Delta_{-v}$. For any $p \in Q_v \cap qQ_{-v}$ and any $1 \leq k \leq n$, the complexes $\Delta_{p,k}$ are evidently path connected.

We work under the added condition that $v(q_i) > 0$ for all i. We show for $1 \leq i \leq n$ and any point $p \in Q_v \cap qQ_{-v}$, there exists a point $p' \in \Delta_{p,1}$ and a point $p'' \in \Delta_{p,1}$ such that $p'q^i \in Q_v \cap qQ_{-v}$ and $p''q^{-i} \in Q_v \cap qQ_{-v}$. Since $v(q_i) > 0$ for all i, every $\Delta_{p,1}^0$ consists of finitely many points. We thus define $p_L \in \Delta_{p,1}^0$ to be the point such that $p_L q_1^{-1} \notin \Delta_{p,1}^0$ and $p_R \in \Delta_{p,1}^0$ such that $p_R q_1 \notin \Delta_{p,1}$.

The proof follows by cases.[3] Let $\epsilon = \pm 1$ be a sign. Suppose at every point $r \in \Delta_{p,1}$ that $rq_i^\epsilon \notin Q_v \cap qQ_{-v}$. Then it is the case that both $0 \leq v(p_L) + \epsilon v_i - v_1 \leq v(q)$ and $0 \leq v(p_R) + \epsilon v_i + v_1 \leq v(q)$. One then shows that this implies either $p_L - e_1 \in Q_v \cap qQ_{-v}$ or $p_R + e_1 \in Q_v \cap qQ_{-v}$ depending on ϵ and the sign of v_1 and v_i. We show the case when $\epsilon = 1$, $v_1, v_i > 0$. This then implies $0 \leq v(p_R) \leq v(p_R) + v_1 \leq v(p_R) + v_i + v_1 \leq v(q)$, hence $p_R + e_1 \in Q_v \cap qQ_{-v}$ which is a contradiction. See figure 2.

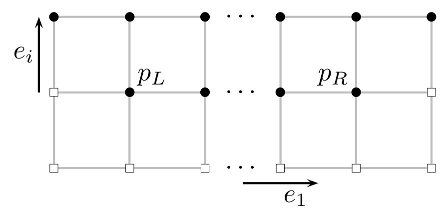

Figure 2: The boxes represent points which are not in $Q_v \cap Q_{-v}$ under the assumption that $r - e_i \notin Q_v \cap qQ_{-v}$ for all $r \in \Delta_{p,1}$. The circular dots are points that we can determine are in $Q_v \cap Q_{-v}$.

Assume that $v(q_i) > 0$ for all i. We then show that for any point $p \in Q_v \cap qQ_{-v}$, there is a path from $(0, \ldots, 0) \in Q_v \cap qQ_{-v}$ into $\Delta_{p,1}$ for p contained in $\Delta_v \cap q\Delta_{-v}$, and hence to p. We prove this by induction on n. If $n = 1$, then $\Delta_{0,1} = \Delta_v \cap q\Delta_{-v}$ which we know is path connected. Now supposing there is a path from 0 to $\Delta_{(p_1,\ldots,p_{n-1},0),1}$, by induction on $|p_n|$ there is a path from $\Delta_{(p_1,\ldots,p_{n-1},0),1}$ to $\Delta_{(p_1,\ldots,p_{n-1},p_n),1}$ and hence to p. The case $|p_n| = 1$ was established in the previous paragraphs. Thus supposing there is a path from $\Delta_{(p_1,\ldots,p_{n-1},0),1}$ into $\Delta_{(p_1,\ldots,p_{n-1},p_n - \mathrm{sgn}(p_n)),1}$, there is a point $p_{\mathrm{sgn}(p_n)} \in \Delta_{(p_1,\ldots,p_{n-1},p_n - \mathrm{sgn}(p_n)),1}$ for which $p_{\mathrm{sgn}(p_n)} + \mathrm{sgn}(p_n) \in Q_v \cap qQ_{-v}$. Hence the result for when $v(q_i) > 0$ for all i.

Now if $v(q_i) = 0$ for some i, then $p + \mathbb{Z}e_{i_j} \subseteq Q_v \cap qQ_{-v}$. Thus given a point $p = (p_1, \ldots, p_n)$, we need only be concerned finding a path from 0 to

$$\sum_{\substack{i \\ v(q_i) \neq 0}} p_i e_i$$

which is then contained in some lower dimensional subspace, on which, v satisfies the conditions of our special case. Thus the result is proven.

$\bowtie$

Lemma 6.1.48: Consider a non-trivial additive character v and suppose θ is chosen as above. Set $\ell = \max \{\mathrm{length}(r) \mid r \in \mathfrak{R}\} \cup \{1\}$. For any $q \in Q$ that satisfies $v(q) \geq \ell \|v\|$, we have $\Gamma_v \cup q\Gamma_{-v} = \Gamma$.

Proof. Like in the previous lemma, we get a region L bounded by 2 parallel hyperplanes which are separated by a distance of $v(q)/\|v\| \geq \ell \geq 1$. It is evident that $Q_v \cup qQ_{-v} = Q$. Now consider a path γ in the 1-skeleton given by a word w of length $\mathrm{length}(w) \leq \ell$. Then if w ever traverses a point a such that $a \notin Q_v$ or $a \notin qQ_{-v}$, the whole path must be contained

[3] I anticipate that one can prove this in a more satisfying way. It seems like it should follow that if one cannot move in the direction q_i^ϵ, then $v_1 = 0$ which contradicts our assumptions.

in qQ_{-v} or Q_v respectively by our choice of q. Thus $\Gamma^1 \subseteq \Gamma_v \cup q\Gamma_{-v}$. We furthermore have all 2-cells (q,r) in $\Gamma_v \cup q\Gamma_{-v}$ as length$(\partial(q,r)) \leq \ell$ by assumption. ⋈

We wish to apply Van Kampen's theorem to Γ_v, $q\Gamma_{-v}$. We record the statement of Van Kampen's theorem as follows:

Theorem 6.1.49: Let X be a path connected space, and suppose $\mathfrak{U} = \{U_\alpha\}$ is an open cover of path connected sets such that finite intersections are still path connected and are in $\mathfrak{U}$. Consider $\mathfrak{U}$ as a category with morphisms given by inclusions of open sets. Then $X = \operatorname{colim}\mathfrak{U}$, and furthermore $\pi_1(X) = \pi_1(\operatorname{colim}\mathfrak{U}) = \operatorname{colim}(\pi_1(\mathfrak{U}))$. In the case with two sets U_α, U_β covering X with intersection $U_{\alpha\beta}$ we have $\pi_1(X) \cong G_\alpha *_{G_{\alpha\beta}} G_\beta$ (where $\pi_1(U_y) = G_y$).

The subspaces Γ_v and $q\Gamma_{-v}$ are not open, but they are deformation retracts of slightly larger open subspaces. Thus we have a push-out diagram

$$\begin{tikzcd} \pi_1(\Gamma_v \cap q\Gamma_{-v}) \arrow[r, "\iota_*"] \arrow[d, "\iota_*"'] & \pi_1(\Gamma_v) \arrow[d, "\iota_*"'] \\ \pi_1(q\Gamma_{-v}) \arrow[r, "\iota_*"] & \pi_1\Gamma = A \end{tikzcd}$$

In particular, this says A is an amalgamated product. However, since A is Abelian, the only way this can happen is if the amalgamated product is trivial, i.e. one of the factors $\pi_1(\Gamma_v)$ or $\pi_1(q\Gamma_{-v})$ maps surjectively onto A. We prove this using the normal form theorem found in [Lyndon, IV.2, p. 187].

Theorem 6.1.50: Consider the pushout diagram

$$\begin{tikzcd} H \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & K \arrow[d] \\ L \arrow[r] & L *_H K. \end{tikzcd}$$

We call a sequence $c_1, \ldots, c_s$, $s \geq 0$ of elements of $L * K$ reduced if: each c_i is in either L or K; for all i, c_i and c_{i+1} do not belong to the same factor; no c_i belongs to $\operatorname{im}\phi_1$ or $\operatorname{im}\phi_2$; if $s = 1$, $c_1 \neq 1$. Then if $c_1, \ldots, c_s$ is a reduced sequence then $c_1 c_2 \cdots c_s \neq 1$ in the amalgamated product $L *_H K$. ⋈

Thus in our case, if neither factor was killed entirely, there would exist $a \in \pi_1(\Gamma_v)$ and $b \in \pi_1(q\Gamma_{-v})$ which are not in the subgroup we are amalgamating over. Then a, b, a^{-1}, b^{-1} is a reduced sequence; hence $aba^{-1}b^{-1} \neq 1$ in A which contradicts A being Abelian. Thus either $\iota_* : \pi_1(\Gamma_v) \twoheadrightarrow A$ or $\iota_* : \pi_1(q\Gamma_{-v}) \twoheadrightarrow A$.

In fact, for any additive character there is a sign $\epsilon = \pm 1$ for which $\iota_* : \pi_1(\Gamma_{\epsilon v}) \twoheadrightarrow A$ is surjective. Suppose $\iota_* : \pi_1(q\Gamma_{-v}) \twoheadrightarrow A$. Then as $\pi_1(\Gamma_{-v}, 1) \cong \pi_1(q\Gamma_{-v}, q) = q \cdot \pi_1(\Gamma_{-v}, 1)$ where the last equality follows by conjugation by the induced action of Q on the fundamental groupoid $\Pi(\Gamma)$. As the fundamental group does not depend on the choice of a basepoint, we have a diagram

$$\begin{tikzcd}
\pi_1(q\Gamma_{-v}, 1) \arrow[rr, "\iota_*"] & & A \\
\pi_1(q\Gamma_{-v}, q) \arrow[u, "\wr"'] & & \\
\pi_1(\Gamma_{-v}, 1) \arrow[u, "\wr"', "q\cdot"] \arrow[uurr, "\iota_*"'] & &
\end{tikzcd}$$

which is not necessarily commutative. Since $\iota_*\pi_1(q\Gamma_{-v}, 1)$ is conjugate to $\iota_*\pi_1(\Gamma_{-v}, 1)$ in $\pi_1(\Gamma, 1) \cong A$, we conclude that $\iota_* : \pi_1(\Gamma_{-v}, 1) \twoheadrightarrow \pi_1(\Gamma)$.

Now because of the Hurewicz natural transformation $h : \pi_1 \to H_1$ we get a commutative diagram:

$$\begin{tikzcd}
\pi_1\left(\Gamma_{\epsilon v}\right) \arrow[r, two heads, "\pi_1 \iota"] \arrow[d, two heads, "h"'] & \pi_1\left(\Gamma\right) = A \arrow[d, "\wr"', "h"] \\
H_1\left(\Gamma_{\epsilon v}\right) \arrow[r, "H_1 \iota"] & H_1\left(\Gamma\right) = A
\end{tikzcd}$$

that forces $H_1\iota = \iota_*$ to be surjective. The homomorphism $H_1\iota$ is also easily seen to be $Q_{\epsilon v}$-linear. All we need to do now is prove the following lemma.

Lemma 6.1.51: For any non-trivial additive character v, $H_1\left(\Gamma_v\right)$ is a finitely generated Q_v-module.

We show that $Z_1(\Gamma_v)$, i.e. the cycles in Γ_v, is finitely generated as a Q_v-module. It is clear that $Z_1(\Gamma_v) = Z_1(\Delta_v) \oplus Z_1(\Omega_v)$. The set $\{(1, m) \mid m \in \mathcal{M}\}$ is finite and generates $Z_1(\Omega_v) = C_1(\Omega_v)$ over Q_v.

As $H_1(\Delta) = 0$, the set

$$\{\partial(1, [t_i, t_j]) \mid 1 \leq i < j \leq n\} = \{(1, t_i) + (q_i, t_j) - (q_j, t_i) - (1, t_j) \mid 1 \leq i < j \leq n\}$$

generates $Z_1(\Delta)$ over Q. We may assume $v(q_i) \geq 0$ for all i, hence $\gamma_{i,j} \in Z_1(\Delta_v)$. We show that they indeed generate $Z_1(\Delta_v)$ over Q_v. Call $V = \langle \gamma_{i,j} \rangle_{Q_v}$.

Consider a cycle $\gamma \in Z_1(\Delta_v)$. It has the form $\sum m_{q,j}(q, t_j)$ where the sum is over all $q \in Q_v$ and satisfies $\sum m_{q,j}\partial(q, t_j) = 0$. From this, we deduce that

$$\sum m_{q,j}\left((qq_j, t_1) - (q, t_1)\right) = 0.$$

Thus

$$\begin{aligned}
q_1\gamma - \gamma &= \sum m_{q,j}\left((qq_1, t_j) - (q, t_j)\right) \\
&= \sum m_{q,j}\left((qq_1, t_j) - (qq_1, t_1) - (q, t_j) + (q, t_1)\right) \\
&= \sum m_{q,j} q\left((1, t_1) + (q_1, t_j) - (q_1, t_1) - (1, t_j)\right) \\
&= \sum_{j>1} m_{q,j} q\gamma_{1,j},
\end{aligned}$$

hence $q_1\gamma-\gamma \in V \subseteq Z_1(\Delta_v)$. Applying the above computation to $q_1\gamma-\gamma$ yields $q_1^2\gamma-\gamma \in Z_1(\Delta_v)$. Inductively we see that for all $m \in \mathbb{N}$ the chain $q_1^m\gamma - \gamma \in V \subseteq Z_1(\Delta_v)$. Alternatively, $\gamma = \sum \lambda_{i,j}\gamma_{i,j}$ where $\lambda_{i,j} \in \mathbb{Z}Q$. As there must exist i such that $v(q_i) > 0$, there is no loss of generality assuming $v(q_1) > 0$. Then as $\mathbb{R}$ is an Archimedean group, there exists m such that $v(q_1^m) = mv(q_1) \geq -v(\lambda_{i,j})$ for all $1 \leq i < j \leq n$, i.e. $v(q_1^m\lambda_{i,j}) \geq 0$. Hence $q_1^m \sum \lambda_{i,j}\gamma_{i,j} \in V$, and $q_1^m\gamma - \gamma \in V$ from which we conclude $\gamma \in V$. Thus $V = Z_1(\Delta_v)$ is a finitely generated $\mathbb{Z}Q_v$-module. With this, the theorem is proven. $\bowtie$

Remark 6.1.52: It is worth noting that the general idea of this proof can be used to show that a metabelian group is FP_2 if and only if it is finitely presented. The details of how to extend the proof can be found in [Bieri], and the basic definitions of FP_n can be found in [Brown, § VIII.5]. It has recently been shown by Mladen Bestvina and Noel Brady [Bestvina] that this is not the case for all groups. They show that right-angled Artin groups satisfy FP_2 but are not finitely presented. For more details on this, see [Charney].

We proceed onward and complete the proof of the main theorem. We do not provide the general proof of the remaining statement as a few computational difficulties arise which hinder the understanding of the main idea. We thus only prove the special case when the extension is the split extension which is formulated in the following theorem.

Theorem 6.1.53: If A is a tame Q-module, then $A \rtimes Q$ is finitely presented.

In order to proceed with the proof, we need to develop some notation for typography's sake. First off, the operation on the split extension $A \rtimes Q$ is given by $(a,q) * (b,r) = \left(a + b \cdot q^{-1}, qr\right)$ because we have switched to the right Q-module A. We also denote for general x, y in any group $x^y := y^{-1}xy$ and $[x,y] = x^{-1}y^{-1}xy$.

Remark 6.1.54: We proceed under the assumption that $Q \cong \mathbb{Z}^n$ because of lemma 43, and $G = A \rtimes Q = A \rtimes \mathbb{Z}^n$.

Definition 6.1.55:

a.) $\mathcal{A}$ is a finite generating set of A;

b.) $\mathcal{T} := \{t_1, ..., t_n\}$ where $t_i = (0, e_i)$ and $\{e_i\}$ is a free basis of $Q \cong \mathbb{Z}^n$;

c.) $F := F(\mathcal{T})$ is the free group with generators from $\mathcal{T}$;

d.) $\overline{F} := \{t_1^{m_1}t_2^{m_2}\cdots t_n^{m_n} \mid m_i \in \mathbb{Z}\} \subseteq F$ is the set of all ordered words in F;

e.) $\widehat{\cdot} : \overline{F} \to Q$ defined by $\widehat{t_i} := e_i$;

f.) $\theta : Q \to \mathbb{Z}^n \subseteq \mathbb{R}^n$ defined by $\theta(e_i) := e_i$;

g.) $\overline{F}_m := \left\{s \in \overline{F} \mid \|\theta(\widehat{s})\| < m\right\}$;

h.) $\mathcal{R}_Q := \{[t_i, t_j] \mid 1 \leq i \leq j \leq n\}$

i.) $\mathcal{K}_m := \left\{[a, b^u] \mid u \in \overline{F}_m,\ a, b \in \mathcal{A}\right\}$;

j.) $\mathcal{K}_\infty := \left\{[a, b^u] \mid u \in \overline{F},\ a, b \in \mathcal{A}\right\}$;

k.) As $S(Q) = \Sigma_A \cup -\Sigma_A = \bigcup_{\lambda \in C(A)} (H_\lambda \cup -H_\lambda)$, and $S(Q)$ is compact, we may choose $\Lambda \subseteq C(A)$ such that it is finite and $S(Q) = \bigcup_{\lambda\in\Lambda} (H_\lambda \cup -H_\lambda)$ holds;

l.) $\mathcal{C}_\Lambda := \left\{a^{-1}\prod_{u\in\overline{F}}\left(a^{\lambda(\widehat{u})^u}\right) \mid a\in\mathcal{A}, \lambda\in\Lambda\right\}$;

m.) $G_m := \langle \mathcal{A}\cup\mathcal{T} \mid \mathcal{R}_Q\cup\mathcal{C}_\Lambda\cup\mathcal{K}_m\rangle$;

n.) $G_\infty := \langle \mathcal{A}\cup\mathcal{T} \mid \mathcal{R}_Q\cup\mathcal{C}_\Lambda\cup\mathcal{K}_\infty\rangle$;

o.) $A_\infty := \mathrm{gp}_{G_\infty}(\mathcal{A}) \triangleleft G_\infty$;

p.) For $\lambda\in\Lambda$ define $L_\lambda := \{\theta(q) \mid q\in\mathrm{supp}(\lambda)\}\in\mathfrak{f}\mathbb{R}^n$

q.) $\mathcal{F} := \{\pm L_\lambda \mid \lambda\in\Lambda\}\in\mathfrak{f}\mathfrak{f}\mathbb{R}^n$.

It is important to note that our choice in b.) is what simplifies the proof of the theorem. In general, we can only pick an element of G which projects onto the basis elements of Q, and need to work a little harder to get things to work out.

Our choice of Λ has the following interpretation: for every nonzero element $x\in\mathbb{R}^n$, there exists an element $\lambda\in\Lambda$ such that either: for all $\overline{v}\in H_\lambda$ we have $v(x)>0$, or for all $\overline{v}\in H_\lambda$ we have $-v(x)>0$.

Observation 6.1.56:

1. $A_\infty = \left\langle a^u \mid a\in\mathcal{A}, u\in\overline{F}\right\rangle = \left\{\alpha^u \mid \alpha\in A, u\in\overline{F}\right\}$.
2. $A_\infty\triangleleft G_\infty$ and $G_\infty/A_\infty = \langle\mathcal{T}\mid\mathcal{R}_Q\rangle$, that is, G_∞ is metabelian.

Proof. (Theorem 53) The proof of the theorem follows easily once we establish that there exists some $m_0>0$ such that $G_{m_0}=G_\infty$. Let us work under this assumption and defer this result to lemma 57.

Define $\psi : G_\infty \twoheadrightarrow A\rtimes Q$ to be the homomorphism defined as the identity on $\mathcal{A}$ and $\mathcal{T}$. One can easily verify that ψ is a well defined homomorphism as all the images of all the relations in G_∞ are satisfied in $A\rtimes Q$. Note that $\ker\pi\circ\psi = A_\infty$. Thus $G_\infty/A_\infty\cong Q$. With this we conclude $\ker\psi\subseteq A_\infty$. As $\mathbb{Z}Q\cong\mathbb{Z}\left[x_1,...,x_n,x_1^{-1},...,x_n^{-1}\right]$ we conclude that $\mathbb{Z}Q$ is Noetherian from Hilbert's Basis Theorem. As $\ker\psi$ is a $\mathbb{Z}Q$-submodule of the finitely generated $\mathbb{Z}Q$-module A_∞, we conclude that $\ker\psi$ is finitely generated as a Q-module because $\mathbb{Z}Q$ is Noetherian. We thus see that a finite generating set $\mathcal{Y}$ of $\ker\psi$ as a Q-module has normal closure in G_∞ equal to $\ker\psi$ as the conjugation action of Q on $\ker\psi$ agrees with the given Q-module structure on A_∞. That is, $\mathrm{gp}_{G_\infty}(\mathcal{Y})=\ker\psi$. Thus

$$\begin{aligned} A\rtimes Q &\cong G_\infty/\,\mathrm{gp}_{G_\infty}(\mathcal{Y}) \\ &\cong \langle\mathcal{A}\cup\mathcal{T}\mid\mathcal{Y}\cup\mathcal{R}_Q\cup\mathcal{K}_{m_0}\cup\mathcal{C}_\Lambda\rangle \end{aligned}$$

which is a finite presentation for $A\rtimes Q$.

$\bowtie$

Lemma 6.1.57: There exists $m_0<\infty$ such that $G_{m_0}=G_\infty$.

Observation 6.1.58: The following relations hold for $u,v,w\in G=A\rtimes Q$:

1. $[u,vw]=[u,w]\left([u,v]^w\right)$;
2. $[uv,w]=\left([u,w]^v\right)[v,w]$;

3. $[u^w, v] = \left[u, v^{w^{-1}}\right]^w$.

Proof. From our previous considerations, it is clear that $\mathcal{F}$ satisfies the conditions of the geometric lemma. Thus there exists a $\rho_0 > 0$ and a function ε which satisfy: $B_{\rho+\varepsilon(\rho)}$ can be taken from B_ρ by $\mathcal{F}$ for all $\rho > \rho_0$. Thus we fix $m_0 > \rho_0$ and set $l := \varepsilon(m_0)$. Thus for all $m \geq m_0$ the ball B_{m+l} can be taken from B_m by $\mathcal{F}$.

We now show that $\phi : G_{m+l} \cong G_m$ for all $m \geq m_0$ from which it will follow inductively that $G_{m_0} \cong G_\infty$. Define ϕ and ϕ^{-1} to be the identity on $\mathcal{A}$ and $\mathcal{T}$. The map ϕ^{-1} is clearly a homomorphism as all of the relations in G_m are in G_{m+l}.

In the following paragraphs, we show that ϕ is a homomorphism by showing that for all $s' \in \overline{F}_{m+l} \setminus \overline{F}_m$ we have $[a, b^{s'}] = 1$ in G_m.

By the Geometric Lemma, for every $x \in B_{m+l} \setminus B_m$ there exists a $\lambda \in \Lambda$ and a sign $\varepsilon = \pm 1$ such that $x + \varepsilon L_\lambda \subseteq B_m$. Observe further that for $s \in \overline{F}$, $\lambda(\widehat{s}) \neq 0$ is equivalent to $\theta(\widehat{s}) \in L_\lambda$. Now let $s' \in \overline{F}_{m+l} \setminus \overline{F}_m$. Thus we can interpret s' in $\mathbb{R}^n$ via $\theta(\widehat{s'})$ and apply the Geometric Lemma. There thus exists a $\lambda \in \Lambda$ and a sign ε such that $\theta(\widehat{s'}) + \varepsilon L_\lambda \subseteq B_m$. In other words, for each $q \in \operatorname{supp}(\lambda)$, we have $\theta(\widehat{s'}) + \varepsilon\theta(q) \in B_m$, yet as θ is a homomorphism, we have $\theta(\widehat{s' q^\varepsilon}) \in B_m$. Thus if $\lambda(\widehat{s}) \neq 0$, the inclusion reduces to $\theta(\widehat{s'\widehat{s}^\varepsilon}) \in B_m$, and thus working modulo $\mathcal{R}_Q$ in F, we conclude that $s's^\varepsilon \in \overline{F}_m$ mod $\mathcal{R}_Q$ (this is another part where trouble arises in the general case).

In order to establish the equation $[a, b^{s'}] = 1$ in G_m we first show:

1'. if $\varepsilon = 1$ then $[a, \left(b^{\lambda(\widehat{s})}\right)^{ss'}] = 1$ in G_m;

2'. if $\varepsilon = -1$ then $[a^{\lambda(s\widehat{s})}, b^{s's^{-1}}] = 1$ in G_m.

Proof.

1'. As $s's \in \overline{F}_m$ we have $[a, b^{s's}] = 1$ in G_m as it is a relation in $\mathcal{K}_m$. We compute using the identities in Observation 58

$$\begin{aligned}
\left[a, \left(b^{\lambda(\widehat{s})}\right)^{ss'}\right] &= \left[a, \left(b^{ss'}\right)\left(b^{\lambda(\widehat{s})-1}\right)^{ss'}\right] \\
&= \left[a, \left(b^{\lambda(\widehat{s})-1}\right)^{ss'}\right]\left[a, b^{ss'}\right]^{\left(b^{\lambda(\widehat{s})-1}\right)^{ss'}} \\
&= \prod_{i=1}^{\lambda(\widehat{s})} \left[a, b^{ss'}\right]^{\left(b^{\lambda(\widehat{s})-1}\right)^{ss'}} \\
&= \prod_{i=1}^{\lambda(\widehat{s})} 1^{\left(b^{\lambda(\widehat{s})-1}\right)^{ss'}} \\
&= 1.
\end{aligned}$$

2'. We likewise compute with the aid of Observation 58

$$
\begin{aligned}
\left[a^{\lambda(\widehat{s})}, b^{s' s^{-1}}\right] &= \left[a, b^{s' s^{-1}}\right]^{a^{\lambda(\widehat{s})-1}} \left[a^{\lambda(\widehat{s})-1}, b^{s' s^{-1}}\right] \\
&= \prod_{i=1}^{\lambda(\widehat{s})} \left[a, b^{s' s^{-1}}\right]^{n-i} \\
&= \prod_{i=1}^{\lambda(\widehat{s})} 1^{a^{n-i}} \\
&= 1
\end{aligned}
$$

⋈

We now prove that the equation $[a, b^{s'}] = 1$ is valid in G_m. We break the proof into two parts: 1. when $\epsilon = 1$; 2. when $\epsilon = -1$.

1. With $\epsilon = 1$, and $\operatorname{supp}(\lambda) = \{\overline{s_1}, \ldots, \overline{s_t}\}$ we compute

$$
\begin{aligned}
\left[a, b^{s'}\right] &= \left[a, \prod_{i=1}^{t} \left(b^{\lambda(\widehat{s_i})}\right)^{s_i s'}\right] \\
&= \prod_{i=0}^{t-1} \left[a, \left(b^{\lambda(\widehat{s_{t-i}})}\right)^{s_{t-i} s'}\right]^{f(i)} \\
&= 1
\end{aligned}
$$

where $f(i) = \prod_{j=l+1-i}^{t} \left(b^{\lambda(\widehat{s_j})}\right)^{s_j s'}$ and $f(0) = 1$. The description of f is not very important and can be ignored as it is clear that some such function exists.

2. With $\epsilon = -1$, and $\operatorname{supp}(\lambda) = \{\overline{s_1}, \ldots, \overline{s_t}\}$ we compute

$$
\begin{aligned}
\left[a, b^{s'}\right] &= \left[\prod_{i=1}^{t} \left(a^{\lambda(\widehat{s_i})}\right)^{s_i}, b^{s'}\right] \\
&= \prod_{i=1}^{t} \left[\left(a^{\lambda(\widehat{s_i})}\right)^{s_i}, b^{s'}\right]^{g(i)} \\
&= \prod_{i=1}^{t} \left[a^{\lambda(\widehat{s_i})}, b^{s' s_i^{-1}}\right]^{s_i g(i)} \\
&= \prod_{i=1}^{t} 1^{s_i g(i)} \\
&= 1
\end{aligned}
$$

where $g(i) = \prod_{j=i+1}^{t} \left(a^{\lambda(\widehat{s_j})}\right)^{s_j}$. Again, the description of g is not important.

Thus all of the relations in G_{m+l} are satisfied in G_m from which it follows that ϕ is a homomorphism, and thus an isomorphism. A simple induction argument proves the claim that $G_{m_0} = G_\infty$. ⋈

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