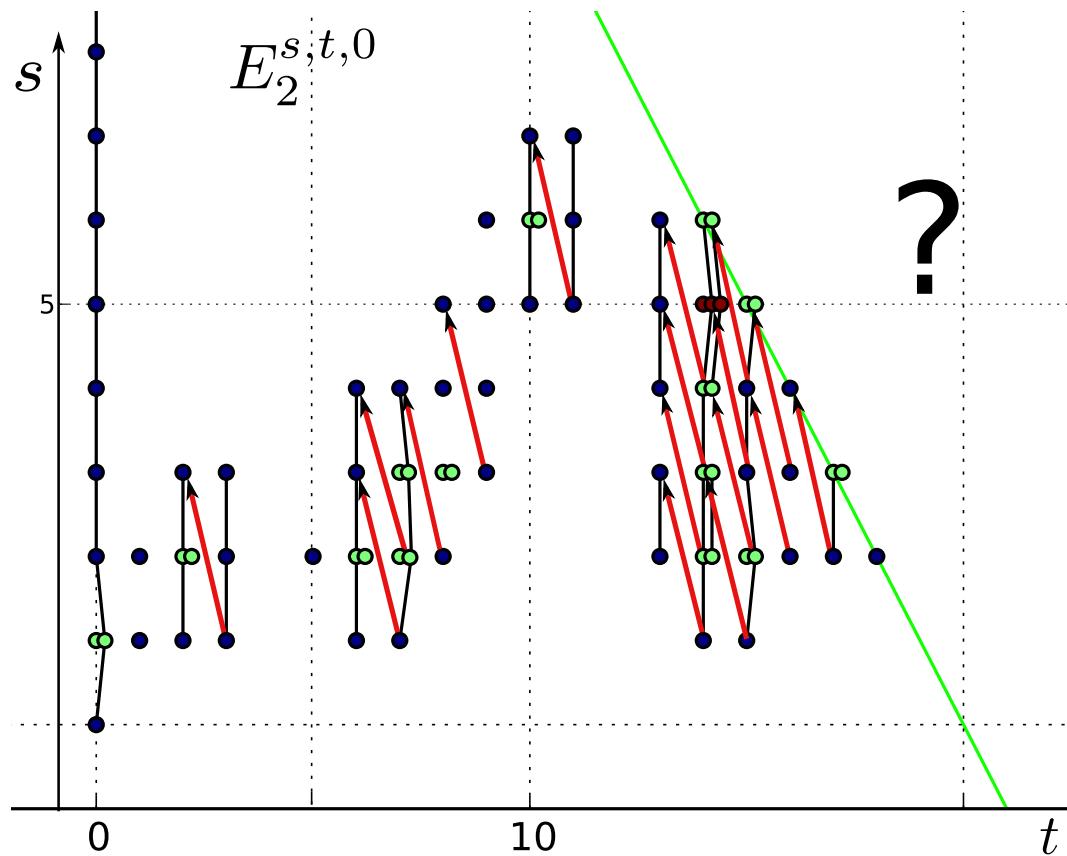


Motivic Adams Spectral Sequence over Finite Fields



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Outline

- Introduction to the motivic stable homotopy category
- $\pi_{n,n}$ by Morel
- Motivic Adams spectral sequence (MASS)
- Computer calculations
- Analysis of E_2 page and differentials
- Computations of stable stems $\pi_{n+3,n}$
- Comparison to topological π_n

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Motivic homotopy theory

- k is a field.
- Sm/k is the category of smooth, separated schemes of finite type over $\text{Spec}(k)$.
- Our “spaces” are Nisnevich sheaves on Sm/k with values in simplicial sets $\text{Spc}(k) = \Delta^{\text{op}}\mathbf{Shv}_{Nis}(\text{Sm}/k)$.
- Put a model structure on $\text{Spc}(k)$ where $X \times \mathbb{A}^1 \rightarrow X$ is a weak equivalence for all spaces X .
- Denote the homotopy category of pointed spaces $\text{Spc}_\bullet(k)$ by $\mathcal{H}_\bullet(k)$.
- Both schemes and simplicial sets can be considered as spaces.

Spheres

- The simplicial circle $S^{1,0} = \Delta^1 / \partial\Delta^1$, pointed at $\partial\Delta^1$.
- The Tate circle $S^{1,1} = \mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$, where $\mathbb{A}^1 \in \text{Sm}/k$ is the affine line, pointed at 1.
- Smash product $X \wedge Y$: pushout of the diagram

$$\begin{array}{ccc} X \vee Y & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \\ \text{Spec}(k) & \dashrightarrow & X \wedge Y \end{array}$$

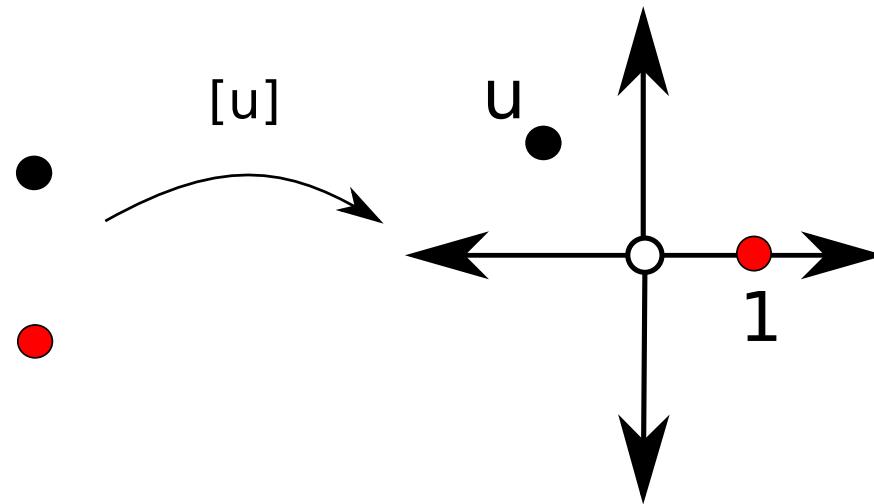
- Mixed spheres $S^{m,n} = (S^{1,0})^{\wedge m-n} \wedge (S^{1,1})^{\wedge n}$.
- There is a weak equivalence $\mathbb{P}^1 \cong S^{2,1} = S_s^1 \wedge \mathbb{G}_m$.
- $S^{0,0} = \text{Spec}(k) \sqcup \text{Spec}(k)$

Motivic Stable Homotopy Theory

- Just as in topology, we construct a category of spectra.
- A $S^{2,1} = \mathbb{P}^1$ -spectrum is a sequence of spaces E_0, E_1, \dots with bonding maps $\sigma_i : \mathbb{P}^1 \wedge E_i \rightarrow E_{i+1}$.
- Sphere spectrum $\mathbb{1}$ with $\mathbb{1}_r = S^{2r,r}$.
- $\pi_{m,n}(E) = \operatorname{colim}_r \mathcal{H}_\bullet((\mathbb{P}^1)^{\wedge r} \wedge S^{m,n}, E_r)$.
- Denote the motivic stable homotopy category by $\mathbf{SH}(k)$.
- $\mathbf{SH}(k)$ is a symmetric, monoidal category with \wedge , triangulated category $[1] = S^{1,0} \wedge -$.
- Cohomology $E^{p,q}(X) = \mathbf{SH}(k)(X, E \wedge S^{p,q})$.
- Homology $E_{p,q}(X) = \mathbf{SH}(k)(\mathbb{1} \wedge S^{p,q}, X \wedge E)$.

Some maps between spheres

- The identity map $1 : \mathbb{1} \rightarrow \mathbb{1}$ and its multiples n .
- A point $u : \text{Spec}(k) \rightarrow \mathbb{G}_m$ determines $[u] \in \mathcal{H}_\bullet(S^{0,0}, S^{1,1})$, which stabilizes $[u] \in \pi_{-1,-1}(\mathbb{1})$.



- Hopf map $\eta : \mathbb{A}^2 \setminus \{0\} \rightarrow \mathbb{P}^1$ defined by $\eta(x, y) = [x, y]$, is weak equivalent to $\eta : S^{3,2} \rightarrow S^{2,1}$. Determines $\eta \in \pi_{1,1}\mathbb{1}$.

Morel's computations

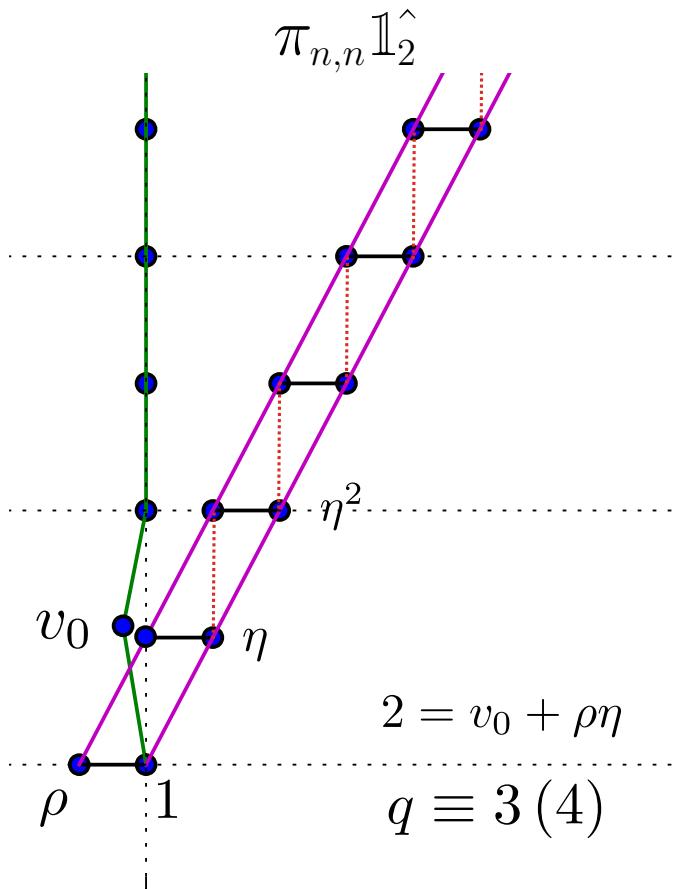
- Connectivity $\pi_{n-i,n} \mathbb{1} = 0$ for all $n \in \mathbb{Z}, i > 0$.
- $\bigoplus \pi_{n,n} \mathbb{1}$ contains 1 , η , and $[u]$ for all $u \in k^\times$.
- Theorem (Morel): $\pi_{n,n} \mathbb{1} \cong K_{-n}^{MW}(k)$.
- $K_*^{MW}(k)$ is the free graded associative algebra with generators $[u]$ of degree 1 for $u \in k^\times$, η of degree -1 satisfying the relations :
 1. $[ab] = [a] + [b] + \eta[a][b]$
 2. $[a][1 - a] = 0$
 3. $[u]\eta = \eta[u]$
 4. $\eta^2[-1] + 2\eta = 0$.

$$\pi_{n,n} \mathbb{1} \text{ over } \mathbb{F}_q$$

- Let \mathbb{F}_q be a finite field with $q = p^r$ odd.
- Morel's computation of $\pi_{n,n}$ shows for $q \equiv 3 \pmod{4}$:

stem	group
$\pi_{n,n}$	$0, n \leq -2$
$\pi_{-1,-1}$	k^\times
$\pi_{0,0}$	$\mathbb{Z} \oplus \mathbb{Z}/2$
$\pi_{n,n}$	$\mathbb{Z}/4, n \geq 1$

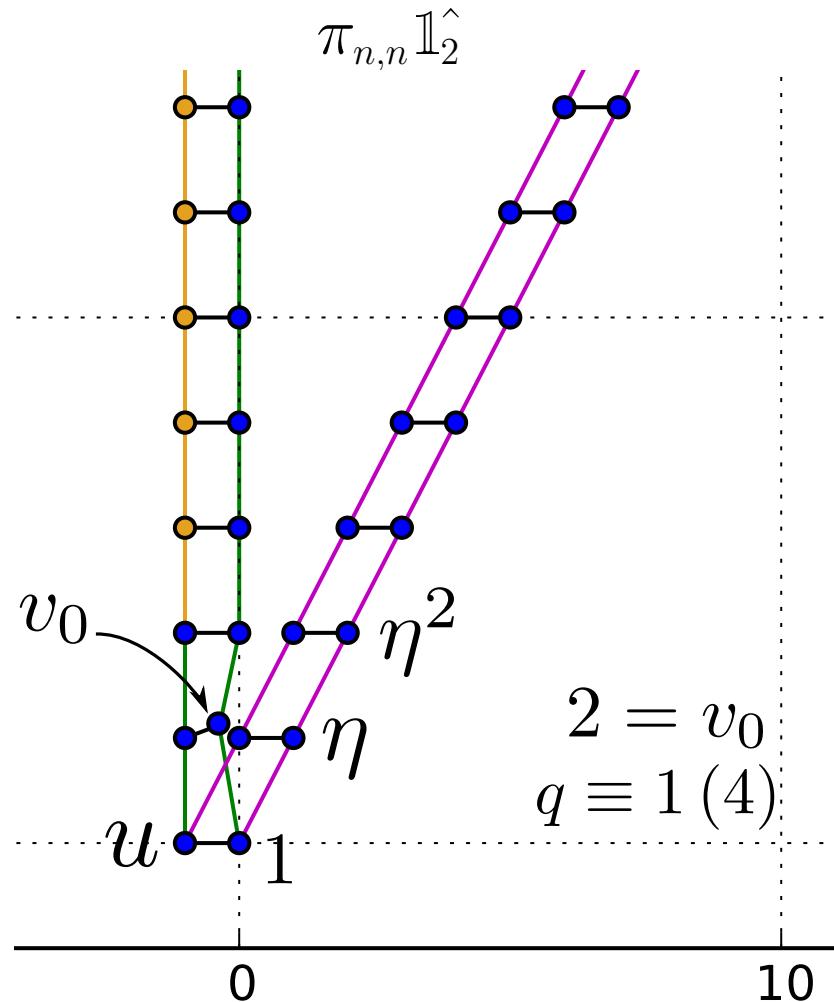
- 2-completion of A :
- $A_2^\wedge = A \otimes \mathbb{Z}_2$
- $\epsilon(q) = \nu_2(q-1)$
- $\mathbb{Z}/(q-1)_2^\wedge = \mathbb{Z}/2^{\epsilon(q)}$



$\pi_{n,n}$ over \mathbb{F}_q with $q \equiv 1 \pmod{4}$

stem	group
$\pi_{n,n}$	$0, n \leq -2$
$\pi_{-1,-1}$	k^\times
$\pi_{0,0}$	$\mathbb{Z} \oplus \mathbb{Z}/2$
$\pi_{n,n}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2, n \geq 1$

- $q = 25, \epsilon(q) = 3$
- $\pi_{-1,-1}\hat{\mathbb{1}}_2 = \mathbb{Z}/8$



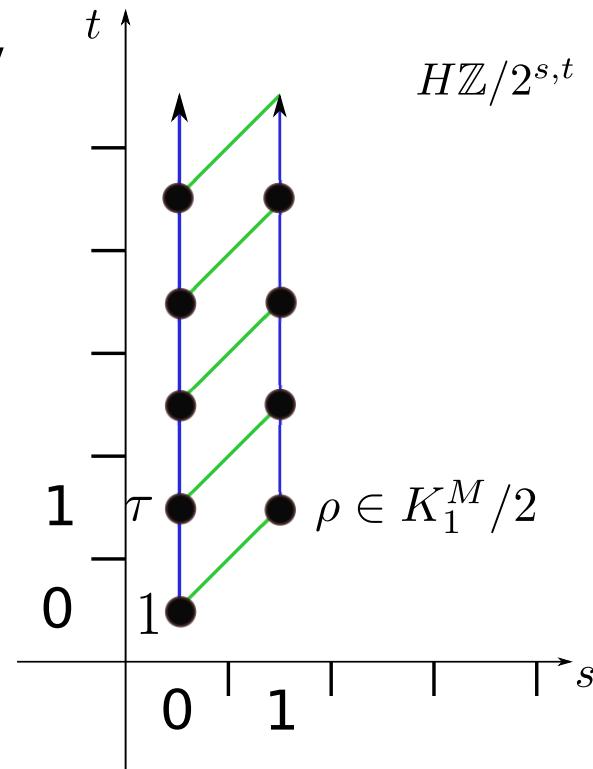
Motivic Adams spectral sequence (MASS)

- There is a MASS

$$E_2^{s,t,u} = \text{Ext}_{\mathcal{A}^{**}}^{s,(t-s,u)}(H\mathbb{Z}/2^{**}X, H\mathbb{Z}/2^{**}\mathbb{1}) \Rightarrow \pi_{t,u} X_2^{\wedge}.$$

- Differentials $d_r : E_r^{s,(t,u)} \rightarrow E_r^{s+r,(t-1,u)}$.

- $H\mathbb{Z}/2$ is motivic cohomology with $\mathbb{Z}/2$ coefficients.
- $H\mathbb{Z}/2^{**}\mathbb{1} = K_*^M/2[\tau]$ over \mathbb{F}_q .
- $q \equiv 3 \pmod{4}$, $H\mathbb{Z}/2^{**} \cong \mathbb{F}_2[\tau, \rho]/\rho^2$, $\rho = [-1] \in K_1^M(\mathbb{F}_q)$
- $q \equiv 1 \pmod{4}$, $H\mathbb{Z}/2^{**} \cong \mathbb{F}_2[\tau, u]/u^2$, u generates $K_1^M(\mathbb{F}_q)/2$.



Motivic steenrod algebra over \mathbb{F}_q

$$E_2^{s,t,u} = \text{Ext}_{\mathcal{A}^{**}}^{s,(t-s,u)}(H\mathbb{Z}/2^{**}X, H\mathbb{Z}/2^{**}\mathbb{1}) \Rightarrow \pi_{t,u} X_2^{\wedge}.$$

- $\mathcal{A}^{**} = H\mathbb{Z}/2^{**}H\mathbb{Z}/2$ the mod 2 Steenrod algebra.
- Generated by Sq^i 's and $x \cup -$ for $x \in H\mathbb{Z}/2^{**}$.
- $\deg \text{Sq}^{2i} = (2i, i)$, $\deg \text{Sq}^{2i-1} = (2i-1, i-1)$.
- If $q \equiv 1 \pmod{4}$, $\text{Sq}^1(\tau) = 0$. The Adem relations are:

$$\text{Sq}^a \text{Sq}^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} \tau^? \text{Sq}^{a+b-j} \text{Sq}^j.$$

- Free left $H\mathbb{Z}/2^{**}$ module on admissible Sq^I .
- Example: $\text{Sq}^2 \text{Sq}^4 = \text{Sq}^6 + \tau \text{Sq}^5 \text{Sq}^1$.

Motivic steenrod algebra over \mathbb{F}_q

- If $q \equiv 3 \pmod{4}$, $\text{Sq}^1(\tau) = \rho$.
- If $a + b \equiv 0 \pmod{2}$, Adem relations are:

$$\text{Sq}^a \text{Sq}^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} \tau^? \text{Sq}^{a+b-j} \text{Sq}^j.$$

If a is odd and b is even, then

$$\begin{aligned} \text{Sq}^a \text{Sq}^b &= \sum_{\substack{j=0 \\ j \text{ even}}}^{[a/2]} \binom{b-1-j}{a-1-2j} \text{Sq}^{a+b-j} \text{Sq}^j \\ &\quad + \sum_{\substack{j=0 \\ j \text{ odd}}}^{[a/2]} \binom{b-1-j}{a-1-2j} \rho \text{Sq}^{a+b-1-j} \text{Sq}^j. \end{aligned}$$

- $\text{Sq}^1 \tau = \tau \text{Sq}^1 + \rho$, but ρ commutes with Sq^i .

Computing $E_2^{s,t,u} = \text{Ext}_{\mathcal{A}^{**}}^{s,(t-s,u)}(H\mathbb{Z}/2^{**}, H\mathbb{Z}/2^{**})$

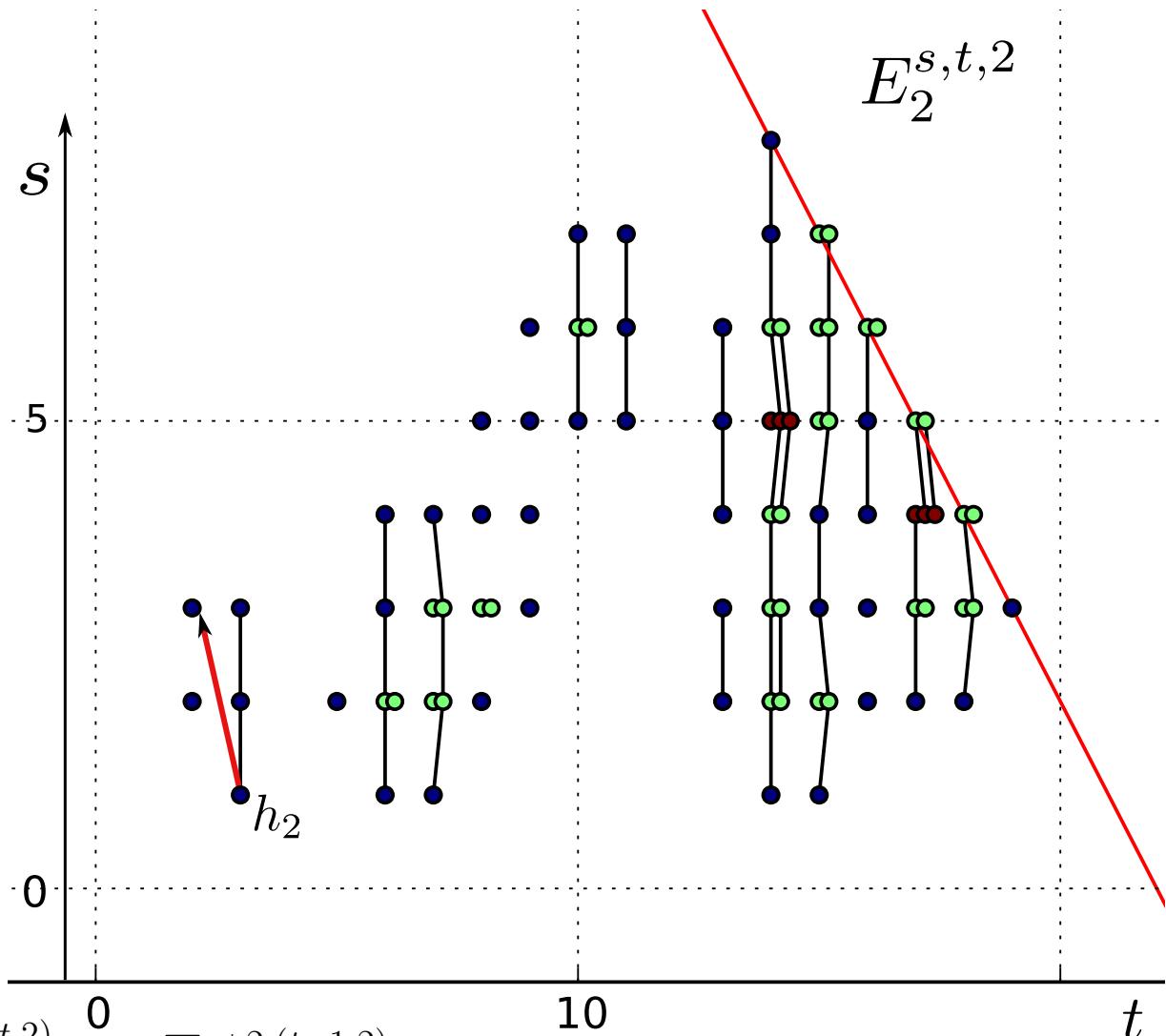
- Produce an explicit resolution of $H\mathbb{Z}/2^{**}$ by free \mathcal{A}^{**} -modules.

$$H\mathbb{Z}/2^{**} \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \dots$$

- Apply $\text{Hom}_{\mathcal{A}^{**}}(-, H\mathbb{Z}/2^{**})$ to F_* , take homology.

$$\begin{array}{ccccccc}
 H\mathbb{Z}/2^{**} & \xleftarrow{f_0} & F_0 & \longleftarrow \ker f_0 & \xleftarrow{f_1} & F_1 & \longleftarrow \dots \\
 1 & \longleftarrow i & Sq^1.i & \longleftarrow h_0 & Sq^1.h_0 & \longleftarrow h2(2,0) \\
 \tau & \longleftarrow \tau.i & Sq^2.i & \longleftarrow h_1 & \tau Sq^3.h_0 + Sq^2.h_1 & \longleftarrow h2(4,2) \\
 \rho & \longleftarrow \rho.i & Sq^3.i & \longleftarrow Sq^1.h_1 \\
 0 & \longleftarrow Sq^j.i & Sq^4.i & \longleftarrow h_2 \\
 & & 0 & \longleftarrow Sq^1.h_0 \\
 & & \tau Sq^3 Sq^1 & \longleftarrow Sq^2.h_2
 \end{array}$$

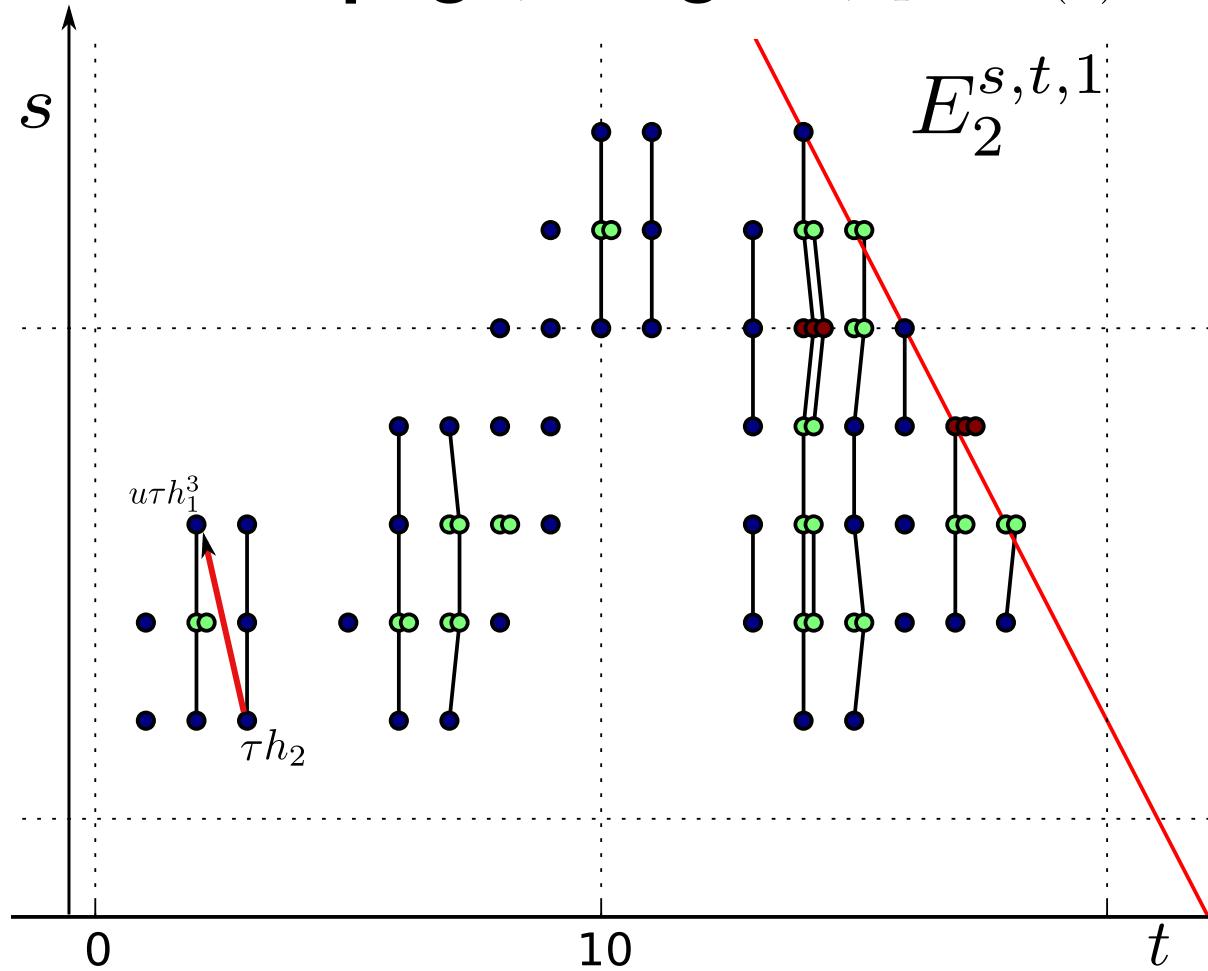
The E_2 page, weight 2, $q \equiv 1 \pmod{4}$



$$d_2 : E_2^{s,(t,2)} \xrightarrow{0} E_r^{s+2,(t-1,2)}$$

$d_2 h_2 = 0$ since $\pi_{2,2} = \mathbb{Z}/2 \oplus \mathbb{Z}/2$

The E_2 page, weight 1, $q \equiv 1 \pmod{4}$



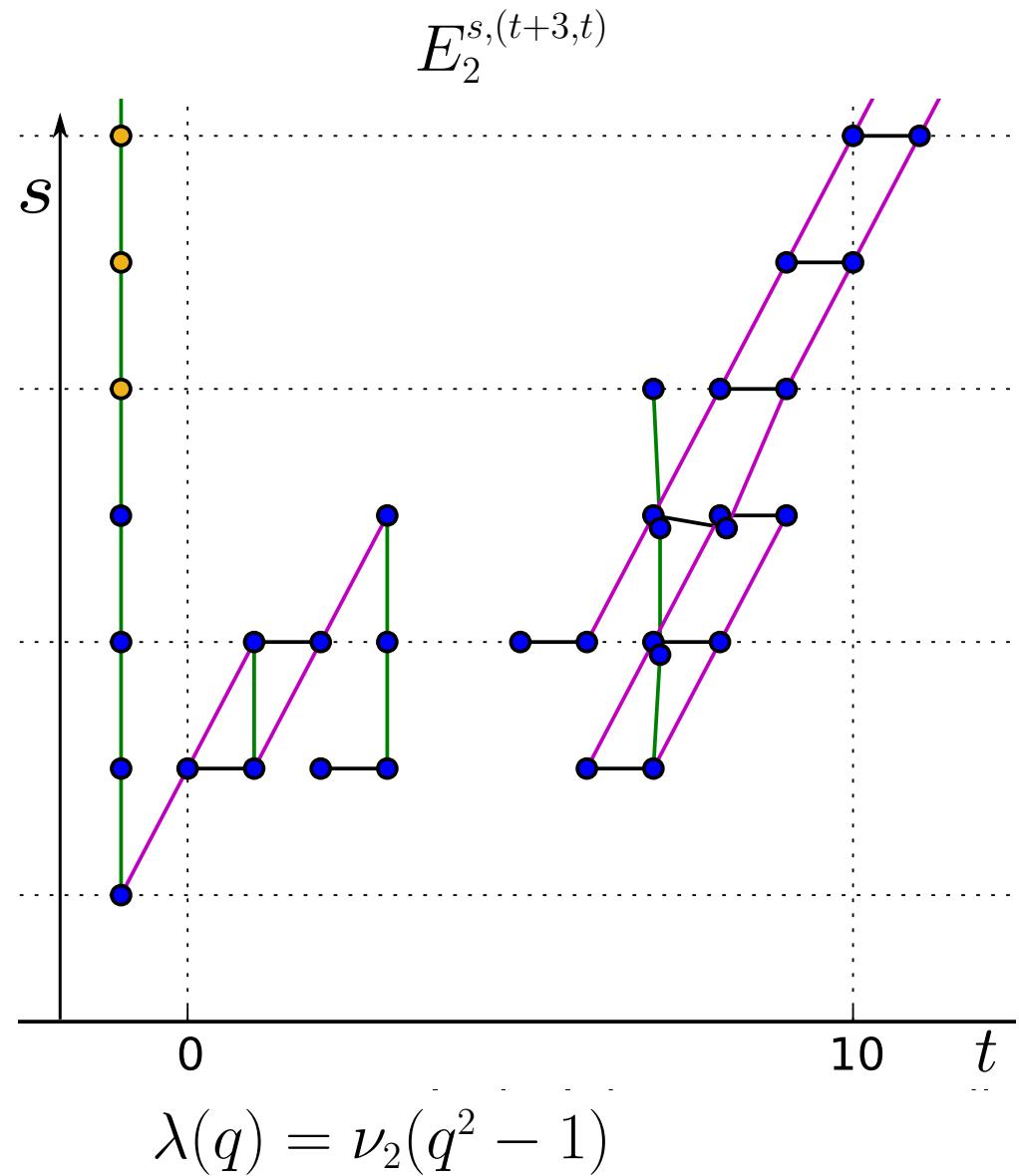
$$\epsilon(q) = \nu_2(q-1), d_{\epsilon(q)}\tau = u.h_0^{\epsilon(q)}$$

E.g., $q = 5$, $\pi_{2,1}\hat{\mathbb{1}_2} = \mathbb{Z}/2 \oplus \mathbb{Z}/4$, $\pi_{3,1}\hat{\mathbb{1}_2} = \mathbb{Z}/4$

E.g., $q = 9$, $\pi_{2,1}\hat{\mathbb{1}_2} = \mathbb{Z}/2 \oplus \mathbb{Z}/8$, $\pi_{3,1}\hat{\mathbb{1}_2} = \mathbb{Z}/8$

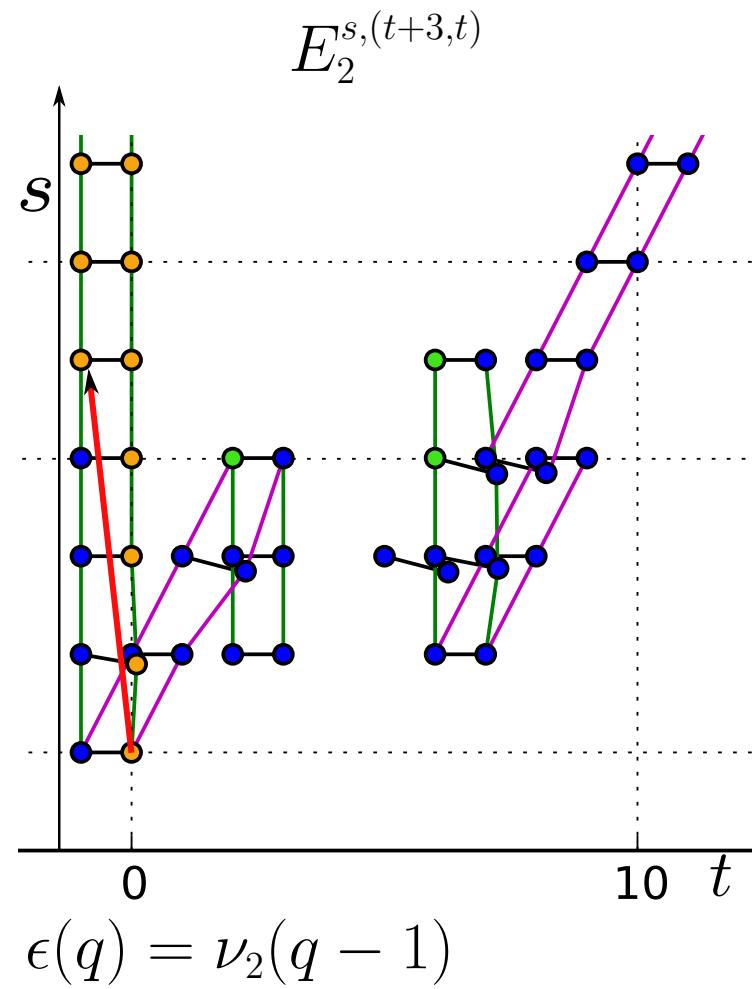
New computations, $q \equiv 3 \pmod{4}$

stem	group
$\pi_{n+3,n}$	$0, n \leq -5$
$\pi_{-1,-4}$	$\mathbb{Z}/2^{\lambda(q)+1}$
$\pi_{0,-3}$	$\mathbb{Z}/2$
$\pi_{1,-2}$	$(\mathbb{Z}/2)^2$
$\pi_{2,-1}$	$(\mathbb{Z}/2)^2$
$\pi_{3,0}$	$\mathbb{Z}/8$
$\pi_{4,1}$	0
$\pi_{5,2}$	$\mathbb{Z}/2$
$\pi_{6,3}$	$(\mathbb{Z}/2)^2$
$\pi_{7,4}$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/16$
$\pi_{8,5}$	$(\mathbb{Z}/4)^2$
$\pi_{9,6}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/4$
$\pi_{n+3,n}$	$\mathbb{Z}/4, n \geq 7$



New computations, $q \equiv 1 \pmod{4}$

stem	group
$\pi_{n+3,n}$	$0, n \leq -5$
$\pi_{-1,-4}$	$\mathbb{Z}/2^{\epsilon(q)+2}$
$\pi_{0,-3}$	$\mathbb{Z}/2$
$\pi_{1,-2}$	$(\mathbb{Z}/2)^2$
$\pi_{2,-1}$	$\begin{cases} (2, 4) & \epsilon(q) = 2 \\ (2, 8) & \epsilon(q) > 2 \end{cases}$
$\pi_{3,0}$	$\mathbb{Z}/8$
$\pi_{4,1}$	0
$\pi_{5,2}$	$\mathbb{Z}/2$
$\pi_{6,3}$	$\begin{cases} (2, 16) & \epsilon(q) > 3 \\ (2, 8) & \epsilon(q) = 3 \\ (2, 4) & \epsilon(q) = 2 \end{cases}$
$\pi_{7,4}$	$(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/16$
$\pi_{8,5}$	$(\mathbb{Z}/2)^4$
$\pi_{9,6}$	$(\mathbb{Z}/2)^3$
$\pi_{n+3,n}$	$(\mathbb{Z}/2)^2, n \geq 7$



Comparison to topology

\mathbb{F}_q topology

stem	group
$\pi_{n,0}$	0 , $n < 0$
$\pi_{0,0}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$
$\pi_{1,0}$	$(\mathbb{Z}/2)^2$
$\pi_{2,0}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/8$
$\pi_{3,0}$	$\mathbb{Z}/8$
$\pi_{4,0}$	0
$\pi_{5,0}$	$\mathbb{Z}/2$
$\pi_{6,0}$	$\mathbb{Z}/16 \oplus \mathbb{Z}/2$
$\pi_{7,0}$	$\mathbb{Z}/16 \oplus (\mathbb{Z}/2)^2$
$\pi_{8,0}$	$(\mathbb{Z}/2)^5$
$\pi_{9,0}$	$(\mathbb{Z}/2)^4$
$\pi_{10,0}$	$\mathbb{Z}/8 \oplus \mathbb{Z}/2$

stem	group
π_n	0 , $n < 0$
π_0	$\mathbb{Z}/2$
π_1	$\mathbb{Z}/2$
π_2	$\mathbb{Z}/2$
$\pi_3,$	$\mathbb{Z}/8$
π_4	0
π_5	0
π_6	$\mathbb{Z}/2$
π_7	$\mathbb{Z}/16$
π_8	$(\mathbb{Z}/2)^2$
π_9	$(\mathbb{Z}/2)^3$
π_{10}	$\mathbb{Z}/2$

Is $\pi_{n,0} = \pi_n \oplus \pi_{n+1}$ for all n ?