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NONSINGULAR AFFINE k^*-SURFACES

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ABSTRACT. Nonsingular affine k^*-surfaces are classified as certain invariant open subsets of projective k^*-surfaces. A graph is defined which is an equivariant isomorphism invariant of an affine k^*-surface. Over the complex numbers, it is proved that the only acyclic affine surface which admits an effective action of the group $\mathbf{C}^*$ is $\mathbf{C}^2$ which admits only linear actions of $\mathbf{C}^*$.

INTRODUCTION

A k^*-surface is a nonsingular two dimensional variety over an algebraically closed field k with an effective action of the algebraic group k^* of units of k.

In this article, we classify affine k^*-surfaces. If $k = \mathbf{C}$, the field of complex numbers, we prove the following.

Theorem A. *The only acyclic affine $\mathbf{C}^*$-surface is $\mathbf{C}^2$ with a linear action of $\mathbf{C}^*$.*

A fixed point x of a k^*-surface X is called elliptic, hyperbolic or parabolic depending on the linear action it induces on k^2, i.e. the representation on the tangent space T_xX. If X is an affine k^*-surface which possesses an elliptic fixed point, then X is equivariantly isomorphic to T_xX. This is an example of a one-fixed-pointed action. Fixed-pointed actions are characterized in [KR] and [BH]. We classify affine k^*-surfaces without elliptic fixed points as follows.

Theorem B. *The affine k^*-surfaces without elliptic fixed points are precisely the differences $V - Y$ where*

(1) *V is a projective k^*-surface without elliptic fixed points,*
(2) *Y is an invariant connected closed curve in V, and*
(3) *$V - Y$ contains no invariant closed curve of V.*

See (3.3) and (4.9) below. A pair (V, Y) satisfying (1)–(3) is called a G-pair.

In [OW], Orlik and Wagreich classify projective k^*-surfaces. These are obtained, by blowing up fixed points, from geometrically ruled k^*-surfaces. An affine k^*-surface without elliptic fixed points can be embedded equivariantly in a projective k^*-surface without elliptic fixed points. Orlik and Wagreich define a graph for these projective surfaces. This graph is utilized to determine which invariant open subsets are affine.

We also define a graph $\Gamma(X)$ for an affine k^*-surface without elliptic fixed points. This graph is shown to be an invariant of the affine k^*-surface X. In

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fact, $\Gamma(X)$ is completely determined by the tangent space representations at isolated (hyperbolic) fixed points, Seifert invariants of nontrivial closed orbits, and the quotient space X/k^*.

The paper is organized as follows.

In the first section we discuss generalities of algebraic group actions on varieties. Results of [KR] and [BH] are interpreted to give a characterization of actions of k^* on affine varieties admitting elliptic or parabolic fixed points. In §2, we describe part of the Orlik–Wagreich classification of projective k^*-surfaces. For the convenience of the reader, the graph Γ_V of a projective k^*-surface V without elliptic fixed points is defined and illustrated. We also prove that the quotient $\pi : V \to V/k^*$ exists for such V.

In §§3 and 4, G-pairs are discussed. The Orlik–Wagreich graph of V is modified to define a graph for G-pairs. The form of the graph of a minimal G-pair is determined in §3. This is used in §4 to characterize the affine k^*-surfaces without elliptic fixed points as differences $V - Y$, for G-pairs (V, Y).

The graph of $\Gamma(X)$ of an affine k^*-surface without elliptic fixed points is defined in §5. We prove that $\Gamma(X)$ is an invariant of X.

In §6 we restrict our attention to surfaces over the complex numbers. A projective $\mathbf{C}^*$-surface V without elliptic fixed points is diffeomorphic to the connected sum of n copies of $\overline{(\mathbf{CP}^2)}$ with the total space of a 2-sphere bundle over a compact 2-manifold. There is a canonical isomorphism between $H_2(V; \mathbf{Z})$ and $\operatorname{Num} V$, the divisors on V modulo numerical equivalence. This is employed to compute the homology of an affine k^*-surface from its graph. The section ends with a proof of Theorem A.

Since the time of this research, the author has become aware of work of K. Fieseler and L. Kaup on this subject. In [FK], the intersection homology of singular, as well as nonsingular, $\mathbf{C}^*$-surfaces is computed. Theorem A could also be deduced from these computations and results in §5 below.

1. Algebraic group actions on varieties

Throughout, k is an algebraically closed field of arbitrary characteristic. A *variety* is an integral separated scheme of finite type over k. The group of units in k is denoted k^*.

We begin by recalling some basic definitions pertinent to the study of algebraic transformation groups. The basic terminology follows that of [Hu]. Quotients of not necessarily affine varieties and 'fixed-pointed' actions of affine varieties are also discussed.

An (affine) *algebraic group* G is an affine variety which is also a group such that the group multiplication and inverse mappings are morphisms. An *action* of G on a variety X is a morphism

$$G \times X \to X , \quad (g, x) \mapsto g \cdot x$$

such that $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ and $1 \cdot x = x$, for all $g_1, g_2 \in G$ and $x \in X$. If $\operatorname{char} k = 0$, the *isotropy subgroup* of a point $x \in X$ is the algebraic subgroup $G_x = \{g \in G : g \cdot x = x\}$. The action is *effective* if $\bigcap_{x \in X} G_x = \{1\}$. If $\operatorname{char} k = p > 0$, the notion of isotropy subgroup is replaced by that of the *stabilizer subscheme* of x, see [MF, p. 3]. A variety with an effective action of G is called a *G-variety*. The subset $Gx = \{y \in X : g \cdot x = y$ for some $g \in G\}$ is called the *orbit* of x. The action induces a map $\sigma : G \to Gx$, $\sigma(g) = g \cdot x$

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with 'kernel' $G_x = \{g \in G : \sigma(g) = x\}$ and thus a bijection $G/G_x \to Gx$. The coset space G/G_x has the structure of an affine variety [Hu, IV]. This allows us to view the orbit Gx as an affine variety. The *fixed set* of the action is the closed subvariety $X^G = \{x \in X : g \cdot x = x \text{ for all } g \in G\}$ of X. A morphism $f : X \to Y$ is *equivariant* if $f(g \cdot x) = g \cdot f(x)$ for all $x \in X$ and $g \in G$.

A (rational) *representation* of an algebraic group G is a finite dimensional vector space V together with a homomorphism $G \to \mathrm{GL}(V)$ which is a morphism of varieties. An action on a vector space V is said to be *linear* if it is given by a representation.

An algebraic group G is *linearly reductive* if every representation of G is completely reducible. An example is the group k^*. Also, if $\operatorname{char} k = 0$, all finite groups are linearly reductive. If $\operatorname{char} k = p > 0$, a finite group H is algebraic if and only if $p \nmid |H|$. Every linearly reductive group is reductive, for the definition of reductive groups and an historical summary we refer the reader to [MF, Appendix I].

Example (1.1). *Finite dimensional representations of* $G = k^*$. Since G is linearly reductive every finite dimensional representation of G decomposes as the direct sum of irreducible representations. The irreducible representations of $G = k^*$ are the one dimensional representations $t \mapsto t^a$, for some integer a. Thus, up to base change, every representation of G on an n-dimensional vector space V has the form

$$G \to \mathrm{GL}(V), \quad t \mapsto \begin{pmatrix} t^{a_1} & 0 & \dots & 0 \\ 0 & t^{a_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & t^{a_n} \end{pmatrix}$$

for some integers $a_1, \dots, a_n$.

This representation will be denoted $V = t^{a_1} + \cdots + t^{a_n}$. An alternate decomposition of V is obtained as $V \cong V^0 \oplus V^+ \oplus V^-$, where $V^0 = \sum_{a_i=0} t^{a_i}$, $V^+ = \sum_{a_i>0} t^{a_i}$ and $V^- = \sum_{a_i<0} t^{a_i}$.

Representations occur naturally at fixed points. If X is an affine variety with an action of an algebraic group G, then there is an action on the ring of regular functions defined by $(g \cdot f)(x) = f(g^{-1} \cdot x)$ for $g \in G$ and $f \in \mathscr{O}(X)$. In fact, with this action, $\mathscr{O}(X)$ is an infinite dimensional representation of G [K, II.2.4]. If $x \in X^G$ and $f \in m_x$, the maximal ideal of x, then $g \cdot f \in m_x$, i.e. the action restricts to m_x. Thus, there is an induced action on (m_x/m_x^2), and on its dual space $(m_x/m_x^2)^*$ which is the tangent space T_xX to X at x. More generally, for arbitrary $x \in X$, this determines a representation of G_x on T_xX.

Definition (1.2). Let X be a variety with an action of $G = k^*$. Then for $x \in X^G$, x is *elliptic* if $T_xX \cong (T_xX)^+$ or $T_xX \cong (T_xX)^-$, *parabolic* if x is not elliptic and $T_xX \cong (T_xX)^0 \oplus (T_xX)^+$ or $T_xX \cong (T_xX)^0 \oplus (T_xX)^-$ and *hyperbolic* otherwise.

Example (1.3). *G-vector bundles.* A *G-vector bundle* is a vector bundle $p : X \to Y$ and an action of G on X which restricts to a linear action on each fiber.

Assume Y is connected and let $p: X \to Y$ be a vector bundle of rank n over Y. There is a covering $\{U_i\}_{i \in \mathscr{I}}$ of Y, isomorphisms $\phi_i : p^{-1}(U_i) \to U_i \times k^n$

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and transition functions $\psi_{i,j} : U_i \cap U_j \to \mathrm{GL}(k^n)$ so that $X \cong \bigcup_{\psi_{i,j}} (U_i \times k^n)$, i.e. for each $i, j \in \mathscr{I}$, $u \in U_i \cap U_j$ and $v \in k^n$, we have $(u, v) \sim (u, \psi_{i,j}(u)v)$.

Let $\theta : G \to \mathrm{GL}(k^n)$ be a representation. For $i \in \mathscr{I}$, define an action of G on $U_i \times k^n$ by $g \cdot (u, v) = (u, \theta(g)v)$ for all $g \in G$ and $(u, v) \in U_i \times k^n$. In order for this to define an action on X, for all $i, j \in \mathscr{I}$ we must have

$$(\dagger) \qquad (u, \theta(g)\psi_{i,j}(u)v) = (u, \psi_{i,j}(u)\theta(g)v)$$

for each $u \in U_i \cap U_j$, $g \in G$ and $v \in k^n$.

One special case which will be of interest is that of k^*-line bundles. If $n = 1$, condition ($\dagger$) is always satisfied. So the one dimensional representation of k^*, $t \mapsto t^a$, determines an action on a line bundle $p : X \to Y$. Moreover, viewing Y as the zero section of p and using (1.8) below, we see that for each $y \in Y$, $T_yX \cong t^0 + t^a$.

Definition (1.4). The (categorical) *quotient* of a variety X by an action of an algebraic group G is a variety Y together with a morphism $\pi : X \to Y$ satisfying:

(1) $\pi(g \cdot x) = \pi(x)$ for all $x \in X$ and $g \in G$.
(2) If $\psi : X \to Z$ is a morphism satisfying $\psi(g \cdot x) = \psi(x)$ for all $x \in X$ and $g \in G$, then there is a unique morphism $\phi : Y \to Z$, such that $\phi \circ \pi = \psi$.

Whenever it does exist, the universal property of (2) guarantees that the quotient $\pi : X \to Y$ is unique up to isomorphism. The quotient variety Y is denoted X/G. If X is affine, then $X/G = \operatorname{Spec} \mathscr{O}(X)^G$.

It is convenient to have a geometric description of the quotient. We begin with

Lemma (1.5). *Let X be a variety with an action of an algebraic group G. Suppose $\psi : X \to Z$ satisfies $\psi(g \cdot x) = \psi(x)$ for all $x \in X$ and $g \in G$.*

(1) *If there is a chain of orbits $Gx = Gx_1, \dots, Gx_n = Gy$ such that $\overline{Gx_i} \cap \overline{Gx_{i+1}} \neq \varnothing$ for $i = 1, \dots, n-1$ then $\psi(x) = \psi(y)$.*
(2) *If $\psi(x) = \psi(y)$ implies that such a chain exists, and if there is a section $s : Z \to X$, then $\psi : X \to Z$ is the quotient.*

Proof. Since ψ is continuous, for each $z \in Z$, $\psi^{-1}(z)$ is closed in X. In particular the closure of each orbit maps to a single point.

To prove (2), let $s : Z \to X$ be a section. We need to verify that $\psi : X \to Z$ satisfies the universal property (1.4)(2) above. Suppose $\phi : X \to W$ is a morphism such that $\phi(g \cdot x) = \phi(x)$ for each $g \in G$ and $x \in X$. Define $f : Z \to W$ to be the composition $\phi \circ s$. For $x \in X$, let $\bar{x} = \psi(x)$ and $y = s(\bar{x})$. Then $\psi(x) = \psi(y)$. So there is a chain of orbits $Gx = Gx_1, \dots, Gx_n = Gy$ such that $\overline{Gx_i} \cap \overline{Gx_{i+1}} \neq \varnothing$ for $i = 1, \dots, n-1$. Thus $f \circ \psi(x) = f(\bar{x}) = \phi \circ s(\bar{x}) = \phi(y) = \phi(x)$ and f is unique with this property. $\square$

If X is an affine variety and G is reductive, the quotient $\pi : X \to X/G$ does exist [MF, p. 27]. In this case, for each $z \in X/G$, $\pi^{-1}(z)$ contains a unique closed orbit, and $\pi(x) = \pi(y)$ if and only if $\overline{Gx} \cap \overline{Gy} \neq \varnothing$ [BH, §1].

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Lemma (1.6). *Let X be an affine variety with an action of a reductive group G. If Z is a nonsingular curve and $\psi : X \to Z$ is a surjective morphism such that $\psi(x) = \psi(y)$ if and only if $\overline{Gx} \cap \overline{Gy} \neq \varnothing$, then $\psi : X \to Z$ is the quotient.*

Proof. Let $\pi : X \to X/G$ be the quotient. Then there is a unique morphism $\phi : X/G \to Z$ such that $\phi \circ \pi = \psi$. First we show that ϕ is bijective. Surjectivity is obvious. Suppose $\phi(\bar{x}) = \phi(\bar{y})$. Let $x \in \pi^{-1}(\bar{x})$ and $y \in \pi^{-1}(\bar{y})$. Then $\psi(x) = \phi \circ \pi(x) = \phi(\bar{x}) = \phi(\bar{y}) = \phi \circ \pi(y) = \psi(y)$, so $\overline{Gx} \cap \overline{Gy} \neq \varnothing$. But this means $\bar{x} = \pi(x) = \pi(y) = \bar{y}$.

Let Y be a projective curve containing X/G such that each point of $Y - X/G$ is regular. Also let W be the unique nonsingular projective curve containing Z. Since ϕ is a bijective morphism of the affine curves X/G and Z, ϕ defines a birational map $\phi' : Y \to W$. Note ϕ' is defined at all but at most finitely many points of Y, and each point at which ϕ' is not defined is a regular point of Y. Since W is nonsingular and projective there is a unique extension of ϕ' to a morphism $\overline{\phi} : Y \to W$, [H1, p. 43]. Since W is nonsingular, the birational morphism $\overline{\phi}$ is an isomorphism. □

Throughout the remainder of this section $G = k^*$ and X is a reduced affine G-variety. If G acts fixed-pointedly on X (see below), then X is a G-vector bundle over the fixed set X^G. This result for algebraic tori is due to Kambayashi and Russell [KR], see also Bass and Haboush [BH]. Our terminology follows that of [BH].

Definition (1.7). Suppose X is a reduced affine variety. An action of a reductive group G on X is *fixed-pointed* if the closure of each orbit contains a fixed point. An action is *one-fixed-pointed* if it is fixed-pointed and there is exactly one fixed point.

Remark. Fixed-pointedness is equivalent to the condition that the composition $X^G \hookrightarrow X \xrightarrow{\pi} X/G$ be an isomorphism, see [BH, 10.0].

From (1.1) we see that a (finite dimensional) representation V is fixed-pointed if and only if V contains an elliptic or parabolic fixed point, and V is one-fixed-pointed if and only if V contains an elliptic fixed point. The following is a useful lemma relating fixed-pointedness of tangent space representations to fixed-pointedness of X. We note that it is valid over fields of arbitrary characteristic so long as the group in question is linearly reductive; this is the case for k^*.

Lemma (1.8) [Lu2, Lemme 1]. *Let X be an affine variety with an action of a linearly reductive group G. Suppose $x \in X$ is a regular point which is fixed by the action. Then there is an equivariant morphism $F : X \to T_xX$ such that $F(x) = 0$ and F is étale.*

A morphism is étale if it is smooth of relative dimension zero [H1, p. 275]. So a morphism as in (1.8) maps an open invariant neighborhood of x onto an open invariant neighborhood of the origin in T_xX, see [H1, III. Exercise 9.1]. The next proposition follows easily from results of Bass and Haboush [BH].

Proposition (1.9). *Let X be a nonsingular affine variety with an action of $G = k^*$.*

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(1) *If X contains an elliptic fixed point x, then X is equivariantly isomorphic to the tangent space representation T_xX and X/G is a point.*
(2) *If X contains a parabolic fixed point then $\pi : X \to X/G$ is a G-vector bundle and X/G is nonsingular.*

Proof. For $x \in X^G$, consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & T_xX \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_x} \\ X/G & \xrightarrow{F/G} & (T_xX)/G \end{array}$$

where F is the morphism of (1.8) and F/G is the induced morphism on quotients.

If x is elliptic, we show that X is one-fixed-pointed. From [BH, 10.6] it will then follow that $X \cong T_xX$. Let U be a neighborhood of x such that $x \in \overline{Gy}$ for $y \in U$. Since F is étale, $V = F(U)$ is an open neighborhood of 0 in T_xX. By (1.5), $\pi(U) = \pi(x)$, so $\pi^{-1}(\pi(x))$ is dense in X. Since π is continuous, $\pi^{-1}(\pi(x))$ is also closed in X. Thus $\pi^{-1}(\pi(x)) = X$. But this means $x \hookrightarrow X \to \pi(x) = X/G$ is an isomorphism. So $X/G \cong \{x\}$ and X is one-fixed-pointed.

Now suppose x is parabolic. Then $T_xX \cong (T_xX)^0 + (T_xX)^\varepsilon$ for $\varepsilon = +$ or $-$, see (1.2). Thus T_xX is fixed-pointed. Since $\mathscr{O}(X)$ is an integral domain, [BH, 11.3] implies X is fixed-pointed. Thus $\pi : X \to X/G$ is a G-vector bundle [BH, 10.3].

We now show that X/G is nonsingular. Let $\bar{x} \in X/G$ and let x be the unique fixed point in $\pi^{-1}(\bar{x})$. Since F is finite and étale at x, F/G is étale at $\bar{x}$ [Lu1, II.1.1]. So it suffices to show that $(T_xX)/G$ is nonsingular at $F/G(\bar{x})$. But, since the action on X is fixed-pointed, so is the action on T_xX. Thus $(T_xX)/G \cong (T_xX)^G$. The later is a linear subspace of T_xX and hence nonsingular. □

The hyperbolic case remains to be studied. From (1.9) we see that any nonsingular affine k^*-variety which contains a hyperbolic fixed point cannot contain an elliptic or parabolic fixed point.

2. Projective surfaces with k^*-actions

A *G-surface* is a nonsingular two dimensional variety with an effective action of an algebraic group G. Throughout $G = k^*$, the multiplicative group of units of an arbitrary fixed algebraically closed field k.

From (1.9) it follows that if X is an affine G-surface which contains an elliptic fixed point x, then $X \cong T_xX \cong k^2$. If X contains no elliptic fixed point, then we will embed X equivariantly into a projective G-surface without elliptic fixed points, see (3.1) below. In this section we describe projective k^*-surfaces without elliptic fixed points. This is part of a broader classification due to P. Orlik and P. Wagreich, see [OW].

If V is a projective G-surface, then V^G is nonempty. Following [OW], we describe the fixed set. For $x \in V^G$, write $T_xV = t^{a(x)} + t^{b(x)}$. Since the action

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is effective, not both $a(x)$ and $b(x)$ are zero. Set

$$F_+(V) = \{x \in V^G : a(x),\ b(x) \geq 0\},$$
$$F_-(V) = \{x \in V^G : a(x),\ b(x) \leq 0\}, \quad \text{and}$$
$$F_h(V) = \{x \in V^G : a(x) > 0 > b(x) \text{ or } a(x) < 0 < b(x)\}.$$

When no confusion will arise we drop the V from the notation and write F_+, F_- and F_h. The points of F_+ are called *sources* and those of F_- *sinks* of the action. Sources and sinks are defined similarly on G-varieties of arbitrary dimension. According to [B2], F_+ and F_- are nonempty distinct irreducible components of V^G. So, for $\varepsilon = +, -$, either F_ε is a point, in which case it is elliptic, or F_ε is an irreducible curve, in which case it consists of parabolic fixed points. Thus, $\dim F_+ = \dim F_- = 1$ if and only if V contains no elliptic fixed point. An orbit O of the G-action is called *ordinary* if $\overline{O} \cap F_+ \neq \varnothing$ and $\overline{O} \cap F_- \neq \varnothing$. If O is an orbit which is not a fixed point and not ordinary, O is called *special.*

We begin with an example. If $p\colon S \to C$ is a geometrically ruled surface which admits two nonintersecting sections,[1] then S admits an action of k^*. The assumption that $p : S \to C$ admits two nonintersecting sections is equivalent to $S = \mathbf{P}(\xi)$ where ξ is a decomposible rank 2 vector bundle over C [H1, V. Exercise 2.2].

Example (2.1). *Standard actions of k^* on geometrically ruled surfaces:* Let $p : \mathscr{L} \to C$ be a line bundle over a nonsingular projective curve C, and let $\mathscr{L}^{-1}$ denote the inverse bundle (i.e. the bundle whose transition functions are the inverses of those of $\mathscr{L}$). As in (1.3), $\mathscr{L}$ and $\mathscr{L}^{-1}$ admit actions of $G = k^*$ determined by the representations t^1 and t^{-1} respectively. In each case the fixed set is the zero section. Write $\mathscr{L}^G = C_0$ and $(\mathscr{L}^{-1})^G = C_\infty$.

Let $f : \mathscr{L} - C_0 \to \mathscr{L}^{-1} - C_\infty$ be the map defined locally by $(u, x) \in U \times k \mapsto (u, x^{-1}) \in U \times k$. It is straightforward to check that f is an equivariant isomorphism. Form $S = \mathscr{L} \cup_f \mathscr{L}^{-1}$, the glueing of $\mathscr{L}$ and $\mathscr{L}^{-1}$ along f. Then $p : \mathscr{L} \to C$ extends to $\bar{p} : S \to C$ and $\bar{p}^{-1}(c) \cong \mathbf{P}^1$ for each $c \in C$. Thus S is a geometrically ruled surface with an action of G.

The fixed set is $S^G \cong C_0 \cup C_\infty$. From (1.3) we see that, for $x \in C_0$, resp. C_∞, $T_xS \cong t^0 + t^1$, resp. $t^0 + t^{-1}$. So $F_+(S) = C_0$ and $F_-(S) = C_\infty$. Also, $F_+(S)^2 = -(F_-(S))^2 = d$ for some $d \in \mathbf{Z}$. Incidentally, d, resp. $-d$, is the degree of $\mathscr{L}$, resp. $\mathscr{L}^{-1}$, as a line bundle. An orbit O of S which is not a fixed point is an orbit of each of $\mathscr{L}$ and $\mathscr{L}^{-1}$, so $\overline{O} \cap F_+(S) \neq \varnothing$ and $\overline{O} \cap F_-(S) \neq \varnothing$. Thus each (nonfixed) orbit of S is ordinary. The map $\bar{p} : S \to C$ satisfies $\bar{p}(x) = \bar{p}(y)$ if and only if there is a z such that $x, y \in \overline{Gz}$. Since there is an obvious section, (1.6) implies $\bar{p} : S \to C$ is the quotient.

This will be called the *standard action* of G on S.

Suppose x is a fixed point of a G-surface V, and let $\phi : \widetilde{V} \to V$ be the blow up at x. Then there is a unique action of G on $\widetilde{V}$ extending the action on $\widetilde{V} - \phi^{-1}(x) \cong V - x$, see [OW, 3.4] for an explicit description. In this case we call $\phi : \widetilde{V} \to V$ the *equivariant blow up* of V at x.

[1] In [OW], $p : S \to C$ is called a $\mathbf{P}^1$ bundle over C.

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If $E \cong \mathbf{P}^1$ is a curve in a G-surface $\widetilde{V}$ with $E \cdot E = -1$, then E is invariant [OW, 1.9]. Thus E is the exceptional curve of an equivariant blow up $\phi : \widetilde{V} \to V$. The process of passing from $\widetilde{V}$ to V is called *blowing down.*

Lemma (2.2). *Let V be a projective G-surface without elliptic fixed points. Suppose*

(1) *the quotient $\pi : V \to V/G$ exists,*
(2) *there is a section $s : V/G \to V$ which maps V/G isomorphically onto $F_+(V)$, and*
(3) *whenever $\pi(x) = \pi(y)$, there is a chain $Gx = Gx_1, \dots, Gx_n = Gy$ of orbits such that $\overline{Gx_i} \cap \overline{Gx_{i+1}} \neq \varnothing$.*

If $\phi : \widetilde{V} \to V$ is the blow up of some $P \in V^G$ then the quotient $\tilde{\pi} : \widetilde{V} \to \widetilde{V}/G$ exists, $V/G = \widetilde{V}/G$ and $\tilde{\pi} = \pi \circ \phi$. Moreover, (2) *and* (3) *are also satisfied for $\tilde{\pi} : \widetilde{V} \to \widetilde{V}/G$.*

Proof. Set $\tilde{\pi} = \pi \circ \phi$. Then there is a section $\tilde{s} : V/G \to \widetilde{V}$ of $\tilde{\pi}$ given by the composition of s with the proper transform $\widetilde{F_+(V)}$ of $F_+(V)$ in $\widetilde{V}$. Since $F_+(V)$ is nonsingular there is such a section, and, since ϕ is equivariant, $\widetilde{F_+(V)} \cong F_+(\widetilde{V})$.

Now suppose $\tilde{\pi}(x) = \tilde{\pi}(y)$. We show there is a chain $Gx = Gx_1, \dots, Gx_n = Gy$ of orbits such that $\overline{Gx_i} \cap \overline{Gx_{i+1}} \neq \varnothing$. Then applying (1.5) we see that $\tilde{\pi} = \pi \circ \phi : \widetilde{V} \to V/G$ is the quotient. Let E be the exceptional curve of ϕ. Set $z = \phi(E)$, $x' = \phi(x)$ and $y' = \phi(y)$.

Case 1. $x' = y'$. In this case, x and y are both in E. Let $x_2 \in E$ be a nonfixed point. Then $Gx = Gx_1, Gx_2, Gx_3 = Gy$ is a chain satisfying $\overline{Gx_i} \cap \overline{Gx_{i+1}} \neq \varnothing$.

Case 2. $x' \neq y'$. Let $Gx' = Gx'_1, \dots, Gx'_n = Gy'$ be a chain of orbits of V satisfying $\overline{Gx_i} \cap \overline{Gx_{i+1}} \neq \varnothing$. If some Gx'_i is a fixed point, then $Gx'_i \in \overline{Gx'_{i-1}} \cap \overline{Gx'_{i+1}}$, and the chain obtained by deleting Gx'_i also satistfies the intersection property. So assume no Gx'_i is a fixed point. Set $x_i = \phi^{-1}(x'_i)$. If $z \notin \bigcup_{i=1}^n \overline{Gx'_i}$, then $Gx = Gx_1, \dots, Gx_n = Gy$ is the required chain. If $z \in \bigcup_{i=1}^n \overline{Gx'_i}$, then $z = \overline{Gx_i} \cap \overline{Gx_{i+1}}$ for some i. Setting $O = E - E^G$, we see that the required chain is

$$Gx = Gx_1, \dots, Gx_i, O, Gx_{i+1}, \dots, Gx_n = Gy. \quad \square$$

Definition (2.3) [OW, 2.4]. Let V be a projective G-surface without elliptic fixed points so that $\dim F_+ = \dim F_- = 1$. The weighted graph $\Gamma(V)$ is defined as follows:

(1) Vertices of $\Gamma(V)$ are v_+, v_- for the curves F_+, F_- and one vertex for each special orbit of V.
(2) Two vertices v and w are linked in $\Gamma(V)$ if the closures of the corresponding curves intersect.
(3) Each vertex is assigned a weight which is the self-intersection number of the closure of the corresponding curve.

Let V be a projective G-surface without elliptic fixed points. By convention, the vertex v_+ corresponding to F_+ will be to the left of the vertex v_- corresponding to F_-. Suppose V contains special orbits $O_1, \dots, O_S$ such that

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$\overline{O}_i \cap \overline{O}_{i+1} \neq \varnothing$ for $i = 1, \dots, S-1$ and $\overline{O}_i \cap \overline{O}_j = \varnothing$ for $j \neq i-1, i, i+1$. Then there is a subgraph $\mathscr{B}$ of $\Gamma(V)$ which has S vertices $v_1, \dots, v_S$ such that (v_i, v_{i+1}) is an edge of $\mathscr{B}$ for $i = 1, \dots, S-1$. If in addition (v_+, v_1) and (v_-, v_S) are edges of $\Gamma(V)$, then we call $\mathscr{B}$ a *branch* of $\Gamma(V)$. The orbits $O_1, \dots, O_S$ are called the *orbits of* $\mathscr{B}$.

Example (2.4). (a) Let $p: S \to C$ be a geometrically ruled surface with a standard action of G. Suppose $(F_+)^2 = d$. From (2.1) it is evident that $\Gamma(S)$ has the form:

$$\overset{d}{\bullet} \qquad \overset{-d}{\bullet}$$

Now suppose O is an ordinary orbit of S. Let $x \in F_+ \cap \overline{O}$ and let $\phi: V \to S$ be the blow up of S at x. Set $E = \phi^{-1}(x)$. Then V is a projective G-surface with exactly two special orbits, $O_1 = E - E^G$ and $O_2 = \phi^{-1}(O)$. Since $\overline{O}$ is a fiber of p, $(\overline{O})^2 = 0$. Thus $\Gamma(V)$ has the form

$$\overset{d-1}{\bullet}\text{———}\overset{-1}{\bullet}\text{———}\overset{-1}{\bullet}\text{———}\overset{-d}{\bullet}$$

Also by (2.1) and (2.2) the quotient $\pi: V \to V/G$ exists and $\pi = p \circ \phi$.

(b) Let Z be a projective G-surface without elliptic fixed points and suppose $\Gamma(Z)$ has a branch $\mathscr{B}$ of the form

$$\overset{I_1}{\bullet}\text{———}\overset{I_2}{\bullet}\ \cdots\ \overset{I_{N-1}}{\bullet}\text{———}\overset{-1}{\bullet}\text{———}\overset{I_{N+1}}{\bullet}\ \cdots\ \overset{I_{S-1}}{\bullet}\text{———}\overset{I_S}{\bullet}$$

Let $O_1, \dots, O_S$ be the orbits of $\mathscr{B}$. Then $(\overline{O}_N)^2 = -1$, so $\overline{O}_N$ is the exceptional curve of a blow up $\phi: Z \to Z'$. Then the corresponding branch of Z' has the form

$$\overset{I_1}{\bullet}\text{———}\overset{I_2}{\bullet}\ \cdots\ \overset{I_{N-2}}{\bullet}\text{———}\overset{I_{N-1}+1}{\bullet}\text{———}\overset{I_{N+1}+1}{\bullet}\text{———}\overset{I_{N+2}}{\bullet}\ \cdots\ \overset{I_{S-1}}{\bullet}\text{———}\overset{I_S}{\bullet}$$

(c) Let $p: S \to \mathbf{P}^1$ be a geometrically ruled surface over the projective line. Suppose $(F_+)^2 = 1$. Then $(F_-)^2 = -1$ and F_- is the exceptional curve of a blow up $\phi: S \to S'$. It is easy to see that $S' = \mathbf{P}^2$. Also, using (1.8), we see that the isolated fixed point $x = \phi(F_-)$ is an elliptic sink of the action.

Suppose C is a nonsingular curve. We say V is a *ruled surface* over C if V is birationally equivalent to $C \times \mathbf{P}^1$. Unless $V \cong \mathbf{P}^2$, this means there is a geometrically ruled surface $p: S \to C$ and a morphism $\phi: V \to S$ such that ϕ is a finite composition of blow ups. In this case, the composition $\psi = p \circ \phi$: $V \to C$ is called a *ruling* of V. For a given ruling $\psi: V \to C$, the *fiber* over a point $x \in C$, denoted ψ_x, is the preimage $\psi^{-1}(x)$. Evidently, each fiber is a connected curve whose irreducible components are each isomorphic to $\mathbf{P}^1$.

Theorem (2.5) [OW, 2.5] (Orlik–Wagreich). *Let V be a projective G-surface without elliptic fixed points. Then there is a geometrically ruled surface $p: S \to F_+(V)$ with a standard action of G and a finite sequence of blow ups at fixed points $\phi: V \to S$, so that $\psi = p \circ \phi: V \to F_+$ is a ruling of V. Moreover, if $\psi_x = \bigcup_{j=1}^{S_x} C_j$ is the decomposition of the fiber over $x \in F_+(V)$ into irreducible components, then we have the following.*

(1) $S_x = 1$ *if and only if ψ_x is the closure of an ordinary orbit of V.*

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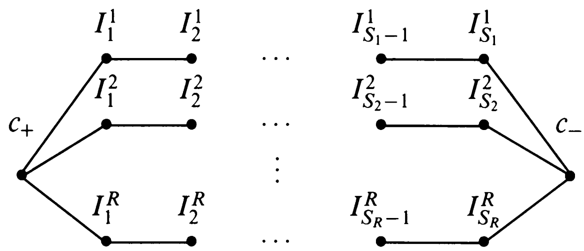
(2) *If* $S_x \geq 2$, *then* $C_1, \ldots, C_{S_x}$ *are the closures of the special orbits of a branch of* $\Gamma(V)$.

(3) *The graph* $\Gamma(V)$ *has the form*



for some integers c_+, c_- *and* I_j^i *satisfying* $I_j^i < 0$ *for all* i *and* j, *and for each* i *there is a* j *with* $I_j^i = -1$.

Corollary (2.6). *Let* V *be a projective* G*-surface without elliptic fixed points. Then the quotient* $\pi: V \to V/G$ *exists. Moreover,* $\pi = p \circ \phi$ *where* $p: S \to F_+(V)$ *is a geometrically ruled surface with a standard action of* G, *and* $\phi: V \to S$ *is a sequence of blow ups at fixed points.*

Proof. This follows from (2.5) and (2.1) upon repeated applications of (2.2). □

Corollary (2.7). *Suppose* V *is a projective* G*-surface without elliptic fixed points and* O *is an orbit of* V, *then* $\overline{O} \cdot \overline{O} = 0$ *if and only if* O *is an ordinary orbit.*

Proof. This is immediate from (2.5)(1) and (2.5)(3). □

3. G-PAIRS

Throughout $G = k^*$. In view of the classification of projective G-surfaces, we study affine G-surfaces via embeddings into projectives, see (3.1) below. Then G-pairs are defined and it is proved that every affine G-surface without elliptic fixed points is $V - Y$ for some (minimal) G-pair (V, Y). The graphs of G-pairs are defined, and the form of the graph of a 'minimal' G-pair is determined.

Proposition (3.1). *If* X *is a nonsingular affine* G*-surface without elliptic fixed points, then there is a projective* G*-surface* V *without elliptic fixed points and an invariant subvariety* $Y \subset V$, *such that* $X \cong V - Y$ *as* G*-surfaces.*

Proof. By the equivariant compactification theorem of Sumihiro [S], one can embed X equivariantly in a complete two dimensional G-variety V_0. Identifying X with its image in V_0, set $Y_0 = V_0 - X$ and $\Sigma_0 = \text{sing}(V_0) \cup \{x \in V_0^G : x \text{ is elliptic}\}$. Note that $\Sigma_0 \subset Y_0$ since X is nonsingular and contains no elliptic fixed point.

The canonical equivariant resolution of V_0 [OW, 3.2] is a complete nonsingular G-surface V without elliptic fixed points and an equivariant morphism $\pi : V \to V_0$ satisfying $\pi|_{V-\pi^{-1}(\Sigma_0)} : V - \pi^{-1}(\Sigma_0) \to V_0 - \Sigma_0$ is an isomorphism. Since V is complete and nonsingular, V is projective. Set $Y = \pi^{-1}(Y_0)$. Then Y is invariant and $X = V_0 - Y_0 \cong V - Y$. □

Definition (3.2). A pair (V, Y) is called a *G-pair* if

(1) V is a nonsingular projective G-surface without elliptic fixed points.

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(2) Y is an invariant connected closed curve of V.
(3) Every invariant closed curve of V meets Y.

Remarks. Since Y is invariant and each component has dimension one, Y consists of some union of fixed curves and closures of ordinary and special orbits of V. Also, since the closure of each ordinary orbit of V meets F_+ and F_-, Y must contain at least one of F_+ or F_-.

Let (V, Y) be a G-pair and suppose E is an exceptional curve of V which is contained in Y. Then E is the exceptional curve of an equivariant blow up $\phi : V \to V'$. Set $Y' = \phi(Y)$. The tangential representation at the fixed point $\phi(E)$ determines the action on E according to [OW, 3.4], this is independent of the type (i.e. elliptic, parabolic or hyperbolic) of the fixed point $\phi(E)$. From this it follows that $\phi(E)$ is an elliptic fixed point if and only if E is pointwise fixed. Also $V - Y = V' - Y'$, so (V, Y) satisfies (3.2)(3) if and only if (V', Y') does. Thus (V', Y') is a G-pair if and only if E is not pointwise fixed. We say (V, Y) is *minimal* if Y contains no exceptional curve E, unless E is pointwise fixed.

Proposition (3.3). *Let X be an affine G-surface without elliptic fixed points. There is a minimal G-pair (V, Y) such that $X \cong V - Y$ as G-surfaces.*

Proof. Let V and Y be as in (3.1) so that $X \cong V - Y$. Since X is affine, Y is connected and has pure codimension one, see [H2, II.3.1] and [H2, II.6.2]. Also, X cannot contain a closed curve of V. Thus (V, Y) is a G-pair. If (V, Y) is minimal, we are done. Otherwise there is an exceptional curve $E_1 \subset Y$ which is not fixed. Let $\phi_1 : V \to V_1$ be the blow down and set $Y_1 = \phi_1(Y)$. Note that (V_1, Y_1) is a G-pair and $X \cong V - Y \cong V_1 - Y_1$. Since E_1 is not fixed and $E_1 \cdot E_1 = -1$, (2.7) implies E_1 is the closure of a special orbit. Thus V_1 has one fewer special orbit than V.

Repeating this process, one obtains a sequence $\{(V_i, Y_i)\}$ of G-pairs such that $X \cong V_i - Y_i$ and V_{i+1} has one fewer special orbit than V_i. But V has only finitely many special orbits by (2.5). Thus (V_n, Y_n) is minimal for some n. $\square$

Definition (3.4). For any G-pair (V, Y) the graph $\Gamma(V, Y)$ is defined as follows:

(1) The vertices of $\Gamma(V, Y)$ are the vertices of $\Gamma(V)$, see (2.3), together with one vertex for each ordinary orbit of V which is contained in Y. Vertices whose corresponding curves are contained in Y are indicated with an open dot ($\circ$). All others are indicated with a closed dot ($\bullet$).
(2) Two vertices are linked by an edge in $\Gamma(V, Y)$ if the closures of the corresponding curves intersect.
(3) Each vertex is assigned a weight which is the self-intersection number of the closure of the corresponding curve.

Suppose (V, Y) is a G-pair. A vertex of $\Gamma(V, Y)$ or $\Gamma(V)$ will be called *fixed* (resp. *ordinary, special*) if the corresponding curve is fixed (resp. ordinary, special). Note that there is a one to one correspondence between the special vertices of $\Gamma(V)$ and the special vertices of $\Gamma(V, Y)$. A *branch* of $\Gamma(V, Y)$ is a subgraph which corresponds to a branch of $\Gamma(V)$. If $\mathscr{B}$ is a branch of $\Gamma(V, Y)$, we write $\mathscr{B}(V)$ for the corresponding branch of $\Gamma(V)$. The *orbits*

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of a branch $\mathscr{B}$ of $\Gamma(V, Y)$ are the orbits of $\mathscr{B}(V)$. The *length* of a branch $\mathscr{B}$ is the number of orbits (or equivalently the number of vertices) of $\mathscr{B}$.

If (V, Y) is a minimal G-pair, the forms of the branches of $\Gamma(V, Y)$ are determined in (3.6) below. First we need the following.

Lemma (3.5). *Let V be a projective G-surface without elliptic fixed points. Suppose $\mathscr{B}$ is a branch of $\Gamma(V)$ of the form*

Then the following hold.

(1) *$\mathscr{B}$ contains two adjacent vertices of weight -1 if and only if $S = 2$.*

(2) *If $I_j = -1$ for $j = 1$ or S, then $I_l = -1$ for some $l \neq j$.*

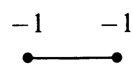
Proof. Let $O_1, \ldots, O_S$ be the orbits of $\mathscr{B}$, so that $(\overline{O}_j)^2 = I_j$. To prove (1), first suppose $S = 2$. Then by (2.5)(3), at least one of $(\overline{O}_1)^2 = -1$ or $(\overline{O}_2)^2 = -1$. Assume without loss of generality that $(\overline{O}_1)^2 = -1$, and let $\phi : V \to V'$ be the blowing down of $\overline{O}_1$. Then $\phi(O_2)$ is an ordinary orbit of V'. Thus $(\overline{\phi(O_2)})^2 = 0$ and $(\overline{O}_1)^2 = (\overline{O}_2)^2 = -1$. To prove the converse, suppose $(\overline{O}_k)^2 = (\overline{O}_{k+1})^2 = -1$ and let $\phi : V \to V'$ be the blowing down of $\overline{O}_k$. Then $(\overline{\phi(O_{k+1})})^2 = 0$. By (2.7) this means $\phi(O_{k+1})$ is an ordinary orbit of V'. So $S = 2$.

The second assertion is proved by induction on S. If $S = 2$, the statement is proved. Suppose $S \geq 3$ and $I_1 = -1$. The proof in the case $I_S = -1$ is similar. Blowing down $\overline{O}_1$ we get a projective G-surface V' whose corresponding branch $\mathscr{B}'$ has the form

Since at least one of the weights of $\mathscr{B}'$ is -1, the induction hypothesis implies $I_j = -1$ for some $j \geq 3$. □

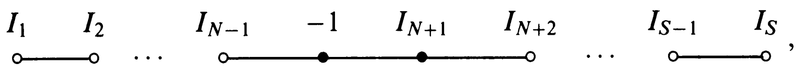
Lemma (3.6). *Let (V, Y) be a minimal G-pair. If $\mathscr{B}$ is a branch of $\Gamma(V, Y)$ of length $S = 2$ then $\mathscr{B}$ has the form*

(1)

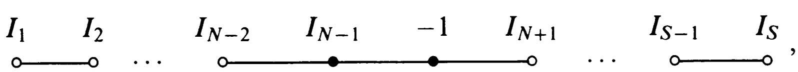


If $\mathscr{B}$ is a branch of $\Gamma(V, Y)$ of length $S \geq 3$ then $\mathscr{B}$ is one of the following:

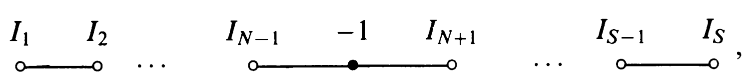
(2)



(3)



(4)



where in (2), (3) *and* (4), *$I_j \leq -2$ for $j \neq N$ and $1 < N < S$.*

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Proof. If the length of $\mathscr{B}$ is two, then by (3.5)(1) the weights are both -1. Minimality implies the the corresponding curves are not contained in Y.

Suppose the length of $\mathscr{B}$ is $S \geq 3$. Let $O_1, \ldots, O_S$ be the orbits of $\mathscr{B}$ and let

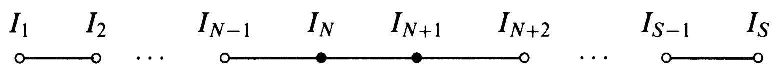
be the corresponding branch of $\Gamma(V)$, so that $(\overline{O}_j)^2 = I_j$.

Since Y is connected and Y contains at least one of F_+ or F_-, if $O_{i_1}, \ldots, O_{i_l}$ are the orbits which are not contained in Y, then $\{O_{i_1}, \ldots, O_{i_l}\} = \{O_k, \ldots, O_{k+l}\}$ for some $k = 1, \ldots, S-l$. Also, since the closure of each orbit meets Y, Y contains all but at most two adjacent orbits O_k and O_{k+1} of $\mathscr{B}$. Let N be such that $I_N = -1$, this is possible by (2.5)(3). Minimality of (V, Y) implies $O_N \not\subset Y$. Now, if Y contains all but exactly one orbit of $\mathscr{B}$, then $\mathscr{B}$ has the form (4). If neither O_k nor O_{k+1} is contained in Y, then either $k = N$ and $\mathscr{B}$ has form (2) or $k = N-1$ and $\mathscr{B}$ has form (3).

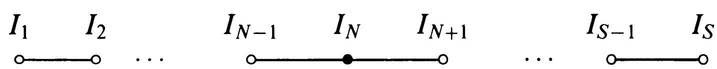
To show that $1 < N < S$, suppose to the contrary that $I_j = -1$ for $j = 1$ or S. Then, by (3.5)(2), $I_l = -1$ for some $l \neq j$. Minimality of (V, Y) implies neither O_j nor O_l is contained in Y. But then O_j and O_l are adjacent orbits with $(\overline{O}_j)^2 = (\overline{O}_l)^2 = -1$. By (3.5)(1) this is only possible if $S = 2$. □

If the closures of orbits intersect, they intersect in a fixed point. Thus the branches of (3.6)(1), (2) and (3) correspond to a fixed point of $V - Y$. On the other hand, the Nth orbit of the branch (3.6)(4) is closed in $V - Y$. This motivates

Definition (3.7). Let (V, Y) be a G-pair and let $\mathscr{B}$ be a branch of $\Gamma(V, Y)$. We say $\mathscr{B}$ is of *type* $\mathscr{F}$ if $\mathscr{B}$ has the form

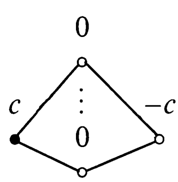


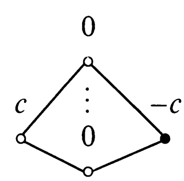
where $1 \leq N < S$. We say $\mathscr{B}$ is of *type* $\mathscr{C}$ if $\mathscr{B}$ has the form:



where $1 < N < S$.

Proposition (3.8). *Let (V, Y) be a G-pair. Set $X = V - Y$. If* $\dim X^G = 1$, *then X is affine, X/G is a nonsingular affine curve isomorphic to X^G, and $\pi : X \to X/G$ has the structure of a G-line bundle. Moreover, V/G is the unique nonsingular projective curve containing X/G. And, if (V, Y) is minimal, then $\Gamma(V, Y)$ is*

(1) 

or (2) 

where c is some integer, and, in each case, the number of ordinary vertices is $|V/G - X/G|$.

Proof. As in the proof of (3.3), after blowing down exceptional curves of V which are contained in Y and are not fixed, we may assume (V, Y) is minimal.

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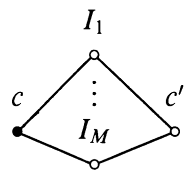
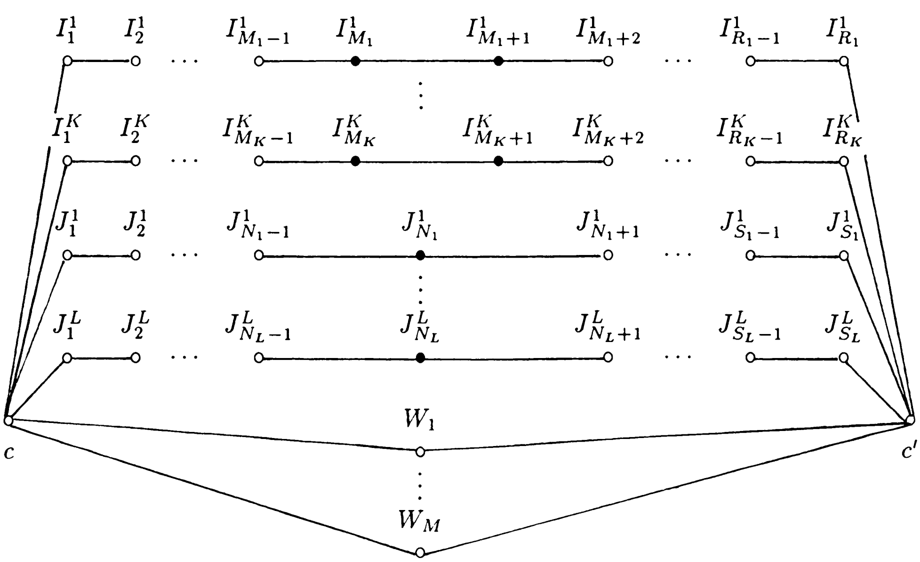
As remarked after (3.2), Y contains at least one of F_+ or F_-. Since $\dim X^G = 1$, Y contains exactly one of the fixed curves F_+ or F_-. We will show that if $F_- \subset Y$ and $F_+ \not\subset Y$, then $\Gamma(V, Y)$ has form (1). A similar proof shows that, if $F_- \not\subset Y$ and $F_+ \subset Y$, then $\Gamma(V, Y)$ has form (2). Connectedness of Y implies that $\Gamma(V, Y)$ cannot contain a branch of the form (3.6)(2) for $1 < N < S$, (3.6)(4) for $1 < N < S$, or of the form (3.6)(3) for $2 < N < S$. Since the closure of each orbit must meet Y, $\Gamma(V, Y)$ cannot contain a branch of the form (3.6)(1), or of the form (3.6)(3) for $N = 2$. Thus $\Gamma(V, Y)$ contains no branches. But, Y contains at least one point of F_+. Thus Y contains at least one ordinary orbit of V. Let $\{O_i\}_{i=1}^M$ be the ordinary orbits of V contained in Y. Then $\Gamma(V, Y)$ has the form



for some integers $I_1, \ldots, I_M$, c and c'. By (2.7), $I_1 = \cdots = I_M = 0$. Also, since V has no special orbits, V is a geometrically ruled surface over F_+ with a standard action of G. It follows from (2.1) that $c' = -c$.

Set $x_i = \overline{O}_i \cap F_+$. Then $X^G = F_+ - \{x_i\}_{i=1}^M$, which is a nonsingular affine curve. Let $\pi : V \to F_+ = V/G$ be the quotient (2.1). Then $\pi|_X : X \to X^G$ has the structure of a G-line bundle over X^G. In particular, X is affine. Since $\pi|_X(x) = \pi|_X(y)$ if and only if $\overline{Gx} \cap \overline{Gy} \neq \varnothing$, from (1.6) it follows that $\pi|_X : X \to X^G$ is the quotient. □

Proposition (3.9). *Let (V, Y) be a minimal G-pair. Set $X = V - Y$. If X^G is finite or empty, then each fixed point of X is hyperbolic, and $\Gamma(V, Y)$ has the form*



where the branches and weights are as follows.

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(1) *There are* $K \geq 0$ *branches of type* $\mathscr{F}$. *For each* $i = 1, \dots, K$, *either* $R_i = 2$, *in which case* $I_1^i = I_2^i = -1$, *or* $R_i \geq 3$ *and there is exactly one* m *with* $I_m^i = -1$. *Moreover, if* $R_i \geq 3$ *and* $I_m^i = -1$, *then* $m = M_i$ *or* M_{i+1}, $1 < m < R_i$, *and* $I_k^i \leq -2$ *for* $k \neq m$.
(2) *There are* $L \geq 0$ *branches of type* $\mathscr{C}$. *For each* $j = 1, \dots, L$, $S_j \geq 3$, $J_{N_j}^j = -1$, $1 < N_j < S_j$, *and* $J_l^j \leq -2$ *for* $l \neq N_j$.
(3) *There are* $M \geq 1$ *ordinary orbits in* Y, *and* $W_1 = \cdots = W_M = 0$.
(4) $c + c' = -(K + L)$.

Proof. Since V contains no elliptic fixed points, neither does X. Since X^G is finite or empty, $F_+, F_- \subset Y$. So X contains no parabolic fixed points. Thus, each fixed point of X is hyperbolic.

By (3.6) each branch of $\Gamma(V, Y)$ has type $\mathscr{F}$ or $\mathscr{C}$. Let K be the number of branches of type $\mathscr{F}$, L the number of branches of type $\mathscr{C}$, and M the number of ordinary vertices of $\Gamma(V, Y)$. Certainly, $K \geq 0$ and $L \geq 0$. The conditions on the weights along these branches follow from (3.6). Connectedness of Y implies $M \geq 1$, and, by (2.7), $W_1 = \cdots = W_M = 0$.

It remains to show $c + c' = -(K + L)$. By (2.5), V is obtained from a geometrically ruled surface S with a standard action of G, by a sequence of blow ups at fixed points. Let

$$V = V_N \xrightarrow{\phi_N} V_{N-1} \to \cdots \to V_1 \xrightarrow{\phi_1} V_0 = S$$

be such a sequence. Also, let b_l be the number of branches of $\Gamma(V_l)$. The proof is by induction on N. If $N = 0$, then $b_0 = 0 = F_+(S)^2 + F_-(S)^2$. Let x be the fixed point which is the center of the lth blow up $\phi_l : V_l \to V_{l-1}$. Set f_{l-1}, resp. f_l, to be the fiber of the ruled surface V_{l-1}, resp. V_l, which contains x, resp. $\phi^{-1}(x)$. A priori, there are three possibilities:

(1) f_{l-1} is irreducible and $x \in F_+(V_{l-1}) \cup F_-(V_{l-1})$,
(2) f_{l-1} is reducible and $x \notin F_+(V_{l-1}) \cup F_-(V_{l-1})$, or
(3) f_{l-1} is reducible and $x \in F_+(V_{l-1}) \cup F_-(V_{l-1})$.

In the first case, f_l has two components, whereas f_{l-1} has only one. So f_l corresponds to a new branch in $\Gamma(V_l)$. Thus, $-b_l = -b_{l-1} - 1 = F_+(V_{l-1})^2 + F_-(V_{l-1})^2 - 1 = F_+(V_l)^2 + F_-(V_l)^2$, where the second equality follows from the induction hypothesis. In the second case, x must be a double point of f_{l-1}. So the blow up introduces no new branches in $\Gamma(V_l)$. Here the induction hypothesis implies

$$-b_l = -b_{l-1} = F_+(V_{l-1})^2 + F_-(V_{l-1})^2 = F_+(V_l)^2 + F_-(V_l)^2.$$

We show that the third case cannot occur. Suppose f_{l-1} is reducible, and consider the branch of $\Gamma(V_{l-1})$ corresponding to f_{l-1}:

$$\overset{I_1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{I_2}{\bullet} \quad \cdots \quad \overset{I_{S-1}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{I_S}{\bullet}$$

If $x \in F_+(V_{l-1})$, then the corresponding branch of $\Gamma(V_l)$ is

$$\overset{-1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{I_1 - 1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{I_2}{\bullet} \quad \cdots \quad \overset{I_{S-1}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{I_S}{\bullet}$$

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Since $I_1 < 0$, (3.5)(2) implies $I_j = -1$ for some $j = 2, \dots, S$. But this is impossible, since the corresponding branch of $\Gamma(V)$ has a unique exceptional curve. Similarly, we obtain a contradiction if $x \in F_-(V_{l-1})$. $\square$

4. Affineness of G-pairs

Let (V, Y) be a G-pair and set $X = V - Y$. We say that (V, Y) is *affine*, if X is affine. In this section, we show that all G-pairs are affine. In view of (3.8), we assume X^G is finite or empty; so that $X^G \subset F_h(V)$. If $x \in X^G$ we construct new G-pairs from (V, Y) by blowing up x and modifying the curve Y. The pairs thus obtained will be affine (minimal) whenever (V, Y) is affine (minimal). We will need a numerical criterion for affineness which we prove now.

Proposition (4.1). *Let V be a G-surface without elliptic fixed points. Suppose Y is an invariant connected subvariety of V. Then $V - Y$ is affine if and only if there is an effective divisor D with* $\operatorname{supp}(D) = Y$ *such that $D \cdot D > 0$ and $D \cdot C > 0$ for each invariant irreducible complete curve C of V.*

Proof. Goodman's criterion for affineness [H2, II.4.2] says an open subset of a projective surface is affine if and only if the difference is the support of an effective ample divisor. A divisor D on a surface is ample if and only if $D^2 > 0$ and $D \cdot H > 0$ for all irreducible complete curves $H \subset V$. This is the Nakai-Moisezon criterion for ampleness, see [H2, p. 365]. ($\Rightarrow$) follows immediately.

Suppose D is an effective divisor of V with $\operatorname{supp}(D) = Y$ such that $D \cdot D > 0$ and $D \cdot C > 0$ for each invariant irreducible complete curve C of V. Let H be an irreducible complete curve in V which is not invariant. For ($\Leftarrow$), we need only show that $D \cdot H > 0$. Since H is not invariant, $H \not\subset Y$. Thus $H \cdot F_i \geq 0$ for each irreducible component F_i of Y and it suffices to show that $H \cap Y \neq \varnothing$.

Let $\pi : V \to V/G$ be the quotient map. We will show that

(1) $H \cap \pi^{-1}(p) \neq \varnothing$ for each $p \in V/G$.
(2) $\pi^{-1}(p) \subset Y$, for some $p \in V/G$.

Consider the restriction $\pi|_H : H \to V/G$. This is a morphism of projective curves and as such $\pi|_H$ is either surjective or the image is a point, see [H1, II.6.8].[2] Since H is not invariant, we must have $\pi|_H(H) = V/G$. So $H \cap \pi^{-1}(p) \neq \varnothing$ for each $p \in V/G$.

To prove (2), note that there is some ordinary orbit O which is not contained in Y. Write $D = \sum n_i D_i + n_+ F_+ + n_- F_-$, where the D_i are prime divisors different from F_+ and F_-. Then $0 < D \cdot \overline{O} = n_+ F_+ \cdot \overline{O} + n_- F_- \cdot \overline{O} = n_+ + n_-$. So at least one of the fixed curves F_+ or F_- is contained in Y. Assume, without loss of generality, that $F_+ \subset Y$. Then $D \cdot F_- > 0$, so $Y \cap F_- \neq \varnothing$. Connectedness of Y implies there are invariant curves $C_1, \dots, C_n$ such that $F_+ \cap C_1 \neq \varnothing$, $C_i \cap C_{i+1} \neq \varnothing$ for $i = 1, \dots, n-1$ and $C_n \cap F_- \neq \varnothing$. Note that the union of these curves maps to a single point $P \in V/G$ and $\pi^{-1}(P) = \bigcup_i C_i$. So Y contains $\pi^{-1}(P)$. $\square$

[2]The nonsingularity of X in the statement of [H1, II.6.8] is not used to prove this assertion.

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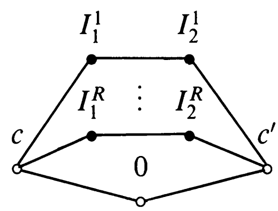
Lemma (4.2). *Let* (V, Y) *be a G-pair and suppose* $\Gamma(V, Y)$ *has the form*



Then $I_j^i = -1$ *and* $V - Y$ *is affine.*

Proof. By (3.5)(1), $I_j^i = -1$. Let O be the ordinary orbit of V which is contained in Y. Set $n = \max(|F_+^2|, |F_-^2|) + 1$ and $D = F_+ + n\overline{O} + F_-$. It's easy to verify that $D^2 > 0$ and $D \cdot C > 0$ for each invariant irreducible complete curve of V. So (4.1) implies $V - Y$ is affine. □

This provides us with a large class of affine surfaces with hyperbolic fixed points. In fact, if $\Gamma(V, Y)$ is as in (4.2), $V - Y$ has R fixed points $x_1, \dots, x_R$, where $x_j = C_1^j \cap C_2^j$ and C_i^j is the closure of the orbit corresponding to the vertex v_i^j with $(C_i^j)^2 = I_i^j$.

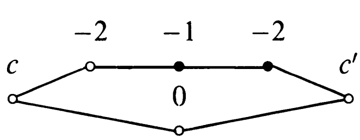
Lemma (4.3) (and definition of O_x^+, O_x^-, C_x^+ and C_x^-). *Let* V *be a projective G-surface and suppose* $x \in F_h(V)$. *Then there are unique orbits* O_x^+ *and* O_x^- *such that* $C_x^+ \cap C_x^- = \{x\}$, x *is a source of* C_x^+ *and* x *is a sink of* C_x^-, *where* $C_x^\varepsilon = \overline{O_x^\varepsilon}$.

Proof. Write $T_x V = t^{-a} + t^b$ for positive integers a and b. Set L_ε to be the one dimensional subrepresentation $(T_x V)^\varepsilon$ for $\varepsilon \in \{+, -\}$. Let U be an invariant affine open neighborhood of x, such a neighborhood exists by [S, Corollary 2]. Let $F\colon U \to T_x U \cong T_x V$ be the equivariant étale morphism such that $F(x) = 0$ from (1.8). Then F maps an invariant open neighborhood W of x bijectively onto an invariant open neighborhood $F(W)$ of 0. Set $H_\varepsilon = F^{-1}(L_\varepsilon) \cap W$. Then $x \in H_\varepsilon$, and, since F is equivariant, x is a source of H_+ and a sink of H_-. Set $O_x^\varepsilon = H_\varepsilon - \{x\}$. □

Now let $\phi\colon \widetilde{V}_x \to V$ be the blow up at x. For $C \subset V$, let $\widetilde{C} = \overline{\phi^{-1}(C - x)}$ denote the proper transform.

Definition (4.4). The *plus*, resp. *minus*, *blow up* of (V, Y) at x is the pair $B_x^\varepsilon(V, Y) = (\widetilde{V}_x, \widetilde{Y} \cup \widetilde{C_x^\varepsilon})$ for $\varepsilon = +$, resp. $-$.

Example (4.5). Suppose (V, Y) is a G-pair whose graph has the form of (4.2) with $R = 1$. Then, $I_1^1 = I_2^1 = -1$. Let $x = C_1 \cap C_2$, where $C_i = \overline{O}_i$ and O_1, O_2 are the orbits of the unique branch of $\Gamma(V, Y)$. Note that $C_1 = C_x^-$ and $C_2 = C_x^+$. Then $\Gamma(\widetilde{V}_x, \widetilde{Y} \cup C_x^-)$ has the form



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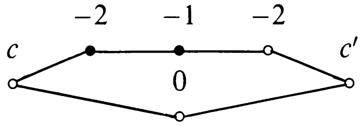
And, $\Gamma(\widetilde{V}_x, \widetilde{Y} \cup C_x^+)$ has the form



This illustrates the effect of plus and minus blow ups on the graph of a G-pair. It is also not difficult to determine, from $\Gamma(V, Y)$, when (V, Y) is $B_x^\varepsilon(V', Y')$ for some G-pair (V', Y') and hyperbolic fixed point $x \in V' - Y'$. In fact, we shall see that this occurs precisely when $\Gamma(V, Y)$ has a branch of type $\mathscr{F}$ with length $S \geq 3$.

Lemma (4.6). *Let (V, Y) be a G-pair and suppose x is a hyperbolic fixed point of $V - Y$. Then, for $\varepsilon \in \{+, -\}$ we have:*

(1) *$B_x^\varepsilon(V, Y)$ is a G-pair.*
(2) *If (V, Y) is affine so is $B_x^\varepsilon(V, Y)$.*
(3) *(V, Y) is minimal if and only if $B_x^\varepsilon(V, Y)$ is minimal.*

Proof. First, Y is connected and $Y \cap C_x^\varepsilon \neq \varnothing$ implies $\widetilde{Y} \cup \widetilde{C_x^\varepsilon}$ is also connected. If H is an irreducible complete curve of $\widetilde{V}_x$, either $H = \widetilde{C}$ for some irreducible complete curve C of V or $H = E_x$ the exceptional curve of $\phi : \widetilde{V}_x \to V$. In the first case $C \cap Y \neq \varnothing$, so $\widetilde{C} \cap \widetilde{Y} \neq \varnothing$. If $H = E_x$, $H \cap \widetilde{C_x^\varepsilon} \neq \varnothing$. It is clear that $\widetilde{Y} \cup \widetilde{C_x^\varepsilon}$ is invariant and closed, and that $\widetilde{V}_x$ contains no elliptic fixed point. This proves that $B_x^\varepsilon(V, Y)$ is a G-pair.

Now suppose (V, Y) is affine. By (4.1) there is an effective divisor D with $\operatorname{supp}(D) = Y$ such that $D^2 > 0$ and $D \cdot C > 0$ for each invariant irreducible curve C of V. Set $H = \widetilde{C_x^\varepsilon}$ and let $n = |H^2| + 1$. Then $D' = n\widetilde{D} + H$ is an effective divisor with support $\widetilde{Y} \cap \widetilde{C_x^\varepsilon}$. Computing intersection numbers we have:

$$\begin{aligned} D' \cdot D' &= n^2\widetilde{D}^2 + 2n\widetilde{D} \cdot H + H^2 \\ &\geq n^2 D^2 + 2n + H^2 > 0, \\ D' \cdot H &= n\widetilde{D} \cdot H + H^2 \geq n + H^2 > 0, \\ D' \cdot E_x &= n\widetilde{D} \cdot E_x + H \cdot E_x = 1. \end{aligned}$$

If F is a complete invariant irreducible curve of $\widetilde{V}_x$ different from H and E_x then $F = \widetilde{C}$ for some complete invariant irreducible curve C of V and

$$D' \cdot F = n\widetilde{D} \cdot F + H \cdot F \geq nD \cdot C > 0.$$

For (3) write $Y = \bigcup_i F_i$, where the F_i are the irreducible components of Y. Then since $x \in V - Y$, $F_i^2 = (\widetilde{F}_i)^2$ for all i. Thus F_i is an exceptional curve of V if and only if $\widetilde{F}_i$ is an exceptional curve of $\widetilde{V}$. This proves $(\Rightarrow)$.

To prove the converse we need only show that $(\widetilde{C_x^\varepsilon})^2 \neq -1$. But, C_x^ε is the closure of a special orbit of V. By (2.5)(3), this means $(C_x^\varepsilon)^2 < 0$. So $(\widetilde{C_x^\varepsilon})^2 = (C_x^\varepsilon)^2 - 1 < -1$. $\square$

We now prove a lemma which will serve as an induction step in (4.8).

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Lemma (4.7). *Let (V, Y) be a minimal G-pair such that each branch of $\Gamma(V, Y)$ has type $\mathscr{F}$, some branch has length $S \geq 3$ and Y contains exactly one ordinary orbit of V. Then there is a minimal G-pair (V', Y'), a fixed point $x \in V' - Y'$ and an $\varepsilon \in \{+, -\}$ such that*

(1) $(V, Y) = B_x^\varepsilon(V', Y')$,
(2) *each branch of* $\Gamma(V', Y')$ *has type* $\mathscr{F}$, *and*
(3) Y' *contains exactly one ordinary orbit of* V'.

Proof. Let $\mathscr{B}$ be a branch of $\Gamma(V, Y)$ of length $S \geq 3$, and let O_i be the orbit corresponding to the ith vertex. Since $\mathscr{B}$ is of type $\mathscr{F}$, there is an N such that $O_N, O_{N+1} \subset V - Y$ and $O_i \subset Y$ for $i \neq N, N+1$. Also, since $S \geq 3$, exactly one of $(\overline{O}_N)^2 = -1$ or $(\overline{O}_{N+1})^2 = -1$.

Assume $(\overline{O}_N)^2 = -1$. We show there is a minimal G-pair (V', Y') and a fixed point $x \in V' - Y'$ such that $(V, Y) = B_x^-(V', Y')$ with (2) and (3) also satisfied. In case $(\overline{O}_{N+1})^2 = -1$, a similar proof will show there is a minimal G-pair (V', Y') and a fixed point $x \in V' - Y'$ such that $(V, Y) = B_x^+(V', Y')$ with (2) and (3) also satisfied.

Let $\phi : V \to V'$ be the blow down of the exceptional curve $\overline{O}_N$. Set $x = \phi(\overline{O}_N)$. Since $S \geq 3$, by (3.6) $N > 1$. Set $O = O_{N-1}$ and $Y' = \overline{\phi(Y - \overline{O})}$. Then $Y' = \overline{\phi(Y) - \phi(\overline{O})} = \overline{\phi(Y) - C_x^-}$ and $(V, Y) = B_x^-(V', Y')$. It is easy to see that (V', Y') is a G-pair, so (4.6) implies (V', Y') is minimal. Properties (2) and (3) are also easily verified. □

Proposition (4.8). *Let (V, Y) be a minimal G-pair. Suppose each branch of $\Gamma(V, Y)$ is of type $\mathscr{F}$ and Y contains exactly one ordinary orbit of V. Then $V - Y$ is affine.*

Proof. We use induction on N the number of special orbits of V. If $N = 2$, there is exactly one branch of $\Gamma(V, Y)$, its length is two and (4.2) implies $V - Y$ is affine. Suppose $N > 2$. If each branch of $\Gamma(V, Y)$ has length two, (4.2) again implies $V - Y$ is affine. Otherwise there is a branch $\mathscr{B}$ of $\Gamma(V, Y)$ of length $S \geq 3$. By (4.7), there is a minimal G-pair (V', Y'), a fixed point $x \in V' - Y'$ and an $\varepsilon \in \{+, -\}$ such that $(V, Y) = B_x^\varepsilon(V', Y')$, each branch of $\Gamma(V', Y')$ is of type $\mathscr{F}$ and Y' contains exactly one ordinary orbit of V'. Notice also that V' has $S - 1$ special orbits. So, by the induction hypothesis, $V' - Y'$ is affine. It follows, from (4.6) that $V - Y$ is affine. □

Theorem (4.9). *If (V, Y) is a G-pair, then $V - Y$ is affine.*

Proof. After blowing down exceptional curves in Y which are not fixed, we may assume (V, Y) is minimal. Set $X = V - Y$. If $\dim X^G = 1$, the theorem is proved (3.8). So suppose X^G is finite or empty. Then $\Gamma(V, Y)$ has the form given in (3.9). Let $\mathscr{B}_1, \dots, \mathscr{B}_L$ be the branches of type $\mathscr{C}$, S_j be the length, and $O_1^j, \dots, O_{S_j}^j$ be the orbits of $\mathscr{B}_j$. Then for each j there is an N_j, such that $1 < N_j < S_j$, $O_{N_j}^j \subset V - Y$ and $O_i^j \subset Y$ for $i \neq N_j$. Let $O_1, \dots, O_M$ be the ordinary orbits of V which are contained in Y, $C_l = \overline{O}_l$ and $C_k^j = \overline{O_k^j}$. Set $Y' = \overline{Y - (\bigcup_{j=1}^L C_{N_j-1}^j \cup \bigcup_{i=2}^M C_i)}$ and $X' = V - Y'$.

We now verify that (V, Y') is a minimal G-pair. Minimality is clear. Suppose, to the contrary, that (V, Y') is not a G-pair. Then there is an irreducible

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invariant closed curve C of V such that $C \cap Y' = \varnothing$. Since $F_+ \cup F_- \subset Y'$, C cannot be F_+, F_- or the closure of an ordinary orbit. Thus C is the closure of a special orbit. But, since (V, Y) is a G-pair, $C \cap Y \neq \varnothing$. So $C \cap Y' = \varnothing$ implies $C \cap C^j_{N_j - 1}$ for some j. But, this means C is one of, $C^j_{N_j-2}$, $C^j_{N_j-1}$ or $C^j_{N_j}$, each of which intersects Y'. Therefore, no such C exists, and (V, Y') is a G-pair.

Next, notice that each branch of $\Gamma(V, Y')$ has type $\mathscr{F}$ and Y' contains exactly one ordinary orbit of V. Thus X' is affine by (4.8). But, $X = X' - H$ where H_j is the closure in X' of $O^j_{N_j-1}$, F_i is the closure in X' of O_i and $H = \bigcup_{j=1}^{L} H_j \cup \bigcup_{i=2}^{M} F_i$. Since H is a finite union of codimension one subvarieties of the affine variety X', [La, p. 120] implies the difference $X = X' - H$ is affine. □

Corollary (4.10). *Let X be an affine G-surface without elliptic fixed points. Let (V, Y) be a G-pair such that $X \cong V - Y$, and let $\pi: V \to V/G$ be the quotient of V. Then $\pi|_X: X \to \pi(X)$ is the quotient of X. In particular, X/G is a nonsingular curve and the points of $V/G - X/G$ are in one-to-one correspondence with the ordinary orbits of V which are contained in Y.*

Proof. Note V/G is nonsingular and $\overline{Gx} \cap \overline{Gy} \neq \varnothing$ (closures in V) implies $\pi(x) = \pi(y)$. So by (1.6) we need only show that $\pi|_X(x) = \pi|_X(y)$ implies $\overline{Gx} \cap \overline{Gy} \neq \varnothing$ (closures in X).

Let x and y be points of X such that $\pi(x) = \pi(y)$. Then x and y lie on the same fiber f of the ruling $\pi : V \to V/G$. If f is irreducible, then $Gx = Gy$ is an ordinary orbit. Otherwise, let $\mathscr{B}$ be the corresponding branch of $\Gamma(V, Y)$. If $\mathscr{B}$ is of type $\mathscr{C}$, then x, $y \in X$ implies $Gx = Gy$. If $\mathscr{B}$ is of type $\mathscr{F}$, then x, $y \in X$ implies either $Gx = Gy$, or Gx and Gy are adjacent orbits. In either case $\overline{Gx} \cap \overline{Gy} \neq \varnothing$ (closures in X). □

5. The graph of an affine G-surface

We continue to work over an algebraically closed field k of arbitrary characteristic. Throughout, $G = k^*$.

Let X be an affine G-surface without elliptic fixed points and let (V, Y) be a minimal G-pair such that $X \cong V - Y$, such a pair exists by (3.3). In this section we show that up to the weights at the fixed vertices, $\Gamma(V, Y)$ is determined by standard invariants of the G-surface X. If $\dim X^G = 1$, this follows from (3.8). In fact, $\Gamma(V, Y)$ is given in (3.8)(1), resp. (3.8)(2), if the fixed curve X^G is a source, resp. sink, of X. So assume X^G is finite or empty. We will show that the branches of $\Gamma(V, Y)$ of type $\mathscr{F}$ are determined by the tangent space representations at the corresponding fixed points, and the branches of type $\mathscr{C}$ are determined by the Seifert invariants, see below, of the corresponding closed orbits of X.

Following [OW, 3.3], we define the Seifert invariant. Denote by μ_α the αth roots of unity in k^*, i.e. $\mu_\alpha = \operatorname{Spec} k[T]/(T^\alpha - 1)$. Note that if $\operatorname{char} k = p$, μ_α is reduced if and only if $p \not| \alpha$. In any characteristic, the only subschemes of k^* which are also subgroups of k^* are the μ_a for $\alpha \geq 1$.

Let x be a nonfixed point of a G-surface V. Then the isotropy subgroup of x is μ_α for some $\alpha > 1$. The action on V induces a representation

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$\sigma : \mu_\alpha \to GL(T_xV)$ of μ_α on the tangent space to x. As in [OW, 3.3], there is a unique integer $\gamma \bmod \alpha$ such that for appropriate coordinates x_1, x_2 on T_xV, $\sigma(t)\cdot(x_1, x_2) = (t^0x_1, t^\gamma x_2)$. In this case, we write $T_xV = t^0 + t^\gamma$ *as a μ_α-surface.* The *Seifert invariant of* x is the pair (α, β) where μ_α is the isotropy subgroup of x, $T_xV = t^0 + t^\gamma$ as a μ_α-surface, $\beta\gamma \equiv 1 \bmod \alpha$ and $0 < \beta < \alpha$.

Suppose $\mathscr{O}$ is an orbit of a G-surface V. It is straightforward to check that for $x, y \in \mathscr{O}$, the Seifert invariant of x is the same as the Seifert invariant of y. Thus we define *the isotropy subgroup of* $\mathscr{O}$ and *the Seifert invariant of* $\mathscr{O}$ to be those of any point $x \in \mathscr{O}$. The isotropy subgroups of orbits of a branch of $\Gamma(V)$ are related to the weights of the corresponding vertices in the following.

Lemma (5.1). *Suppose V is a projective G-surface without elliptic fixed points. Let $\mathscr{B}$ be a branch of $\Gamma(V)$ of the form*

$$\overset{I_1}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{I_2}{\bullet}\qquad\cdots\qquad\overset{I_{S-1}}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{I_S}{\bullet}$$

Let $O_1, \dots, O_S$ be the orbits of $\mathscr{B}$. Let μ_{α_i} be the isotropy subgroup of O_i, and let $\alpha_0 = \alpha_{S+1} = 0$. Then the following hold.

(1) *If $I_m = -1$, then $\alpha_m = \alpha_{m-1} + \alpha_{m+1}$.*
(2) *If $S = 2$, or if there is a unique m with $I_m = -1$ and $I_i \le -2$ for $i \ne m$, then $\alpha_i = 1$ if and only if $i = 1$ or $i = S$.*
(3) $(\alpha_i, \alpha_{i+1}) = 1$ *for* $i = 1, \dots, S-1$.

Proof. Since $I_m = -1$, O_m is the exceptional curve of a blow up $\psi : V \to W$. By (1.8), $T_xW = t^{-\alpha_{m-1}} + t^{\alpha_{m+1}}$, where $x = \psi(\overline{O}_m)$. From [OW, 3.4], it follows that $\alpha_m = \alpha_{m-1} + \alpha_{m+1}$.

To prove (2), let $x = O_1 \cap F_+$. By (2.1), $T_xV = t^0 + t^1$, so $\alpha_1 = 1$. Similarly, $\alpha_S = 1$. The conditions on the weights of $\mathscr{B}$ imply that V is obtained by a sequence of blow ups at hyperbolic fixed points, $\phi : V \to V'$, where the corresponding branch of V' has length two. The converse follows from (1) by induction on the number of blow ups in ϕ.

The third statement is also proved by induction on the number of blow ups along $\mathscr{B}$. If $S = 2$, $\alpha_1 = \alpha_S = 1$. If $S \ge 3$, let n be such that $I_n = -1$ and $1 < n < S$. Blowing down O_n, by the induction hypothesis we have $(\alpha_{n-1}, \alpha_{n+1}) = 1$. Thus, by (1) $(\alpha_{n-1}, \alpha_n) = (\alpha_{n-1}, \alpha_{n-1} + \alpha_{n+1}) = (\alpha_{n-1}, \alpha_{n+1}) = 1$. Similarly, $(\alpha_n, \alpha_{n+1}) = 1$. $\square$

We say an orbit of a G-surface is *trivial*, if its isotropy subgroup is either G or $\{1\}$, and *nontrivial* otherwise. Suppose (V, Y) is a minimal G-pair, and suppose that the fixed set of $X = V - Y$ is finite or empty. From (3.9) it is clear that there is a one-to-one correspondence between the branches of $\Gamma(V, Y)$ of type $\mathscr{F}$ and the fixed points of X. Since ordinary orbits are trivial, from (3.9) and (5.1)(2) it follows that there is a one-to-one correspondence between the branches of $\Gamma(V, Y)$ of type $\mathscr{C}$ and the nontrivial closed orbits of X.

Proposition (5.2). *Let X be an affine G-surface without elliptic fixed points, and suppose that X^G is finite or empty. If (V_1, Y_1) and (V_2, Y_2) are both minimal G-pairs such that $X \cong V_1 - Y_1 \cong V_2 - Y_2$ and $F_+(V_1)\cdot F_+(V_1) = F_+(V_2)\cdot F_+(V_2)$,*

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then $\Gamma(V_1, Y_1) = \Gamma(V_2, Y_2)$. *Moreover,*

(1) *the branches of type* $\mathscr{F}$ *are determined by the tangent space representations at the fixed points of* X,

(2) *the branches of type* $\mathscr{C}$ *are determined by the Seifert invariants of the nontrivial closed orbits of* X, *and*

(3) *the number of ordinary vertices is* $|Z - X/G|$, *where* Z *is the unique nonsingular projective curve containing* X/G.

Proof. Let K be the number of fixed points and L the number of nontrivial closed orbits of X. Then each of $\Gamma(V_1, Y_1)$ and $\Gamma(V_2, Y_2)$ has K branches of type $\mathscr{F}$, one for each fixed point of X, and L branches of type $\mathscr{C}$, one for each nontrivial closed orbit of X. By (3.9)(4), $F_+(V_1)^2 + F_-(V_1)^2 = -(K+L) = F_+(V_2)^2 + F_-(V_2)^2$. So, $F_+(V_1)^2 = F_+(V_2)^2$ implies $F_-(V_1)^2 = F_-(V_2)^2$. It follows from (4.10), that the number of ordinary vertices of $\Gamma(V_1, Y_1)$ is the same as the number of ordinary vertices of $\Gamma(V_2, Y_2)$, and that this number is $|Z - X/G|$, where Z is the unique nonsingular projective curve containing X/G.

Let (V, Y) be any minimal G-pair such that $X \cong V - Y$. We now show that the branches of $\Gamma(V, Y)$ of type $\mathscr{F}$ are determined by the tangent space representations at the fixed points of X, and the branches of $\Gamma(V, Y)$ of type $\mathscr{C}$ are determined by the Seifert invariants of the nontrivial closed orbits of X.

If $X^G \neq \varnothing$, fix $x \in X^G$ and write $T_xX = t^{-a} + t^b$ for positive integers a and b. Let $\mathscr{B}$ be the corresponding branch of $\Gamma(V, Y)$ of type $\mathscr{F}$. Then O_x^- and O_x^+, cf. (4.3), are the adjacent orbits of $\mathscr{B}$ which lie in X. Using (1.8), we see that the isotropy subgroup of O_x^- is μ_a and the isotropy subgroup of O_x^+ is μ_b. From (5.1)(2), it follows that $a = b = 1$ if and only if the length of $\mathscr{B}$ is two. So, in this case, $\mathscr{B}$ is

$$\overset{-1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{-1}{\bullet}$$

Otherwise, exactly one of C_x^- and C_x^+, cf. (4.3), has self-intersection -1 in V. If $(C_x^-)^2 = -1$, then (5.1)(1) implies $a > b$. In this case, the Seifert invariants, cf. [OW, 3.3], of O_x^- can be computed to be (a, η), where $0 < \eta < a$, $(a, \eta) = 1$ and $\eta b \equiv 1 \bmod a$. Let $[a_1, \dots, a_k]$ denote the continued fraction:

$$a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_k}}}$$

From [OW, (3.5)(3)] it follows that $\mathscr{B}$ has the form

$$\overset{-b_1}{\circ}\!\!-\!\!\!-\!\!\overset{-b_2}{\circ}\ \cdots\ \overset{-b_{n-1}}{\circ}\!\!-\!\!\!-\!\!\overset{-1}{\bullet}\!\!-\!\!\!-\!\!\overset{-b_{n+1}}{\bullet}\!\!-\!\!\!-\!\!\overset{-b_{n+2}}{\circ}\ \cdots\ \overset{-b_{s-1}}{\circ}\!\!-\!\!\!-\!\!\overset{-b_s}{\circ}$$

where the weights are uniquely determined by the continued fraction representations $\frac{a}{a-\eta} = [b_1, \dots, b_{n-1}]$ and $\frac{a}{\eta} = [b_s, \dots, b_{n+1}]$.

If $(C_x^+)^2 = -1$, then (5.1)(2) implies $b > a$. In this case, the Seifert invariants of O_x^+ are (b, ν), where $0 < \nu < b$, $(b, \nu) = 1$ and $-a\nu \equiv 1 \bmod b$.

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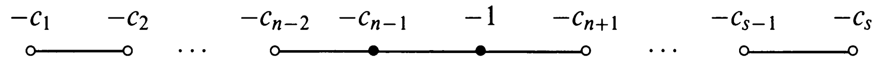
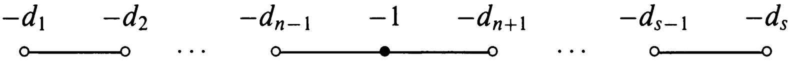
Here [OW, (3.5)(3)] implies $\mathscr{B}$ has the form



where $\frac{b}{b-\nu} = [c_1, \dots, c_{n-1}]$ and $\frac{b}{\nu} = [c_s, \dots, c_{n+1}]$.

If O is a nontrivial closed orbit of X with Seifert invariants (α, β), then the corresponding branch of $\Gamma(V, Y)$ of type $\mathscr{C}$ is determined from [OW, (3.5)(3)] to have the form



where $\frac{\alpha}{\alpha-\beta} = [d_1, \dots, d_{n-1}]$ and $\frac{\alpha}{\beta} = [d_s, \dots, d_{n+1}]$. $\square$

Suppose X is an affine G-surface without elliptic fixed points and (V, Y) is a minimal G-pair with $X \cong V - Y$. Let C be the closure of an ordinary orbit of V which is contained in Y. Then, for $\varepsilon = +$ or $-$, blowing up $C \cap F_\varepsilon$ followed by blowing down the proper transform of C yields another G-pair (V', Y') with $X \cong V' - Y'$, $F_\varepsilon(V') \cdot F_\varepsilon(V') = F_\varepsilon(V) \cdot F_\varepsilon(V) - 1$, and $F_{-\varepsilon}(V') \cdot F_{-\varepsilon}(V') = F_{-\varepsilon}(V) \cdot F_{-\varepsilon}(V) + 1$. Thus, repeating this process if necessary, we may assume that $F_+(V) \cdot F_+(V) = 0$.

Definition (5.3). If X is an affine G-surface without elliptic fixed points, then the graph $\Gamma(X)$ is defined to be $\Gamma(V, Y)$, for any minimal G-pair (V, Y) with $X \cong V - Y$ and $F_+(V) \cdot F_+(V) = 0$. This is independent of the choice of such a G-pair.

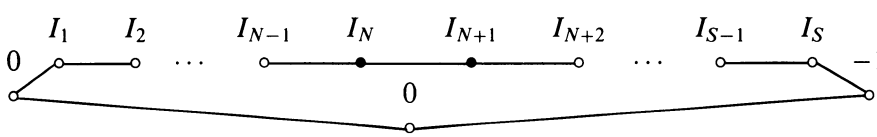
In (5.5) below, we see that the hyperbolic representations of k^* on k^2 can be identified from their graphs and quotients. For this, we need the following.

Lemma (5.4). *Let $\phi : Z \to k^2$ be the blow up of k^2 at $(0, 0)$. Then $Z - \widetilde{L} \cong k^2$, where L is any line in k^2 through $(0, 0)$.*

Proof. Embed k^2 in $\mathbf{P}^2$ as $k^2 = \mathbf{P}^2 - L_\infty$, where $L_\infty = \{[x : y : 0]\}$, and $[x : y : z]$ are the usual homogeneous coordinates on $\mathbf{P}^2$. Let H be the closure in $\mathbf{P}^2$ of the line L through $(0, 0)$. Let $\hat{\phi} : \hat{Z} \to \mathbf{P}^2$ be the blow up of $\mathbf{P}^2$ at $[0 : 0 : 1]$, and $\hat{E} = \hat{\phi}^{-1}([0 : 0 : 1])$. Then $\hat{Z}$ is a geometrically ruled surface over $\mathbf{P}^1$ with two nonintersecting sections, $\hat{E}$ and $\widetilde{L}_\infty$. Also, $\widetilde{H}$ is a fiber of the ruling. Thus $Z - \widetilde{L} \cong \hat{Z} - \widetilde{H} - \widetilde{L}_\infty$. But the latter is a line bundle over k and hence isomorphic to k^2. $\square$

Remark. Suppose (V, Y) is a G-pair such that $V - Y \cong k^2$, and $x \in V - Y$ is a hyperbolic fixed point. Then (5.4) implies that the plus and minus blow ups at x, $B_x^\varepsilon(V, Y) = (\widetilde{V}, \widetilde{Y} \cup C_x^\varepsilon)$, satisfy $\widetilde{V} - (\widetilde{Y} \cup C_x^\varepsilon) \cong k^2$.

Lemma (5.5). *Let (V, Y) be a minimal G-pair and let $X = V - Y$. Suppose $X/G \cong k$ and $\Gamma(X)$ has the form*



Then X is equivariantly isomorphic to a hyperbolic representation of k^* on k^2.

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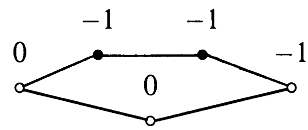
Proof. As in the proof of (4.8), (V, Y) is obtained, by plus and minus blow ups, from a G-pair (V_0, Y_0) whose graph has the form



Since $X/G \cong k$, $V/G \cong V_0/G \cong \mathbf{P}^1$. Let O be the ordinary orbit of V_0 which lies in Y_0, and set $x = F_- \cap \overline{O}$. From its graph we see that (V_0, Y_0) is obtained from $(\mathbf{P}^1 \times \mathbf{P}^1, F_+ \cup \overline{O})$ by blowing up some $y \in F_-$ and removing the line $F_- - \{x\}$. But,

$$\mathbf{P}^1 \times \mathbf{P}^1 - (F_+ \cup \overline{O}) \cong k^2 .$$

Since (V, Y) is obtained from (V_0, Y_0) by plus and minus blow ups, by the preceding remark, $V - Y \cong k^2$. Every action of k^* on k^2 is linear by [B1, p. 123], so $X \cong k^2$ with a hyperbolic representation. □

6. Homology of $\mathbf{C}^*$-surfaces

Throughout this section all varieties are over the complex numbers and $G = \mathbf{C}^*$. The integral homology of an affine $\mathbf{C}^*$-surface with hyperbolic fixed points is computed in terms of its graph. The rational homology of fixed point free $\mathbf{C}^*$-surfaces is also computed. Our main result is that the only acyclic affine $\mathbf{C}^*$-surface is the complex plane with a linear action.

Let V be a nonsingular projective surface. Denote by $\operatorname{Pic}^n V$, resp. $\operatorname{Pic}^a V$, the subgroup of $\operatorname{Pic} V$ consisting of divisors numerically, resp. algebraically, equivalent to zero. The quotient groups are denoted $\operatorname{Num} V = \operatorname{Pic} V/\operatorname{Pic}^n V$ and $\operatorname{NS} V = \operatorname{Pic} V/\operatorname{Pic}^a V$; the latter is called the Néron-Severi group of V. Viewing V as a complex manifold one can consider the integral singular cohomology $H^*(V; \mathbf{Z})$. There is a natural map $\nu \colon \operatorname{NS} V \to H^2(V; \mathbf{Z})$ which maps a divisor to its fundamental class. From the exponential sequence, [H1, p. 446], it follows that ν is an isomorphism if and only if $H^2(V; \mathscr{O}_V) = 0$. Since $\operatorname{Num} V$ is the free part of $\operatorname{NS} V$, [H3, 3.1], $\nu : \operatorname{Num} V \to H^2(V; \mathbf{Z})$ is an isomorphism if and only if $H^2(V; \mathscr{O}_V) = 0$ and $H^2(V; \mathbf{Z})$ is torsion free.

If V is a projective G-surface, then V is ruled. So $H^2(V; \mathbf{Z})$ is free, and the geometric genus, which is the rank of $H^2(V; \mathscr{O}_V)$, is zero. Thus, in this case, $\nu : \operatorname{Num} V \to H^2(V; \mathbf{Z})$ is an isomorphism. By Poincaré duality $H^2(V; \mathbf{Z}) \cong H_2(V; \mathbf{Z})$. Thus, $\operatorname{Num} V \cong H_2(V; \mathbf{Z})$. This will be used to obtain relations among homology classes.

We now find relations in $\operatorname{Num} V$. If C and D are numerically equivalent, we write $C \sim_n D$. The setting for (6.1) and (6.2) is the following.

(i) V is a projective G-surface such that $\Gamma(V)$ has the form of (2.5)(3).
(ii) $O_1^i, \dots, O_{S_i}^i$ are the orbits of the ith branch of $\Gamma(V)$, and $C_k^i = \overline{O_k^i}$.
(iii) α_k^i is such that $\mu_{\alpha_k^i}$ is the isotropy subgroup of O_k^i, and $\alpha_0^i = \alpha_{S_i+1}^i = 0$.
(iv) $\Phi : V \to S$ is a sequence of equivariant blow ups from a geometrically ruled surface S with a standard action of G.

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Lemma (6.1). *Let f be the closure of an ordinary orbit of V. Then in* $\operatorname{Num} V$

$$C_1^i \sim_n f - \sum_{k=2}^{S_i} \alpha_k^i C_k^i$$

for each $i = 1, \ldots, R$.

Proof. In S, any two fibers are numerically equivalent. So, $\Phi(f) \sim_n \Phi(C_1^i)$. Pulling back to V, $f \sim_n \sum_{m=2}^{S_i} m_k^i C_k^i$, for some m_k^i. Clearly $m_1^i = 1$. Using (5.1)(1) and induction on the number of blow ups in Φ, one can show $m_k^i = \alpha_k^i$. □

Lemma (6.2). *Suppose $F_+(S)^2 = F_-(S)^2 = 0$. Suppose also that Φ factors as $V \xrightarrow{\phi} V_0 \xrightarrow{\psi} S$ where $\Gamma(V_0)$ has R branches of length two, each blow up of ψ occurs along F_-, and each blow up of ϕ is centered at a hyperbolic fixed point which is contained in an exceptional curve. Then the following hold.*

(1) *There are unique positive integers γ_k^i such that*

$$F_- \sim_n F_+ - \sum_{i=1}^{R} \sum_{k=2}^{S_i} \gamma_k^i C_k^i.$$

(2) *If $(C_2^i)^2 = -1$, then $\gamma_2^i = \cdots = \gamma_{S_i}^i = 1$.*

(3) *If $(C_m^i)^2 = -1$ and $1 < m < S_i$, then $\gamma_m^i = \gamma_{m-1}^i + \gamma_{m+1}^i$, $\alpha_{m-1}^i \gamma_m^i - \alpha_m^i \gamma_{m-1}^i = 1$ and $\alpha_m^i \gamma_{m+1}^i - \alpha_{m+1}^i \gamma_m^i = 1$.*

Proof. Let γ_k^i be the multiplicity of C_k^i in the pullback of $F_-(S)$ to V. We induct on the number, N, of blow ups in ϕ. If $N = 0$, then $V = V_0$. The condition $F_+(S)^2 = F_-(S)^2 = 0$ implies that $F_+(S) \sim_n F_-(S)$ in S. Note that the R blow ups of $\psi : V_0 \to S$ are centered at points of $F_-(S)$ which lie on distinct fibers. It follows that, in pulling back to $V = V_0$, we have

$$(\dagger) \qquad F_+ = \psi^*(F_+(S)) \sim_n \psi^*(F_-(S)) = F_- + \sum_{i=1}^{R} C_2^i.$$

Then $\gamma_1^i = 0$ and $\gamma_2^i = 1$. So, (1) and (2) hold when $N = 0$.

If $N = 1$, let $(C_2^n)^2 = -1$ be the exceptional curve of the blow up $\phi : V \to V_0$. From (†), it follows that in V, we have

$$F_- \sim_n F_+ - (C_2^n + C_3^n) - \sum_{\substack{i=1 \\ i \neq n}}^{R} C_2^i.$$

Then $\gamma_1^i = 0$, $\gamma_2^i = 1$, and $\gamma_3^n = 1$. So, (1) and (2) hold when $N = 1$. Noting that $\alpha_1^n = \alpha_3^n = 1$ and $\alpha_2^n = 2$, (3) is easily verfied.

If $N > 1$, let C_m^l be the exceptional curve of the last blow up $\phi_N : V \to V_{N-1}$ of ϕ. By the induction hypothesis for (1), in V_{N-1} we have

$$\phi_N(F_-) \sim_n \phi_N(F_+) - \left[\sum_{\substack{k=2 \\ k \neq m}}^{S_l} \gamma_k^l \phi_N(C_k^l) + \sum_{\substack{i=1 \\ i \neq l}}^{R} \sum_{k=2}^{S_i} \gamma_k^i \phi_N(C_k^i) \right].$$

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Pulling back to V, we have

$$F_- \sim_n F_+ - \left[\sum_{\substack{k=2 \\ k\neq m}}^{S_l} \gamma_k^l C_k^l + (\gamma_{m-1}^l + \gamma_{m+1}^l) C_m^l + \sum_{\substack{i=1 \\ i\neq l}}^{R} \sum_{k=2}^{S_i} \gamma_k^i C_k^i \right].$$

Then $\gamma_m^l = \gamma_{m-1}^l + \gamma_m^l$. So, (1) and the first part of (3) are proved. If $m = 2$, then every blow up along the l^{th} branch occured at the hyperbolic fixed point in (the image of) C_1^l. Since $\gamma_1^l = 0$ and by induction $\gamma_3^l = \cdots = \gamma_{S_1}^l = 1$, (2) follows. To prove the remaining parts of (3), recall that from (5.1)(1), $\alpha_m^1 = \alpha_{m-1}^1 + \alpha_{m+1}^1$. Thus

$$\begin{aligned} \alpha_{m-1}^l \gamma_m^l - \alpha_m^l \gamma_{m-1}^l &= \alpha_{m-1}^l(\gamma_{m-1}^l + \gamma_{m+1}^l) - (\alpha_{m-1}^l + \alpha_{m+1}^l)\gamma_{m-1}^l \\ &= \alpha_{m-1}^l \gamma_{m+1}^l - \alpha_{m+1}^l \gamma_{m-1}^l = 1, \end{aligned}$$

where the last equality holds by the induction hypothesis. Similarly, $\alpha_m^1 \gamma_{m+1}^1 - \alpha_{m+1}^1 \gamma_m^1 = 1$. □

Let X be an affine G-surface. We will compute the integral homology $H_*(X; \mathbf{Z})$, if $X^G \neq \varnothing$, and the rational homology $H_*(X; \mathbf{Q})$, if $X^G = \varnothing$. If X contains an elliptic or parabolic fixed point, the homology of X is easily obtained from (1.9). We record this here. The reduced homology $H_*(X, x; R)$ is denoted $\widetilde{H}_*(X; R)$.

Lemma (6.3). *Let X be an affine G-surface.*

(1) *If X contains an elliptic fixed point, then X/G is a point, $X \cong \mathbf{C}^2$ and X is acyclic.*

(2) *If X contains a parabolic fixed point, then X/G is an M-punctured compact 2-manifold of genus g, for some $M \geq 1$ and $g \geq 0$, and*

$$\widetilde{H}_q(X; \mathbf{Z}) = \begin{cases} \mathbf{Z}^{2g+M-1} & \text{if } q = 1, \\ 0 & \text{if } q \neq 1. \end{cases}$$

Henceforth we assume the following.

(a) X is an affine G-surface which contains no elliptic or parabolic fixed points.

(b) (V, Y) is a minimal G-pair such that $X \cong V - Y$.

(c) $\Gamma(V, Y)$ is as in (3.9) with $c = F_+^2 = 0$.

(d) g is the genus of F_+.

(e) The orbits of the ith branch of type $\mathscr{F}$ are $U_1^i, \ldots, U_{R_i}^i$, and $C_k^i = \overline{U_k^i}$.

(f) The orbits of the jth branch of type $\mathscr{C}$ are $O_1^j, \ldots, O_{S_j}^j$, and $D_l^j = \overline{O_l^j}$.

(g) $f_1, \ldots, f_M$ are the closures of the ordinary orbits of V contained in Y.

Generators for the homology groups of V and Y are given in the next two lemmas. Submanifolds are identified with their corresponding fundamental classes in $H_q(V; \mathbf{Z})$ and $H_q(Y; \mathbf{Z})$.

Lemma (6.4). *Set $A = 2 + \sum_{i=1}^K (R_i - 1) + \sum_{j=1}^L (S_j - 1)$.*

(1) *$H_1(V; \mathbf{Z}) \cong \mathbf{Z}^{2g}$, generated by the meridians and longitudes of F_+.*

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(2) *$H_2(V;\mathbf{Z}) \cong \mathbf{Z}^A$, generated by F_+, f_1, $\{C_k^i : 2 \le k \le R_i\}_{i=1}^K$ and $\{D_l^j : 2 \le l \le S_j\}_{j=1}^L$.*

(3) *$H_3(V;\mathbf{Z}) \cong \mathbf{Z}^{2g}$.*

Moreover, the relations given in (6.1) *and* (6.2) *hold in $H_2(V;\mathbf{Z})$.*

Proof. Let $\phi : V \to S$ be a sequence of blow ups where S is a geometrically ruled surface with a standard action of G. Topologically, V is obtained from the S^1-bundle S over the 2-manifold F_+, of genus g, by taking connected sums with $\overline{(\mathbf{CP}^2)}$, one for each blow up. Thus $H_*(V;\mathbf{Z})$ and its generators are as stated. It follows from (c) and (3.9) that the blow ups can be performed so as to satisfy the hypotheses of (6.2). Since numerical and homological equivalence agree in V, the relations in (6.1) and (6.2) also hold in $H_2(V;\mathbf{Z})$. □

Lemma (6.5). *Set $B = 2 + M + \sum_{i=1}^K (R_i - 2) + \sum_{j=1}^L (S_j - 1)$.*

(1) *$H_1(Y;\mathbf{Z}) \cong \mathbf{Z}^{4g+M-1}$, generated by the meridians and longitudes F_+ and F_-, together with $\{\sigma_s\}_{s=1}^{M-1}$ such that $i_1(\sigma_s) = 0$ in $H_1(V;\mathbf{Z})$.*

(2) *$H_2(Y;\mathbf{Z}) \cong \mathbf{Z}^B$ generated by*

$$F_+,\ F_-,\ \{f_m\}_{m=1}^M,\ \{C_k^i : k \ne M_i,\ M_i+1\}_{i=1}^K,\ \text{and } \{D_l^j : l \ne N_j\}_{j=1}^L.$$

Proof. Observe that Y has the homotopy type of a wedge of F_+, F_-, $(M-1)$ circles, and a collection of 2-spheres, one for each C_k^i and D_l^j which lies in Y. The curve σ_s can be realized as a loop connecting the poles of the 2-spheres f_s and f_{s+1}. With this choice it is clear that $i_1(\sigma_s) = 0$ for $s = 1, \dots, M-1$. □

Lemma (6.6). *The homomorphism $i_1 \colon H_1(Y;\mathbf{Z}) \to H_1(V;\mathbf{Z})$ is surjective with kernel a free abelian group of rank $2g + M - 1$.*

Proof. This follows immediately from (6.4)(1) and (6.5)(1). □

Let B_V, resp. B_Y, be the basis for $H_2(V;\mathbf{Z})$, resp. $H_2(Y;\mathbf{Z})$, given in (6.4)(2), resp. (6.5)(2). Consider the homomorphism $i_2 \colon H_2(Y;\mathbf{Z}) \to H_2(V;\mathbf{Z})$. If $C \in B_V \cap B_Y$, then $i_2(C) = C$. Thus, in analyzing i_2, we disregard these common generators. Let H_V, resp. H_Y, be the subgroup of $H_2(V;\mathbf{Z})$, resp. $H_2(Y;\mathbf{Z})$, generated by $B_V \cap B_Y$. Set $H_2(V;\mathbf{Z}) = H_2'(V) \oplus H_V$ and $H_2(Y;\mathbf{Z}) = H_2'(Y) \oplus H_Y$. Then $i_2' : H_2'(Y) \to H_2'(V)$ satisfies $\ker i_2 = \ker i_2'$ and $\operatorname{coker} i_2' = \operatorname{coker} i_2$.

A basis for $H_2'(Y)$, resp. $H_2'(V)$, is $B_Y - (B_V \cap B_Y)$, resp. $B_V - (B_V \cap B_Y)$. The elements of these bases are now given explicitly.

First, of the curves which are not the closures of the special orbits, $f_2, \dots, f_M$ and F_- are in $B_Y - (B_V \cap B_Y)$.

Next, consider the curves D_l^j which are the closures of the special orbits of branches of type $\mathscr{C}$. The jth such branch has the form

$$\overset{J_1^j}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{J_2^j}{\circ}\ \cdots\ \overset{J_{N_j-1}^j}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{J_{N_j}^j}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{J_{N_j+1}^j}{\circ}\ \cdots\ \overset{J_{S_j-1}^j}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{J_{S_j}^j}{\circ}$$

From (3.6) we know $1 < N_j < S_j$. Thus $D_1^j \in B_Y - (B_V \cap B_Y)$ and $D_{N_j}^j \in B_V - (B_V \cap B_Y)$.

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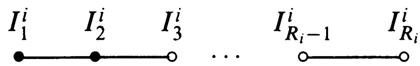
The situation for the C_k^i is more complicated. If $C_1^i \not\subset Y$, then the corresponding branch of type $\mathscr{F}$ has the form

(†)



So $C_1^i \notin B_V \cup B_Y$. But $C_2^i \in B_V - (B_V \cap B_Y)$. By reordering the branches if necessary, we may choose n so that the first n branches of type $\mathscr{F}$ have the form (†). Explicitly, choose n so that $0 \leq n \leq K$, $C_1^i \not\subset Y$ for $i \leq n$, and $C_1^i \subset Y$ for $i > n$. Then for $i > n$, the ith branch of type $\mathscr{F}$ has the form

where $M_i > 1$. So for $i > n$, $C_1^i \in B_Y - (B_V \cap B_Y)$ and $C_{M_i}^i$ and $C_{M_i+1}^i$ are in $B_V - (B_V \cap B_Y)$.

Thus ordered bases B_Y' and B_V' for $H_2'(Y)$ and $H_2'(V)$ respectively are

$$B_Y' = \langle D_1^1, \ldots, D_1^L, F_-, C_1^{n+1}, \ldots, C_1^K, f_2, \ldots, f_M \rangle$$

and

$$B_V' = \langle D_{N_1}^1, \ldots, D_{N_L}^L, C_2^1, \ldots, C_2^n, C_{M_{n+1}}^{n+1}, C_{M_{n+1}+1}^{n+1}, \ldots, C_{M_K}^K, C_{M_K+1}^K \rangle.$$

Lemma (6.7). *In $H_2'(V)$ we have the following relations.*

(1) $D_1^j = -\alpha_j D_{N_j}^j$, *where μ_{α_j} is the isotropy subgroup of $O_{N_j}^j$.*

(2) *For $i > n$, $C_1^i = -a_i C_{M_i}^i - b_i C_{M_i+1}^i$, where μ_{a_i} is the isotropy subgroup of $U_{M_i}^i$ and μ_{b_i} is the isotropy subgroup of $U_{M_i+1}^i$. By* (5.1)(3), $(a_i, b_i) = 1$.

(3) $f_2 = \cdots = f_M = 0$.

(4) *There are unique positive integers δ_j, σ_i and τ_i such that $a_i\tau_i - b_i\sigma_i = 1$ and*

$$F_- = -\left[\sum_{j=1}^{L} \delta_j D_{N_j}^j + \sum_{r=1}^{L} C_2^r + \sum_{i=n+1}^{K} (\sigma_i C_{M_i}^i + \tau_i C_{M_i+1}^i)\right].$$

Proof. The first two statements follow from (6.1). Since any two fibers are homologically equivalent, (3) holds. By (6.2)(1) there exist unique positive integers δ_j, δ_r', σ_i and τ_i such that

$$F_- = -\left[\sum_{j=1}^{L} \delta_j D_{N_j}^j + \sum_{r=1}^{L} \delta_r' C_2^r + \sum_{i=n+1}^{K} (\sigma_i C_{M_i}^i + \tau_i C_{M_i+1}^i)\right].$$

By (6.2)(2), $\delta_1' = \cdots = \delta_n' = 1$, and by (6.2)(3) $a_i\tau_i - b_i\sigma = 1$. □

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Lemma (6.8). *With respect to the ordered bases* B'_Y *and* B'_V, *the matrix for* $i'_2 : H'_2(Y) \to H'_2(V)$ *is given by* $-\mathscr{M}^t$ *where*

$$\mathscr{M} = \left[\begin{array}{ccc|ccc|ccccc}
\alpha_1 & & & & & & & & & & \\
 & \ddots & & & 0_{L\times n} & & & & 0_{L\times 2(K-n)} & & \\
 & & \alpha_L & & & & & & & & \\
\hline
\delta_1 & \cdots & \delta_L & 1 & \cdots & 1 & \sigma_{n+1} & \tau_{n+1} & \cdots & \sigma_K & \tau_K \\
\hline
 & & & & & & a_{n+1} & b_{n+1} & & & \\
 & 0_{(K-n)\times L} & & & 0_{(K-n)\times n} & & & a_{n+2} & b_{n+2} & & \\
 & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & a_K & b_K \\
\hline
 & 0_{(M-1)\times L} & & & 0_{(M-1)\times n} & & & & 0_{(M-1)\times 2(K-n)} & &
\end{array}\right]$$

and the integers α_j, δ_j, σ_i, τ_i, a_i *and* b_i *are as in* (6.7).

Proof. This follows immediately from (6.7). □

Lemma (6.9). *If* $K \geq 1$, *then* $\ker i_2 \cong \mathbf{Z}^{M-1}$ *and*

$$\operatorname{coker} i_2 \cong \mathbf{Z}^{K-1} \oplus \bigoplus_{j=1}^{L} \mathbf{Z}/\alpha_j\mathbf{Z}.$$

If $K = 0$, *then* $\ker i_2 \cong \mathbf{Z}^M$ *and* $\operatorname{coker} i_2 \subset \bigoplus_{j=1}^{L} \mathbf{Z}/\alpha_j\mathbf{Z}$.

Proof. First assume $K \geq 1$. Consider the submatrix

$$\mathscr{M}' = \left[\begin{array}{ccc|ccccc}
1 & \cdots & 1 & \sigma_{n+1} & \tau_{n+1} & \cdots & \sigma_K & \tau_K \\
\hline
 & & & a_{n+1} & b_{n+1} & & & \\
 & 0_{(K-n)\times n} & & & a_{n+2} & b_{n+2} & & \\
 & & & & & \ddots & \ddots & \\
 & & & & & & a_K & b_K
\end{array}\right]$$

of the matrix $\mathscr{M}$ of (6.8).

Suppose $n > 0$, then since $(a_i, b_i) = 1$ for all $i = n+1, \dots, K$, by (6.7)(2), column operations reduce $\mathscr{M}'$ to

$$\mathscr{M}'' = \left[\begin{array}{ccc|c}
1 & & & \\
 & \ddots & & 0_{(K-n+1)\times(K-1)} \\
 & & 1 &
\end{array}\right].$$

If $n = 0$, then since also

$$\begin{vmatrix} \sigma_i & \tau_i \\ a_i & b_i \end{vmatrix} = \sigma_i b_i - \tau_i a_i = -1,$$

by (6.7)(4), one can again reduce the matrix $\mathscr{M}'$ to $\mathscr{M}''$. In either case column

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operations reduce $\mathscr{M}$ to

$$\left[\begin{array}{ccc|c|c} \alpha_1 & & & & \\ & \ddots & & 0_{L\times(K+1-n)} & 0_{L\times(K-1)} \\ & & \alpha_L & & \\ \hline & & & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & \\ & 0_{(K+1-n)\times L} & & & 0_{(K+1-n)\times(K-1)} \\ \hline & 0_{(M-1)\times L} & & 0_{(M-1)\times(K+1-n)} & 0_{(M-1)\times(K-1)} \end{array}\right],$$

from which the result for $K \geq 1$ follows.

If $K = 0$, then

$$\mathscr{M} = \left[\begin{array}{ccc} \alpha_1 & & \\ & \ddots & \\ & & \alpha_L \\ \hline \delta_1 & \cdots & \delta_L \\ \hline & 0_{(M-1)\times L} & \end{array}\right],$$

from which it follows that $\ker i_2 \cong \mathbf{Z}^M$ and $\operatorname{coker} i_2 \subset \bigoplus_{j=1}^{L} \mathbf{Z}/\alpha_j\mathbf{Z}$. $\square$

Proposition (6.10). *Let X be an affine $\mathbf{C}^*$-surface with*

(1) *$K \geq 1$ hyperbolic fixed points,*
(2) *$L \geq 0$ closed orbits with nontrivial isotropy subgroups $\mu_{\alpha_1}, \ldots, \mu_{\alpha_L}$,*
(3) *and quotient space $X/\mathbf{C}^*$.*

Let Z be the unique nonsingular projective curve containing $X/\mathbf{C}^*$, g the genus of Z, $M = |Z - X/G|$ and N the rank of $H_1(X/\mathbf{C}^*; \mathbf{Z})$. Then $N = 2g + M - 1$ and

$$\widetilde{H}_q(X; \mathbf{Z}) \cong \begin{cases} \mathbf{Z}^N \oplus \bigoplus_{j=1}^{L} \mathbf{Z}/\alpha_j\mathbf{Z}, & \text{if } q = 1, \\ \mathbf{Z}^{N+K-1}, & \text{if } q = 2, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. Let (V, Y) be a minimal G-pair such that $X = V - Y$. Then $Z = V/G$, which is a compact 2-manifold of genus g. Since the cardinality of $Z - X/G$ is M, $H_1(X/G; \mathbf{Z}) \cong \mathbf{Z}^{2g+M-1}$.

To compute $H_*(X; \mathbf{Z})$, we relate this to $H_*(V, Y)$. The later is then computed using the homology exact sequence for the pair (V, Y). All homology and cohomology groups have integral coefficients; the $\mathbf{Z}$ will be dropped from the notation.

Let T be a neighborhood of Y of which Y is a deformation retract. Then $X' = V - T$ is a 4-manifold with boundary, $\partial X'$, and X' is homotopy equivalent to X. By excision and Poincaré duality $H_s(V, Y) \cong H_s(V, T) \cong$

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$H_s(X', \partial X') \cong H^{4-s}(X') \cong H^{4-s}(X)$. Thus, by the universal coefficient theorem, for $0 \le s \le 3$ we have

$$\begin{aligned} H_s(V, Y) &\cong \operatorname{Hom}(H_{4-s}(X), \mathbf{Z}) \oplus \operatorname{Ext}(H_{3-s}(X), \mathbf{Z}) \\ &\cong \mathbf{Z}^{\operatorname{rank}(H_{4-s}(X))} \oplus \operatorname{torsion}(H_{3-s}(X)). \end{aligned}$$

Solving for the free and torsion parts of $H_q(X)$, for $1 \le q \le 3$

$$(\dagger) \qquad H_q(X) \cong \mathbf{Z}^{\operatorname{rank}(H_{4-q}(V, Y))} \oplus \operatorname{torsion}(H_{3-q}(V, Y)).$$

Consider the following portion of the exact sequence for the pair (V, Y):

$$\begin{aligned} 0 \longrightarrow H_3(V) \longrightarrow H_3(V, Y) &\xrightarrow{\partial_3} H_2(Y) \xrightarrow{i_2} H_2(V) \\ &\longrightarrow H_2(V, Y) \xrightarrow{\partial_2} H_1(Y) \xrightarrow{i_1} H_1(V) \longrightarrow 0. \end{aligned}$$

Exactness at the ends holds since $H_3(Y) = 0$ and i_1 is surjective (6.6). Extending the sequence, it is easy to see that $H_1(V, Y) = 0$. Since $H_{s-1}(Y)$ is free, $H_s(V, Y) \cong \ker \partial_s \oplus \operatorname{image} \partial_s$ for $s = 2, 3$. Applying (6.6) and (6.9),

$$H_2(V, Y) \cong \operatorname{coker} i_2 \oplus \ker i_1 \cong \mathbf{Z}^{N+K-1} \oplus \bigoplus_{j=1}^{L} \mathbf{Z}/\alpha_j\mathbf{Z},$$

and, by (6.4) and (6.9),

$$H_3(V, Y) \cong H_3(V) \oplus \ker i_2 \cong \mathbf{Z}^N.$$

Thus, by ($\dagger$),

$$H_1(X) \cong \mathbf{Z}^N \oplus \bigoplus_{j=1}^{L} \mathbf{Z}/\alpha_j\mathbf{Z} \qquad \text{and} \qquad H_2(X) \cong \mathbf{Z}^{N+K-1}.$$

The homology groups $H_q(X)$ vanish for $q \ge 3$, by [M, 7.1]. □

Proposition (6.11). *Let X be an affine $\mathbf{C}^*$-surface with $X^{\mathbf{C}^*} = \varnothing$. Let Z be the unique nonsingular projective curve containing $X/\mathbf{C}^*$, let g be the genus of Z and $M = |Z - X/\mathbf{C}^*|$. Then the rational homology of X is given by*

$$\widetilde{H}_q(X; \mathbf{Q}) = \begin{cases} \mathbf{Q}^{2g+M}, & \text{if } q = 1, \\ \mathbf{Q}^{2g+M-1}, & \text{if } q = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. As in the proof of (6.10) we have $H_q(X; \mathbf{Z}) = 0$ for $q \ge 3$, and

$$\begin{aligned} \operatorname{rank}(H_q(X; \mathbf{Z})) &= \operatorname{rank}(H_{4-q}(V, Y; \mathbf{Z})) \\ &= \begin{cases} 2g + \operatorname{rank}(\ker i_2), & \text{if } q = 1, \\ \operatorname{rank}(\operatorname{coker} i_2) + \operatorname{rank}(\ker i_1), & \text{if } q = 2. \end{cases} \end{aligned}$$

By (6.6), $\operatorname{rank}(\ker i_1) = 2g + M - 1$, and, by (6.9), $\operatorname{rank}(\ker i_2) = M$ and $\operatorname{rank}(\operatorname{coker} i_2) = 0$. Thus,

$$\widetilde{H}_q(X; \mathbf{Q}) = \widetilde{H}_q(X; \mathbf{Z}) \otimes \mathbf{Q} = \begin{cases} \mathbf{Q}^{2g+M}, & \text{if } q = 1, \\ \mathbf{Q}^{2g+M-1}, & \text{if } q = 2, \\ 0, & \text{otherwise.} \end{cases} \quad \square$$

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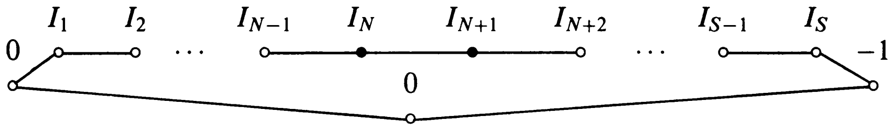
Theorem (6.12). *Let X be an affine $\mathbf{C}^*$-surface. If X is acyclic, then X is equivariantly isomorphic to $\mathbf{C}^2$ with a linear action of $\mathbf{C}^*$.*

Proof. If $X^{\mathbf{C}^*} = \varnothing$, then by (6.11) $\operatorname{rank}(H_1(X; \mathbf{Z})) > 0$. Thus, $X^{\mathbf{C}^*}$ is nonempty.

Case 1. *X contains an elliptic fixed point.* In this case, by (1.9), X is isomorphic to the tangent space representation at the fixed point.

Case 2. *X contains a parabolic fixed point.* The acyclic condition implies that in (6.3) we must have $g = 0$ and $M = 1$. But then $X/\mathbf{C}^* = \mathbf{C}$. So, by (1.9), X is a G-vector bundle over $\mathbf{C}$. The only such surface is $\mathbf{C}^2$, and an action on a G-vector bundle is linear.

Case 3. *X contains a hyperbolic fixed point.* Since $H_1(X; \mathbf{Z}) = 0$, in (6.10), we must have $g = 0$, $M = 1$, and $L = 0$. Hence $X/\mathbf{C}^* = \mathbf{C}$ and X contains no nontrivial closed orbit. Also, since $H_2(X; \mathbf{Z}) = 0$, $K = 1$. Thus X contains a unique fixed point and the graph $\Gamma(X)$ has the form:



By (5.5), X must be $\mathbf{C}^2$ with a linear action of $\mathbf{C}^*$ and a hyperbolic fixed point. □

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