Precise bounds for finite time blow-up of solutions to very general one space-dimensional nonlinear Neumann problems

Kurt Bryan,[*] Michael S. Vogelius,[†]

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Abstract

In this paper we analyze the asymptotic finite time blow-up of solutions to the heat equation with a nonlinear Neumann boundary flux in one space dimension. We perform a detailed examination of the nature of the blow-up, which can occur only at the boundary, and we provide tight upper and lower bounds for the blow-up rate for "arbitrary" nonlinear functions F, subject to very mild restrictions.

Keywords: blow-up, heat equation, nonlinear Neumann boundary condition.

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1 Introduction: the basic problem

Let $\Omega = (0,1) \subset \mathbb{R}$ and $u(x,t)$ a solution to the heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \ , \ \ 0 < x < 1, \ \ 0 < t < T \ \ , \tag{1}$$

for some $T > 0$, with the nonlinear Neumann boundary conditions

$$\begin{aligned} -\frac{\partial u}{\partial x}(0,t) &= F(u(0,t)) \ , \ \ 0 < t < T \ \ , & (2)\\ \frac{\partial u}{\partial x}(1,t) &= F(u(1,t)) \ , \ \ 0 < t < T \ \ , & (3) \end{aligned}$$

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[*] Department of Mathematics, Rose-Hulman Institute of Technology, Terre Haute, IN 47803.

[†] Department of Mathematics, Rutgers University, New Brunswick, NJ 08903.

and initial condition

$$u(x,0) = f(x)\ ,\quad 0 < x < 1\quad . \tag{4}$$

Precise assumptions about the function F will be stated later, but in general F should be strictly increasing and grow superlinearly. There is a higher space-dimensional version of the problem in which $u(x,t)$ satisfies $\frac{\partial u}{\partial t} - \triangle u = 0$ for $0 < t < T$ and $x \in \Omega \subset \mathbb{R}^N$, $N \geq 2$, with Neumann boundary condition $\frac{\partial u}{\partial \mathbf{n}} = F(u)$ on $\partial\Omega$.

The solution $u(x,t)$ will frequently "blow up"—that is, become unbounded—in finite time. In particular, it is well known, in any space dimension, that if the initial condition f is of one sign then blow-up in finite time is assured (if the function f changes sign then the solution need not blow up; see for instance [2].) Early results on blow-up for the heat equation with nonlinear boundary conditions were obtained in [9] and [13], where the authors demonstrate the inevitability of blow-up for certain types of nonlinear boundary conditions and initial data, as well as for variations of the heat equation itself. Considerable work has been done on the problem (1)-(4) to determine, for example, where in $\overline{\Omega}$ the function u will blow up (in general, on some subset of $\partial\Omega$), to provide upper and lower bounds on the time at which blow-up will occur, and to provide upper and lower bounds for the solution near blow-up. Most of this work has been done for nonlinear functions F of rather specific forms, typically $F(u) = u^p$ for $p > 1$ or $F(u) = e^u$. Many variations of the above basic model have also been examined. The articles [1, 5, 8] provide a more complete survey of results concerning the blow-up of solutions to the heat equation with these types of nonlinear boundary conditions.

In many papers, for example [4, 6, 7, 10, 11], special attention is given to the problem in one space dimension; see also [3]. In this case equations (1)-(4) can be distilled down to a pair of, or even a single, nonlinear integral equation(s). Results from the theory of nonlinear integral equations can thus be of use. In particular, [12] provides a nice survey of recent results for these types of integral equations. However, sharp results on the asymptotic growth of the solution near blow-up—whether for the heat equation or the integral equation formulation—exist only for the power law case $F(u) = u^p$, the exponential case $F(u) = e^u$, or simple variations of these.

As already indicated, the present paper considers only the one space-dimensional case. In the first part, Section 3, we present a simple proof of the well known fact that initial data of one sign leads to finite time blow-up, and we provide an upper bound on the time at which blow up must occur. In Section 4 and subsections we establish quite sharp upper and lower bounds on the growth (in time) of solutions near blow-up. The bounds are in terms of the functions $F(u)/u$ and $F'(u)$, or rather, their inverses. Our upper and lower bounds for the solution behavior near blow-up are new as far as the generality of F's are concerned. For F of polynomial or exponential type our bounds coincide with already known bounds.

2 Assumptions about F and an integral equation formulation

We begin by making some very general assumptions about F, namely that

$$F \in C^2(\mathbb{R}),\ F \text{ is odd } (F(-u) = -F(u)),\ F'(0) = 1, \text{ and } F''(u) > 0 \text{ for } u > 0. \tag{A1}$$

Note that the last assumption implies that $F'(u)$ is strictly increasing for $u > 0$. The assumption that $F'(0) = 1$ is for convenience; we really need only that $F'(0) > 0$. By the odd symmetry we have $F(0) = 0$. We shall also require that F grow superlinearly in the sense that

$$u\tfrac{F'(u)}{F(u)} \geq 1 + \delta_1, \quad u \geq M, \tag{A2}$$

for some positive constants M and δ_1. We note that

$$\frac{d}{du}\left(\frac{F(u)}{u^{1+\delta}}\right) = u^{\delta} F(u) \frac{u\tfrac{F'(u)}{F(u)} - (1+\delta)}{u^{2+2\delta}} \quad ,$$

and so according to assumption (A2) (and the fact that F is odd, with $F(u) > 0,\ u > 0$) it follows that

$$\frac{F(u)}{|u|^{1+\delta}} \ ,\ |u| \geq M \ , \qquad \text{is strictly increasing for any } \delta < \delta_1 \ , \\ \text{and non-decreasing for } \delta = \delta_1 \ . \tag{5}$$

The facts that $F(0) = 0$ and $F''(u) > 0$ for $u > 0$ are easily seen to imply that $F(u) < uF'(u)$ when $u > 0$, and therefore

$$\frac{F(u)}{|u|} \quad \text{is strictly increasing on all of } \mathbb{R} \ . \tag{6}$$

From (5) and (6) it follows that

$$\frac{|F(u)|}{|u|^{1+\delta_1}} \geq c > 0. \tag{7}$$

The facts that $F(0) = 0$ and $F'(0) = 1$ ensure that the quantity $|F(u)|/|u|^{1+\delta_1}$ tends to ∞ as u limits to 0. We note that the strict monotonicity of $F'(u)$ and $F(u)/u$ for $u > 0$ imply the existence of well defined (positive valued) inverses for these functions, say, for arguments larger than 1.

Define $u_0(t) := u(0,t)$ and $u_1(t) := u(1,t)$. A standard argument involving the fundamental solution to the heat equation and integration by parts shows that (provided u is sufficiently regular) the pair $(u_0(t), u_1(t))$ satisfy the coupled nonlinear integral equations

$$\begin{aligned} u_0(t) &= \int_0^t K_1(t-s)u_1(s)\,ds + \int_0^t K_3(t-s)F(u_1(s))\,ds \\ &\qquad + \int_0^t K_2(t-s)F(u_0(s))\,ds + \frac{1}{\sqrt{\pi t}}\int_0^1 e^{-\frac{x^2}{4t}} f(x)\,dx \qquad (8) \\ u_1(t) &= \int_0^t K_1(t-s)u_0(s)\,ds + \int_0^t K_3(t-s)F(u_0(s))\,ds \\ &\qquad + \int_0^t K_2(t-s)F(u_1(s))\,ds + \frac{1}{\sqrt{\pi t}}\int_0^1 e^{-\frac{(x-1)^2}{4t}} f(x)\,dx \qquad (9) \end{aligned}$$

where

$$K_1(t) = \frac{e^{-\frac{1}{4t}}}{2\sqrt{\pi}t^{3/2}} \quad , \quad K_2(t) = \frac{1}{\sqrt{\pi t}} \quad , \quad K_3(t) = \frac{e^{-\frac{1}{4t}}}{\sqrt{\pi t}} \quad . \qquad (10)$$

The solution $u(x,t)$ to the initial-boundary value problem (1)-(4) is then given by

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi t}}\int_0^1 e^{-\frac{(x-y)^2}{4t}} f(y)\,dy + \frac{1}{2}\int_0^t \frac{e^{-\frac{(x-1)^2}{4(t-s)}}}{\sqrt{\pi(t-s)}}F(u_1(s))\,ds \\ &+ \frac{1}{2}\int_0^t \frac{e^{-\frac{x^2}{4(t-s)}}}{\sqrt{\pi(t-s)}}F(u_0(s))\,ds + x\int_0^t \frac{e^{-\frac{x^2}{4(t-s)}}}{4\sqrt{\pi}(t-s)^{3/2}}u_0(s)\,ds \\ &+ (1-x)\int_0^t \frac{e^{-\frac{(x-1)^2}{4(t-s)}}}{4\sqrt{\pi}(t-s)^{3/2}}u_1(s)\,ds \quad . \qquad (11) \end{aligned}$$

for $0 < x < 1$ and $0 < t < T$. A standard contraction mapping argument shows that for any $f \in C^0[0,1]$ the equations (8)-(9) possess a unique continuous solution on an interval $[0,T)$, for some $T > 0$, and in fact both u_0 and u_1 belong to $C^\alpha[0,T)$ for any $\alpha < 1$. If f is non-negative then u_0, u_1, and $u(x,t)$ are non-negative; indeed, if $f(x) \geq 0$ and f is not identically zero, a simple maximum principle argument applied to (1)-(4) shows that for any fixed time $t_0 > 0$ we have $u(x,t_0) \geq f_0 > 0$ for some constant f_0 and all $x \in [0,1]$ (provided $u(x,t_0)$ exists).

3 Upper bound on the blow-up time

Let $H(u)$ be an anti-derivative for $1/F(u)$ for $u > 0$. From the assumptions (A1) and (A2) (in particular, the consequence (5)) it's easy to see that $H(u)$ is strictly increasing and has a finite limit as $u \to \infty$. By addition of a suitable constant we may assume that $\lim_{u\to\infty} H(u) = 0$. The function H is invertible, with $H(u) < 0$ for $u > 0$. Also, since $F(0) = 0$ and $F'(0) = 1$ we find that

$$H(u) = \ln(u) + O(1) \tag{12}$$

for u near 0. The inverse function, H^{-1}, is strictly increasing with $\lim_{z\to 0^-} H^{-1}(z) = \infty$.

Let "erf" denote the error function

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2}\, dx \quad , \quad z > 0 \quad ,$$

and let f_0 be a fixed positive number.

We note that $\mathrm{erf}(z) = \frac{2z}{\sqrt{\pi}} + O(z^3)$ for z near zero, and therefore $\mathrm{erf}(1/(2\sqrt{t}))$ behaves asymptotically like $1/\sqrt{\pi t}$ as $t \to \infty$. As a consequence of this and (12) it now follows that

$$-\sqrt{\pi t} H(f_0 \mathrm{erf}(1/(2\sqrt{t}))) \quad \text{behaves asymptotically like} \quad \frac{\sqrt{\pi t}}{2} \ln t \quad ,$$

as $t \to \infty$. From the fact that $\mathrm{erf}(z)$ limits to 1 as $z \to \infty$ we also easily see that

$$-\sqrt{\pi t} H(f_0 \mathrm{erf}(1/(2\sqrt{t}))) \quad \text{behaves asymptotically like} \ -H(f_0)\sqrt{\pi t} \quad ,$$

as $t \to 0_+$. Consequently the equation

$$t = -\sqrt{\pi t} H(f_0 \mathrm{erf}(1/(2\sqrt{t}))) \tag{13}$$

has at least one positive solution, and indeed it has a smallest positive solution t_1. We proceed to establish the following result.

Proposition 3.1 *Let F satisfy assumptions (A1) and (A2). Suppose $f(x) \geq f_0$ for some positive constant f_0, and let u_0, u_1 denote the solutions to (8)-(9). Let t_1 denote the smallest positive solution to (13). Then there exists $0 < t^* \leq t_1$ such that*

(1) *u_0 and u_1 are both in $C^\alpha[0, t^*)$ for some $\alpha > 1/2$,*
(2) *at least one of the functions u_0 and u_1 fails to be bounded as t approaches t^* .*

Proof: The contraction mapping argument referred to in the previous section establishes the existence of continuous solutions u_0 and u_1 on some interval $[0, t)$ with $t > 0$. Moreover, u_0 and u_1 must belong to $C^\alpha[0, t)$ for any $\alpha < 1$ on such an interval $[0, t)$. Define

$$T = \sup\{t > 0 \ : \ u_i \in C^\alpha[0, t) \cap L^\infty(0, t)\} \quad .$$

If T is finite then at least one of these two functions must fail to be in $L^\infty(0,T)$ (otherwise the contraction mapping argument yields existence beyond T). There are potentially two possibilities: (1) $T \leq t_1$, or (2) $T > t_1$. In the first scenario we know that at least one of the functions u_0 and u_1 is in $L^\infty(0,t)$ for any $t < T$ *but not* in $L^\infty(0,T)$. This partially verifies the proposition with $t^* = T \leq t_1$. We now proceed to show that the second scenario cannot occur, thus completing the proof of the proposition.

Let's focus on the left end of the domain ($x = 0$) and the function $u_0(t)$, and suppose we are in the second scenario. Consider equation (8). Since the functions u_0 and u_1 are positive and bounded we see that the integrals involving u_1 are positive and finite for any $t < T$. As a result we can drop these terms to obtain

$$u_0(t) \;\geq\; \int_0^t K_2(t-s)F(u_0(s))\,ds + \frac{1}{\sqrt{\pi t}}\int_0^1 e^{-\frac{x^2}{4t}} f(x)\,dx \tag{14}$$

for $0 < t < T$; the function u_1 has completely dropped out. We shall show that the above inequality alone implies that u_0 blows up at t_1 at the latest – a contradiction to the basic assumption of the second scenario ($T > t_1$). This will complete the proof of Proposition 3.1. Using the monotonicity of the error function we can bound

$$\frac{1}{\sqrt{\pi t}}\int_0^1 e^{-\frac{x^2}{4t}} f(x)\,dx \geq \frac{f_0}{\sqrt{\pi t}}\int_0^1 e^{-\frac{x^2}{4t}}\,dx = f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t}}\right) > f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t_1}}\right) \quad ,$$

for $0 \leq t < t_1$. From (14) we then have

$$u_0(t) \;>\; \int_0^t K_2(t-s)F(u_0(s))\,ds + f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t_1}}\right).$$

for $0 \leq t < t_1$. After insertion of the formula (10) for K_2 this yields

$$\begin{aligned} u_0(t) \;&>\; \frac{1}{\sqrt{\pi}}\int_0^t \frac{F(u_0(s))}{\sqrt{t-s}}\,ds + f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t_1}}\right) \\ &\geq\; \frac{1}{\sqrt{\pi t}}\int_0^t F(u_0(s))\,ds + f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t_1}}\right) \\ &\geq\; \frac{1}{\sqrt{\pi t_1}}\int_0^t F(u_0(s))\,ds + f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t_1}}\right) \quad , \end{aligned} \tag{15}$$

for $0 \leq t < t_1$. Let

$$v(s) = \int_0^s F(u_0(t))\,dt \quad ,$$

so that $v'(s) = F(u_0(s))$ and $u_0(s) = F^{-1}(v'(s))$. The estimate (15) now reads

$$F^{-1}(v'(s)) > \frac{1}{\sqrt{\pi t_1}} v(s) + f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t_1}}\right) \quad , \quad 0 \le s < t_1 \quad .$$

Since F is strictly increasing this implies $v'(s) > F[v(s)/\sqrt{\pi t_1} + f_0 \cdot \mathrm{erf}(1/(2\sqrt{t_1}))]$, which we may rewrite

$$\frac{v'(s)}{F\left[v(s)/\sqrt{\pi t_1} + f_0 \cdot \mathrm{erf}(1/(2\sqrt{t_1}))\right]} > 1 \quad , \quad 0 \le s < t_1 \quad . \tag{16}$$

As before, let $H(u) < 0$ be the anti-derivative of $1/F(u)$, $u > 0$, that satisfies $\lim_{u\to\infty} H(u) = 0$. Since $v(0) = 0$ integration of the inequality (16) from 0 to $t < t_1$ yields

$$\sqrt{\pi t_1} H\left[\frac{v(t)}{\sqrt{\pi t_1}} + f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t_1}}\right)\right] - \sqrt{\pi t_1} H\left[f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t_1}}\right)\right] > t \quad ,$$

so that

$$H\left[\frac{v(t)}{\sqrt{\pi t_1}} + f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t_1}}\right)\right] > \frac{t}{\sqrt{t_1 \pi}} + H\left[f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t_1}}\right)\right] \quad . \tag{17}$$

Note that the right hand side of (17) must be negative, and thus lies in the domain of definition of H^{-1}. We now apply H^{-1} (strictly increasing) to both sides of (17) to arrive at

$$v(t) > \left[H^{-1}\left[\frac{t}{\sqrt{t_1 \pi}} + H\left(f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t_1}}\right)\right)\right] - f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t_1}}\right)\right] \sqrt{\pi t_1} \quad . \tag{18}$$

As $t < t_1$ approaches t_1 the expression

$$\frac{t}{\sqrt{t_1 \pi}} + H\left(f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t_1}}\right)\right)$$

approaches

$$\sqrt{\frac{t_1}{\pi}} + H\left(f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t_1}}\right)\right) = 0 \quad .$$

Since $H^{-1}(z)$ tends to ∞ as $z \to 0^-$ it follows that

$$H^{-1}\left[\frac{t}{\sqrt{t_1 \pi}} + H\left(f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t_1}}\right)\right)\right] \to \infty \quad , \qquad \text{as } t \to t_1 \text{ from below.}$$

The inequality (18) implies that $v(t)$ becomes unbounded as t approaches t_1 from below, and so $u_0(t)$ also becomes unbounded as t approaches t_1 from below. Indeed, a direct combination of (15) and (18) yields the following lower bound for $u_0(t)$

$$\begin{aligned} u_0(t) &> \frac{1}{\sqrt{\pi t_1}} v(t) + f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t_1}}\right) \\ &> H^{-1}\left[\frac{t}{\sqrt{t_1 \pi}} + H\left(f_0 \cdot \mathrm{erf}\left(\frac{1}{2\sqrt{t_1}}\right)\right)\right] \quad , \quad 0 \le t < t_1 \quad . \end{aligned}$$

This establishes the impossibility of the second scenario ($T > t_1$), and completes the proof of the proposition.

Remark 1 In Proposition 3.1 we can relax the condition $f(x) \geq f_0$ to $f(x) \geq 0$ with f not identically 0. To see this simply note that if $f(x) \geq 0$ and is not identically 0 then the maximum principle guarantees that $u(x, \epsilon) \geq f_\epsilon$, where f_ϵ is a positive constant, for some fixed $\epsilon > 0$. We may now apply Proposition 3.1 with ϵ as the initial time. t_1 thus gets replaced by $t_1 + \epsilon$, where t_1 is the smallest positive solution to $t = -\sqrt{\pi t} H(f_\epsilon \cdot \mathrm{erf}(1/(2\sqrt{t})))$. Proposition 3.1 also holds if $f(x) \leq 0$ with f not identically 0. To prove this we simply replace u by $-u$, and note that, due to the odd symmetry of F, this function solves the same initial boundary value problem as u only with f replaced by $-f$.

Remark 2 As noted in the introduction, if the initial condition f changes sign then blow-up need not occur. In [2] we show that sign-changing solutions with certain symmetries may in fact decay to zero if f is "small enough", while other solutions with the same symmetries must blow-up in finite time.

4 Asymptotic bounds near blow-up

In the following two sections we provide a more detailed study of the asymptotic structure of solutions to equations (1)-(4) near the time of blow-up. The nonlinearity F is of a very general superlinear nature, with the very mild assumptions (A1) and (A2) described in Section 2. From the the representation formula (11) for $u(x, t)$ in terms of the initial data and $u(0, t)$, $u(1, t)$, it is clear that blow-up at time t^* always implies blow-up at one or both boundary points. As we shall see later (after establishing an upper bound for endpoint blow-up) the solution always stays bounded in the interior, even when boundary blow-up occurs. In general, blow-up will happen at one boundary point, and the solution will remain bounded at the other.

In the remainder of this section we shall assume that $u_0(t) := u(0,t)$ and $u_1(t) := u(1,t)$ are sufficiently smooth and defined on $[0,t^*)$, that u_1 becomes unbounded near $t^* < \infty$, but that u_0 remains bounded (the pathological case in which blow-up appears simultaneously at both endpoints requires a slightly different analysis which we do not present here.) For simplicity we assume that u_1 attains arbitrarily large positive values as t approaches t^*. From the integral formulation (9) we know that $u_1 \in C^\alpha[0,t^*)$ is a solution to

$$u_1(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{F(u_1(s))}{\sqrt{t-s}}\, ds + q(t) \quad , \tag{19}$$

where q is a C^1 function on the closed interval $[0,t^*]$. In Appendix A we prove the following monotonicity result.

Proposition 4.1 *Let $\phi(t)$ be a function in $C^\alpha[0,t^*)$ for some $\alpha > 1/2$, $0 < t^* < \infty$, and suppose ϕ satisfies an integral equation of the form*

$$P(\phi(t)) = \int_0^t \frac{H(s,\phi(s))}{\sqrt{t-s}}\, ds + q(t)$$

where $P \in C^1(-\infty,\infty)$ is strictly increasing, $q \in C^1[0,t^*]$, and $H \in C^1([0,t^*) \times (-\infty,\infty))$. Assume that $H(s,z)$ is nondecreasing in each argument with

$$\lim_{z\to\infty} H(s,z) = \infty$$

for each fixed $s \in [0,t^*)$, and assume that

$$\limsup_{t\to t^*} \phi(t) = \infty.$$

For $R \in (\inf \phi, \infty)$ let $t_R \in [0,t^*)$ be defined as

$$t_R = \inf\{\ t \in [0,t^*) \ :\ \phi(t) = R\ \} \quad .$$

Then there exists R such that $\phi(t)$ is strictly increasing for $t_R \le t < t^*$.

Remark 3 Proposition 4.1, in combination with the fact that u_1 attains arbitrarily large positive values as t approaches t^*, shows that u_1 is strictly increasing in some interval (t_1,t^*). By changing the initial time, if necessary, we may without loss of generality assume that u_1 is positive and strictly increasing on the whole interval $(0,t^*)$.

We now proceed to establish upper and a lower bounds for the blow-up of behavior of u_1 near t^*. In brief, we shall prove that

$$K_1 (F')^{-1}(C_1/\sqrt{t^*-t}) \leq u_1(t) \leq K_2 G^{-1}\left(C_2/(\sqrt{t^*-t})^\gamma\right) \tag{20}$$

for constants K_i and C_i, and γ larger than, but arbitrarily close to 1. The function G is defined as $G(u) = F(u)/u$. Note that as part of the assumption (A1), $F(0) = 0$ and $F''(u) > 0$ for $u > 0$. It follows immediately that $G \leq F'$, and therefore due to the strict monotonicity of F' and G (both are increasing for positive arguments) we obtain $(F')^{-1} \leq G^{-1}$ for positive arguments. However, let us note that the bounds (20) are still true even when $F(0) \neq 0$ and provided only $F(u), F'(u)$, and $F''(u)$ are positive for u sufficiently large; see Remark 6. We suspect that the presence of $\gamma > 1$ in the upper bound is a technical artifact, but we are presently unable to derive the bound with $\gamma = 1$.

4.1 Lower bound for growth

Proposition 4.2 *Let F satisfy assumptions (A1) and (A2). Let $u_0(t)$ and $u_1(t)$ be $C^\alpha[0,t^*)$ solutions to equations (8)-(9) for some $0 < t^* < \infty$. Suppose that $u_0(t)$ remains bounded and that $u_1(t)$ attains arbitrarily large positive values for t near t^*. Let K^* denote the constant $K^* = 1 - \limsup_{z\to\infty} \frac{F(z)}{zF'(z)} > 0$, and let $0 < K < K^*$, and $0 < C < \sqrt{\pi}/2$. Then*

$$u_1(t) \geq K (F')^{-1}(C/\sqrt{t^*-t}) \quad ,$$

for t near t^*.

Proof: As discussed in Remark 3, we may without loss of generality assume that $u_1(t)$ is positive and strictly increasing for $0 < t < t^*$. For $0 < t < t+h < t^*$ we have from equation (19)

$$\begin{aligned}
u_1(t+h) &= \frac{1}{\sqrt{\pi}} \int_0^{t+h} \frac{F(u_1(s))}{\sqrt{t+h-s}}\, ds + q(t+h) \\
&= \frac{1}{\sqrt{\pi}} \int_0^{t} \frac{F(u_1(s))}{\sqrt{t+h-s}}\, ds + \frac{1}{\sqrt{\pi}} \int_t^{t+h} \frac{F(u_1(s))}{\sqrt{t+h-s}}\, ds + q(t+h) \\
&\leq \frac{1}{\sqrt{\pi}} \int_0^{t} \frac{F(u_1(s))}{\sqrt{t-s}}\, ds + F(u_1(t+h)) \frac{1}{\sqrt{\pi}} \int_t^{t+h} \frac{1}{\sqrt{t+h-s}}\, ds + q(t+h) \\
&= u_1(t) + 2\frac{\sqrt{h}}{\sqrt{\pi}} F(u_1(t+h)) + q(t+h) - q(t) \quad ,
\end{aligned}$$

where the inequality follows from $0 \le F(u_1(s)) \le F(u_1(t+h))$ for $0 \le s \le t+h$. We now use the fact that $|q(t+h) - q(t)| \le C_q h$ to obtain

$$u_1(t+h) - 2\frac{\sqrt{h}}{\sqrt{\pi}} F(u_1(t+h)) \le u_1(t) + C_q h \quad , \tag{21}$$

for $0 < t < t+h < t^*$. Note that since $q \in C^1[0, t^*]$ we can choose C_q independently of t and h. Let G_h denote the function

$$G_h(x) = x - 2\frac{\sqrt{h}}{\sqrt{\pi}} F(x) \quad .$$

For a given $z > u_1(t)$ consider that (unique) h such that $u_1(t+h) = z$. Equation (21) shows that

$$G_h(z) \le u_1(t) + C_q h.$$

The above inequality leads to a lower bound for h (e.g., when $h = 0$ the inequality is violated, for it becomes $z \le u_1(t)$). For fixed $z > u_1(t)$ the quantity $G_h(z)$ is strictly decreasing in h, and limits to $-\infty$ as h increases, so there is a unique $h_z > 0$ for which

$$G_{h_z}(z) = u_1(t) + C_q h_z. \tag{22}$$

This h_z provides a lower bound for the value of h for which $u_1(t+h) = z$: in other words, $h_z \le h$. We can solve equation (22) for h_z to arrive at

$$h_z = \frac{F^2(z)}{C_q^2 \pi} (2 + p - 2\sqrt{1+p})$$

with $p = \frac{C_q \pi (z - u_1(t))}{F^2(z)}$. A simple (but asymptotically sharp) lower bound on h_z can be obtained from the fact that $2 + p - 2\sqrt{1+p} \ge \frac{p^2}{2(p+2)}$ for all $p \ge 0$. We obtain

$$\begin{aligned} h_z &\ge \frac{F^2(z)}{C_q^2 \pi} \frac{p^2}{2(p+2)} \\ &= \frac{\pi (z - u_1(t))^2}{4F^2(z) + 2C_q \pi (z - u_1(t))} \quad . \end{aligned}$$

Due to the superlinearity of F we obtain, for any $C_0 < 1$ (and sufficiently large values of $u_1(t)$) that

$$h \ge h_z \ge C_0 \frac{\pi (z - u_1(t))^2}{4F^2(z)} \tag{23}$$

for any $z \geq u_1(t)$, where $u_1(t+h) = z$. The constant C_0 is independent of t and h.

Let $v = u_1^{-1}$, which is well-defined, since u_1 is strictly increasing. Using the notation $u_1(t) = z_1$ we have $v(z) - v(z_1) = h$, and the inequality (23) may be written

$$v(z) - v(z_1) \geq C_1 \frac{(z - z_1)^2}{F^2(z)} \tag{24}$$

with $C_1 = \frac{C_0 \pi}{4}$. This inequality holds for any sufficiently large z_1 and all $z \geq z_1$. We know that u_1 blows up monotonically as $t \to t^*$, and so v is strictly increasing, with

$$\lim_{z \to \infty} v(z) = t^*.$$

For $\epsilon > 0$ sufficiently small we now choose $z_1 = u_1(t^* - \epsilon)$ so that $v(z_1) = t^* - \epsilon$. Since $v(z) < t^*$, $z_1 \leq z$, the estimate (24) now yields

$$C_1 \frac{(z - z_1)^2}{F^2(z)} \leq v(z) - v(z_1) < \epsilon \tag{25}$$

for all $z \geq z_1$. We get as much out of the estimate (25) as possible by taking that $z \geq z_1$ which maximizes the left hand side. It's easy to see that there is a unique such z, for the left side of (25) equals 0 when $z = z_1$, tends to zero as $z \to \infty$, and has a stationary point $z = z_1^*$ that satisfies

$$z_1^* - \frac{F(z_1^*)}{F'(z_1^*)} = z_1, \tag{26}$$

easily found by differentiating the left side of (25). That the equation (26) has a unique solution $z_1^* > z_1$ is a consequence of the fact that the function $z - \frac{F(z)}{F'(z)}$, $z_1 \leq z$, is strictly increasing (a consequence of $F'' > 0$) with

$$z_1 - \frac{F(z_1)}{F'(z_1)} < z_1 \quad , \text{ and } \quad \lim_{z \to \infty} z - \frac{F(z)}{F'(z)} = \infty \quad .$$

The latter assertion is a consequence of assumption (A2), which implies

$$\frac{F(z)}{zF'(z)} \leq \frac{1}{1 + \delta_2} < 1 \tag{27}$$

for some $\delta_2 > 0$, and so

$$z - \frac{F(z)}{F'(z)} = z \left(1 - \frac{F(z)}{zF'(z)} \right) \to \infty \quad \text{as } z \to \infty \quad . \tag{28}$$

From (27) and (28) it furthermore follows that with K^* as defined in the statement of the proposition

$$K^* = 1 - \limsup_{z\to\infty} \frac{F(z)}{zF'(z)} \geq \frac{\delta_2}{1+\delta_2} > 0 \quad ,$$

and that, for any $K < K^*$

$$Kz_1^* < z_1^* - \frac{F(z_1^*)}{F'(z_1^*)} = z_1$$

for z_1 sufficiently large. Of course this implies that

$$z_1^* \leq \frac{z_1}{K} \quad , \tag{29}$$

for z_1 sufficiently large. If we use $z = z_1^*$ in (25) then (26) gives

$$C_1 \frac{1}{(F'(z_1^*))^2} \leq \epsilon \quad ,$$

which in combination with (29), and the fact that F' is increasing, yields

$$C_1 \frac{1}{(F'(z_1/K))^2} \leq \epsilon$$

for z_1 sufficiently large. We conclude that, with $C = \sqrt{C_1}$,

$$z_1 \geq K(F')^{-1}(C/\sqrt{\epsilon}) \quad , \tag{30}$$

for z_1 sufficiently large. Inequality (30) yields a lower bound on the growth of u_1 near t^* if we note that $z_1 = u_1(t^* - \epsilon)$, namely

$$u_1(t^* - \epsilon) \geq K(F')^{-1}(C/\sqrt{\epsilon}) \quad , \quad \text{for } \epsilon \text{ sufficiently small} \ ,$$

or

$$u_1(t) \geq K(F')^{-1}(C/\sqrt{t^* - t}) \quad , \quad \text{for } t \text{ near } t^* \quad .$$

Here $C = \sqrt{C_1} = \sqrt{C_0\pi}/2 < \sqrt{\pi}/2$ can be arbitrarily close to $\sqrt{\pi}/2$, and $K < K^*$ can be arbitrarily close to K^*. This proves Proposition 4.2.

Remark 4 To illustrate the above bound, let us consider a couple of typical examples. With $F(u) = \sinh(u)$ it's easy to check that $K^* = 1$. In this case Proposition 4.2 yields a lower bound of the form

$$u_1(t) \geq K \operatorname{arccosh}(C/(t^* - t)^{1/2}) = -\frac{K}{2} \ln(t^* - t) + O(1) \quad ,$$

for any $K < 1$.

For $F(u) = |u|^{p-1}u + u$ with $p > 1$, Proposition 4.2 gives

$$u_1(t) \geq K_p (t^* - t)^{-1/(2(p-1))} \quad ,$$

for some positive constant K_p, smaller than, but arbitrarily close to $(\sqrt{\pi}/2)^{1/(p-1)} p^{-p/(p-1)} (p-1)$. It is easy to check that K_p must approach 0 as $p \to 1$, and that K_p may be picked arbitrarily close to 1 as $p \to \infty$.

4.2 Upper bound for growth

In order to establish an upper bound on the solution near blow-up we shall make one more assumption on the behavior of F, in addition to those of Section 2. We shall require that

$$\text{There exists } M > 0 \text{ such that } F'(u)/F(u) \text{ is non-increasing for } u \geq M. \tag{A3}$$

This assumption and those of Section 2 are all satisfied, for example, by $F(u) = |u|^{p-1}u + u$, $p > 1$, and $F(u) = \sinh(u)$. Assumption (A3) implies that $F'(u)/F(u) \leq K_2$ for $M \leq u$ and some constant K_2. Integration then yields $F(u) \leq Ce^{K_2 u}$ for $M \leq u$, and from continuity we obtain $F(u) \leq C_2 e^{K_2 u}$ for some constant C_2 and all $u > 0$. In combination with (7) we conclude that our assumptions about F imply the existence of constants $K_1 > 1$ and positive constants K_2, C_1, C_2 so that

$$C_1 u^{K_1} \leq F(u) \leq C_2 e^{K_2 u} \text{ for all } u > 0.$$

The apriori assumptions about the solution u near t^* are as in the previous section; that is, we assume that $u_0(t) := u(0, t)$ and $u_1(t) := u(1, t)$ are smooth and defined on $[0, t^*)$, that u_1 becomes unbounded near t^*, but that u_0 remains bounded. For simplicity we assume that u_1 attains arbitrarily large positive values as t approaches t^*. Finally we recall that $u_1 \in C^\alpha[0, t^*)$ (any $\alpha < 1$) satisfies the integral equation (19) where q is a C^1 function on the closed interval $[0, t^*]$.

Proposition 4.3 *Let the assumptions be as in Proposition 4.2, and additionally suppose that (A3) holds. Then for any $\gamma > 1$ and any $K > 1$ there exists a constant C such that*

$$u_1(t) \leq K G^{-1}\left(C/(\sqrt{t^* - t})^\gamma\right)$$

for t near t^*. Here $G(u) = F(u)/u$.

Proof: As argued in Remark 3, we may without loss of generality assume that $u_1(t)$ is positive and strictly increasing for $0 < t < t^*$. Let b_n, $n \geq 0$, be a sequence that limits to infinity, starting with $b_0 > u_1(0)$. Indeed, let us suppose that we have $b_n = b(n)$ for some strictly increasing function $b(x)$ of a real variable x. Let $a_n \in (0, t^*)$ be chosen (uniquely) so that $u_1(a_n) = b_n$. Note that $0 < a_0 < a_1 < \ldots < a_n < a_{n+1} < \ldots$ with $a_n \to t^*$, as $n \to \infty$. We then have

$$\begin{aligned}
b_n &= u_1(a_n) \\
&= \int_0^{a_n} \frac{F(u_1(s))\,ds}{\sqrt{a_n - s}} + q(a_n) \\
&= \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} \frac{F(u_1(s))\,ds}{\sqrt{a_n - s}} + q(a_n) \\
&\geq \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} \frac{F(u_1(a_k))\,ds}{\sqrt{a_n - s}} + q(a_n) \\
&= \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} \frac{F(b_k)\,ds}{\sqrt{a_n - s}} + q(a_n) \\
&= 2\sum_{k=0}^{n-1} F(b_k)(\sqrt{a_n - a_k} - \sqrt{a_n - a_{k+1}}) + q(a_n) \\
&\geq 2F(b_{n-1})\sqrt{a_n - a_{n-1}} - Q \quad , \qquad (31)
\end{aligned}$$

where we use that u is strictly increasing, and $q(a_n) \geq -Q$, and drop all but the $k = n-1$ term in the last sum. A little rearrangement of (31) yields

$$a_n - a_{n-1} \leq \frac{(b_n + Q)^2}{4F^2(b_{n-1})} \quad ,$$

and a straightforward telescoping summation then shows

$$a_n - a_m \leq \frac{1}{4} \sum_{k=m+1}^{n} \frac{(b_k + Q)^2}{F^2(b_{k-1})} \quad .$$

We let n tend to infinity to find

$$t^* - a_m \quad \leq \quad \frac{1}{4} \sum_{k=m+1}^{\infty} \frac{(b_k + Q)^2}{F^2(b_{k-1})} \quad .$$

From now on we take $b_k = b_0 d^k$ for some fixed $d > 1$, and so based on the previous estimate

$$\begin{aligned} t^* - a_m &\leq \frac{1}{4}\sum_{k=m+1}^{\infty} \frac{(b_k+Q)^2}{F^2(b_{k-1})} \\ &= \frac{1}{4}\sum_{k=m+1}^{\infty} \frac{(db_{k-1}+Q)^2}{F^2(b_{k-1})} \\ &= \frac{1}{4}\sum_{k=m}^{\infty} \frac{(db_k+Q)^2}{F^2(b_k)} \quad , \quad m \geq 0 \quad . \end{aligned}$$

Since Q is fixed and $b_k \to \infty$, as $k \to \infty$, it follows that, given any $D > 1$,

$$t^* - a_m \leq \frac{d^2 D}{4} \sum_{k=m}^{\infty} \frac{b_k^2}{F^2(b_k)} \tag{32}$$

for m sufficiently large. Define

$$\phi(m) = \sum_{k=m}^{\infty} \frac{b_k^2}{F^2(b_k)} = b_0^2 \sum_{k=m}^{\infty} \frac{d^{2k}}{F^2(b_0 d^k)} \quad . \tag{33}$$

Note that $\phi > 0$, and that ϕ is strictly decreasing in m, with $\phi(m) \to 0$ as $m \to \infty$. Of course, ϕ is defined only on the non-negative integers, but we may extend ϕ to a function with the same properties on the non-negative reals. From (32) we have

$$t^* - a_m \leq \frac{d^2 D}{4}\phi(m) = \frac{d^2 D}{4}\phi(b^{-1}(u(a_m))) \quad ,$$

since $b^{-1}(u(a_m)) = b^{-1}(b_m) = m$. Divide both sides above by $d^2 D/4$ and apply the strictly decreasing function $\psi = b \circ \phi^{-1}$ to both sides (which reverses the inequality) to find

$$u(a_m) \leq \psi\left(\frac{4}{d^2 D}(t^* - a_m)\right) \tag{34}$$

for m sufficiently large. We note that $\psi(s) \to \infty$ as $s \to 0^+$. Given ϵ sufficiently small we have $t^* - \epsilon \in [a_{m-1}, a_m)$ for some $m \geq 1$ (with $m \to \infty$ as $\epsilon \to 0$). The estimate (34) then yields (recall $u(a_m) = b_m$)

$$
\begin{aligned}
u(t^* - \epsilon) \leq u(a_m) &= \frac{b_m}{b_{m-1}} u(a_{m-1}) \\
&\leq d\psi\left(\frac{4}{d^2 D}(t^* - a_{m-1})\right) \\
&\leq d\psi\left(\frac{4\epsilon}{d^2 D}\right) \quad , \qquad (35)
\end{aligned}
$$

for m sufficiently large, or ϵ sufficiently small. Here we can take any $D > 1$ (though D closer to 1 may require smaller ϵ). It only remains to bound ψ. Comparison to an integral shows that, with ϕ defined by (33), we have

$$
b_0^2 \int_m^\infty \frac{d^{2x}}{F^2(b_0 d^x)}\, dx \leq \phi(m) \leq b_0^2 \int_{m-1}^\infty \frac{d^{2x}}{F^2(b_0 d^x)}\, dx
$$

where we make use of $z/F(z)$ strictly decreasing for z large. The change of variables $y = b_0 d^x$ yields

$$
\frac{1}{\ln(d)} \int_{b_0 d^m}^\infty \frac{y}{F^2(y)}\, dy \leq \phi(m) \leq \frac{1}{\ln(d)} \int_{b_0 d^{m-1}}^\infty \frac{y}{F^2(y)}\, dy \quad . \qquad (36)
$$

Now $\psi = b \circ \phi^{-1}$, so that $\psi^{-1} = \phi \circ b^{-1}$, that is, $\psi^{-1}(p) = \phi(\log_d(p/b_0))$. In conjunction with (36) this yields bounds

$$
\frac{1}{\ln(d)} \int_p^\infty \frac{y}{F^2(y)}\, dy \leq \psi^{-1}(p) \leq \frac{1}{\ln(d)} \int_{p/d}^\infty \frac{y}{F^2(y)}\, dy \quad . \qquad (37)
$$

Lemma B.1 of Appendix B asserts the existence of constants C_1 and $0 < \beta \leq 1$, (dependent upon F and d, of course) such that

$$
\int_{p/d}^\infty \frac{y}{F^2(y)}\, dy \leq C_1 \left(\frac{p^2}{F^2(p)}\right)^\beta \quad \text{for all } p > 1 \quad ,
$$

which upon combination with the last estimate of (37) gives the upper bound

$$
\psi^{-1}(p) \leq \frac{C_1}{\ln(d)} \left(\frac{p^2}{F^2(p)}\right)^\beta \quad . \qquad (38)
$$

The constant C_1 may be taken arbitrarily close to $\frac{1}{2\delta_1}$, where δ_1 is the constant from the superlinearity assumption (A2). If we define $G(z) = F(z)/z$ and $L(z) = \frac{C_1}{\ln(d)} \frac{1}{z^{2\beta}}$ then inequality (38) can be written $\psi^{-1}(p) \leq L(G(p))$ for p sufficiently large. Since ψ is strictly decreasing in its argument, and since $L(G(p)) \to 0$ as $p \to \infty$ (note $L(G(p))$ is also strictly decreasing in p), we conclude that $\psi(z) \leq G^{-1}(L^{-1}(z))$ for z sufficiently close to zero. This yields

$$\psi(z) \leq G^{-1}\left(\frac{C_2}{(\sqrt{z})^{1/\beta}}\right)$$

where $C_2 = \left(\frac{C_1}{\ln(d)}\right)^{1/(2\beta)}$. In combination with (35) this gives

$$u(t^* - \epsilon) \leq dG^{-1}\left(\frac{C_2 D^{1/(2\beta)} d^{1/\beta}}{2^{1/\beta}(\sqrt{\epsilon})^{1/\beta}}\right) \quad ,$$

or

$$u(t) \leq dG^{-1}\left(\frac{C_2 D^{1/(2\beta)} d^{1/\beta}}{2^{1/\beta}(\sqrt{t^* - t})^{1/\beta}}\right) \quad .$$

This is exactly an estimate of the type asserted in the statement of this proposition, with $\gamma = 1/\beta > 1$, $K = d > 1$ and $C = \frac{(C_1 D)^{1/(2\beta)} d^{1/\beta}}{(\ln d)^{1/(2\beta)} 2^{1/\beta}}$. As d and D approach 1, γ approaches 1, K approaches 1 and C approaches $\frac{C_1^{1/2}}{2(\ln d)^{1/2}}$, which may be picked arbitrarily close to $\frac{1}{2\sqrt{2}(\delta_1 \ln d)^{1/2}}$. These last observations all follow from Remark 8 of Appendix B.

Remark 5 As an example, we may consider the function $F(u) = |u|^{p-1}u + u$ for $p > 1$. In this case $G(u) = u^{p-1} + 1$, $u > 0$, so that $G^{-1}(z) = (z-1)^{1/(p-1)} \leq C'/z^{1/(p-1)}$ for any $C' > C$ and z sufficiently large. The upper bound on $u_1(t)$ then becomes

$$u_1(t) \leq K'(t^* - t)^{-\gamma/(2(p-1))}$$

for a suitable constant K'. This bound is (modulo γ) of the same form as the lower bound in Remark 4.

As another example, consider $F(u) = \sinh(u)$. Here $G(u) = \sinh(u)/u$ and $G^{-1}(z) = \ln(z) + o(\ln(z))$ as $z \to \infty$. We then obtain the bound

$$u_1(t) \leq -\frac{\gamma K}{2}\ln(t^* - t) + O(\ln|\ln(t^* - t)|) \quad ,$$

where γ and K can be taken arbitrarily close to 1.

Remark 6 For the validity of Propositions 4.2 and 4.3 it is irrelevant whether F is odd and whether $F'(0) = 1$. More precisely, the assumption (A1) may there be replaced by the condition that $F \in C^2(\mathbb{R})$, with F, F', and F'' being positive for sufficiently large argument.

4.3 Boundedness of the solution in the interior

In this section we show that for each fixed $x \in (0,1)$ the solution $u(x,t)$ remains bounded for $0 \le t < t^*$. We suppose that u_0 and u_1 are both defined on $[0,t^*)$, and that u_1 (or for that matter, both u_1 and u_0) blows up at $t = t^*$ in a manner controlled by the estimate in Proposition 4.3. As per Remark 3 we may assume that u_1 grows monotonically as $t \to t^*$. According to (7), the strictly increasing function $G(u) = F(u)/u$ satisfies $cu^{\delta_1} \le G(u)$ for $u \ge 0$, where $\delta_1 > 0$ is the constant from the superlinearity condition (A2), so that $G^{-1}(u) \le \tilde{c}u^{1/\delta_1}$ for some constant $\tilde{c}$. The upper bound from Proposition 4.3 then guarantees that for some constant C

$$u_1(t) \le C/(\sqrt{t^*-t})^{\gamma/\delta_1} \quad \text{for } t \text{ close to } t^*, \tag{39}$$

where γ can be chosen arbitrarily close to 1.

We also need a bound on $F(u_1(t))$, and so we shall introduce one additional assumption on F, namely that for fixed $K > 1$, sufficiently close to 1,

$$\text{there exist constants } C_K, A_K \text{ and } M_K \text{ such that } F(Ku) \le C_K F(u)^{A_k} \text{ for } u > M_K. \tag{A4}$$

This condition is, in some sense, another type of exponential bound on the growth of F, and is easily checked for any specific function, e.g., it holds for $F(x) = |x|^{p-1}x + x$ or $F(x) = \sinh(ax)$. With $K > 1$ being the constant from Proposition 4.3 we have

$$\begin{aligned} F(u_1(t)) &= u_1(t)G(u_1(t)) \\ &\le u_1(t)G\left(KG^{-1}\left(\frac{C}{(t^*-t)^{\gamma/2}}\right)\right). \end{aligned} \tag{40}$$

Note that K may be chosen arbitrarily close to 1. We thus also have, for any $z \ge M_K$,

$$\begin{aligned} G(KG^{-1}(z)) &= \frac{F(KG^{-1}(z))}{KG^{-1}(z)} \\ &\le C_K \frac{(F(G^{-1}(z)))^{A_K}}{KG^{-1}(z)} \\ &= \tilde{C}_K z^{A_k} (G^{-1}(z))^{A_K-1} \end{aligned} \tag{41}$$

with $\tilde{C}_K = C_K/K$. Here we make use of the assumption (A4) and the fact that $F(G^{-1}(z)) = zG^{-1}(z)$. If we combine the estimates (40), (41), and (39) (using $z = C/(\sqrt{t^*-t})^{\gamma}$) we obtain

$$F(u_1(t)) \le \frac{\tilde{C}}{(\sqrt{t^*-t})^{A_K\gamma(1+1/\delta_1)}} \quad . \tag{42}$$

From (11) we know that $u(x,t)$ $0 < x < 1$, $0 < t < t^*$ may be represented as

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi t}} \int_0^1 e^{-\frac{(x-y)^2}{4t}} f(y)\,dy + \frac{1}{2}\int_0^t \frac{e^{-\frac{(x-1)^2}{4(t-s)}}}{\sqrt{\pi(t-s)}} F(u_1(s))\,ds \\ &+ \frac{1}{2}\int_0^t \frac{e^{-\frac{x^2}{4(t-s)}}}{\sqrt{\pi(t-s)}} F(u_0(s))\,ds + x\int_0^t \frac{e^{-\frac{x^2}{4(t-s)}}}{4\sqrt{\pi}(t-s)^{3/2}} u_0(s)\,ds \\ &+ (1-x)\int_0^t \frac{e^{-\frac{(x-1)^2}{4(t-s)}}}{4\sqrt{\pi}(t-s)^{3/2}} u_1(s)\,ds \quad . \end{aligned}$$

It is clear that the integral involving the initial condition f stays bounded (in any norm) as t approaches t^*. The estimates (39) and (42) show that $u_1(s)$ and $F(u_1(s))$ grow at most like a negative power of $t^* - s$ as $s \to t^*$. At any fixed interior point $0 < x_0 < 1$ the kernels in the corresponding two integrals (and all their derivatives) behave like $e^{-c/(t-s)}$ for s near t, with $c > 0$. For this reason it follows immediately that the integrals involving $u_1(s)$ and $F(u_1(s))$ stay bounded as $t \to t^*$ for any fixed $x = x_0 \in (0,1)$. The same argument applies to the integrals involving $u_0(s)$ and $F(u_0(s))$ if u_0 blows up in a matter controlled by the estimate of Proposition 4.3. In summary we have proven:

> Under the additional assumption (A4), the solution $u(x_0,t)$ stays bounded (in any norm) in a neighborhood of any fixed interior point $x_0 \in (0,1)$, as $t \to t^*$.

A Appendix: A monotonicity result

The goal of this section is to give a proof of Proposition 4.1. Before we do so it will be useful to establish the following two closely related lemmata.

Lemma A.1 *Let $\phi(t)$ be a function in $C^\alpha[0,t^*)$ for some $\alpha > 1/2$, $0 < t^* < \infty$, and suppose ϕ satisfies an integral equation of the form*

$$P(\phi(t)) = \int_0^t \frac{H(s,\phi(s))}{\sqrt{t-s}}\,ds + q(t)$$

where $P \in C^1(-\infty,\infty)$ is strictly increasing, $q \in C^1[0,t^*]$, and $H \in C^1([0,t^*) \times (-\infty,\infty))$. Assume that $H(s,z)$ is nondecreasing in each argument and that

$$\lim_{z\to\infty} H(s,z) = \infty$$

for each fixed $s \in [0,t^*)$. *Suppose additionally that* $\limsup_{t\to t^*}\phi(t) = \infty$. *For* $R \in (\inf\phi,\infty)$ *let* $t_R \in [0,t^*)$ *be defined as*

$$t_R = \inf\{\ t\in[0,t^*)\ :\ \phi(t) = R\ \}\quad .$$

Then for any sufficiently large R *there exists* $h_R > 0$ *so that*

$$\phi(t_R) < \phi(t_R + h) \quad \textit{for } 0 < h \le h_R\quad .$$

We may use a common value $h_R = h^*_I > 0$ *for all* R *(sufficiently large but) in a bounded interval* I.

Proof: For convenience define an operator Q as

$$Q(\phi)(t) = \int_0^t \frac{H(s,\phi(s))}{\sqrt{t-s}}\,ds + q(t)\quad .$$

Given that the function P is strictly increasing we can establish the lemma by showing that $Q(\phi)(t_R+h) - Q(\phi)(t_R) > 0$ when $h > 0$, that is,

$$\begin{aligned} 0 \ &<\ \int_0^{t_R+h}\frac{H(s,\phi(s))}{\sqrt{t_R+h-s}}\,ds + q(t_R+h) - \int_0^{t_R}\frac{H(s,\phi(s))}{\sqrt{t_R-s}}\,ds - q(t_R) \\ &=\ \int_{t_R}^{t_R+h}\frac{H(s,\phi(s))}{\sqrt{t_R+h-s}}\,ds + \int_0^{t_R} H(s,\phi(s))\left(\frac{1}{\sqrt{t_R+h-s}} - \frac{1}{\sqrt{t_R-s}}\right)ds \\ &\qquad + q(t_R+h) - q(t_R)\quad . \end{aligned}$$

This inequality can be written

$$\begin{aligned} \int_0^{t_R} H(s,\phi(s))\left(\frac{1}{\sqrt{t_R-s}} - \frac{1}{\sqrt{t_R+h-s}}\right)ds + q(t_R) - q(t_R+h) \\ < \int_{t_R}^{t_R+h}\frac{H(s,\phi(s))}{\sqrt{t_R+h-s}}\,ds\quad . \end{aligned} \tag{43}$$

Since $q \in C^1[0,t^*]$ we have $|q(t_R) - q(t_R+h)| \le C_q h$ for some constant C_q independent of t_R and h. Inequality (43), and hence $Q(\phi)(t_R+h) - Q(\phi)(t_R) > 0$, will hold if we obtain

$$\int_0^{t_R} H(s,\phi(s))\left(\frac{1}{\sqrt{t_R-s}} - \frac{1}{\sqrt{t_R+h-s}}\right)ds + C_q h < \int_{t_R}^{t_R+h}\frac{H(s,\phi(s))}{\sqrt{t_R+h-s}}\,ds\quad , \tag{44}$$

for $0 < h < h_R$. We will establish, under suitable circumstances, an upper bound I_1 on the left side of (44), and a lower bound I_2 on the right side of (44), with $I_1 < I_2$. This will establish (44) and the lemma.

We first obtain a lower bound for the integral on the right in (44). Let R be any positive value in interval $(\inf \phi, \infty)$, and let t_R be as in the statement of the lemma. For some constant $C_{\alpha,R}$ we have

$$\frac{|\phi(t) - \phi(t_R)|}{|t - t_R|^\alpha} \leq C_{\alpha,R} \quad ,$$

for all t in a neighborhood of t_R. The constant $C_{\alpha,R}$ is uniformly bounded for R in any compact subinterval of $(\inf \phi, \infty)$. In particular, for all sufficiently small h we have

$$|\phi(t_R + h) - R| \leq C_{\alpha,R} h^\alpha \quad ,$$

since $\phi(t_R) = R$. As a consequence $\phi(t_R + h) \geq R - C_{\alpha,R} h^\alpha$, and for $s \in [t_R, t_R + h]$ we have

$$\phi(s) \geq R - C_{\alpha,R} h^\alpha \quad .$$

For sufficiently small h the integral on the right side in (44) is then bounded below by

$$\begin{aligned} \int_{t_R}^{t_R+h} \frac{H(s, \phi(s))}{\sqrt{t_R + h - s}}\, ds &\geq \int_{t_R}^{t_R+h} \frac{H(t_R, R - C_{\alpha,R} h^\alpha)}{\sqrt{t_R + h - s}}\, ds \\ &= 2\sqrt{h} H(t_R, R - C_{\alpha,R} h^\alpha) \quad . \end{aligned}$$

Here we have made use of the fact that H is nondecreasing in each argument. Since H is C^1 we find that

$$\int_{t_R}^{t_R+h} \frac{H(s, \phi(s))}{\sqrt{t_R + h - s}}\, ds \geq 2\sqrt{h} \left[H(t_R, R) - \tilde{C}_{\alpha,R} h^\alpha \right] \tag{45}$$

for some constant $\tilde{C}_{\alpha,R}$. The constant $\tilde{C}_{\alpha,R}$ is uniformly bounded for R in any compact subinterval of $(\inf \phi, \infty)$. To obtain an upper bound on the integral on the left side in (44) we begin with

$$\begin{aligned} &\int_0^{t_R} H(s, \phi(s)) \left(\frac{1}{\sqrt{t_R - s}} - \frac{1}{\sqrt{t_R + h - s}} \right) ds \\ &\leq \int_0^{t_R} H(t_R, R) \left(\frac{1}{\sqrt{t_R - s}} - \frac{1}{\sqrt{t_R + h - s}} \right) ds \\ &= 2H(t_R, R)(\sqrt{t_R} + \sqrt{h} - \sqrt{t_R + h}) \quad , \end{aligned} \tag{46}$$

where the above inequality follows from $\phi(t) \leq R$ for $t \leq t_R$ and the properties of H. It is easy to see that

$$\sqrt{1+x} - 1 \geq \frac{1}{4}x$$

for all $0 \leq x \leq 8$. With $x = h/t$ this inequality becomes $\sqrt{1+h/t} - 1 \geq \frac{h}{4t}$ for $0 \leq h \leq 8t$. Multiplication by $-2\sqrt{t}$ and addition of $2\sqrt{h}$ to both sides yields

$$2(\sqrt{t} + \sqrt{h} - \sqrt{t+h}) \leq 2\sqrt{h} - \frac{h}{2\sqrt{t}} \tag{47}$$

for $0 \leq h \leq 8t$. By insertion of (47) into the right side of (46) we obtain the upper bound

$$\int_0^{t_R} H(s, \phi(s)) \left(\frac{1}{\sqrt{t_R - s}} - \frac{1}{\sqrt{t_R + h - s}} \right) ds \leq H(t_R, R) \left(2\sqrt{h} - \frac{h}{2\sqrt{t_R}} \right) \tag{48}$$

for suitably small h. Here we use that t_R is bounded away from 0, for instance $t^*/2 < t_R < t^*$, for R sufficiently large. If we make use of the upper and lower bounds (48) and (45) we see that the inequality (44) (and thus $Q(\phi)(t_R + h) > Q(\phi)(t_R)$) will be established if

$$C_q h + H(t_R, R) \left(2\sqrt{h} - \frac{h}{2\sqrt{t_R}} \right) < 2\sqrt{h} \left[H(t_R, R) - \tilde{C}_{\alpha,R} h^{\alpha} \right] \quad . \tag{49}$$

A cancelation and a bit of algebra show that (49) is equivalent to

$$\left(\frac{H(t_R, R)}{2\sqrt{t_R}} - C_q \right) h > 2\tilde{C}_{\alpha,R} h^{\alpha+1/2} \quad . \tag{50}$$

Due to the monotonicity of $H(\cdot, R)$, and since $0 \leq t_R < t^*$, we obtain (50) if we establish that for fixed $t_1 \in [0, t^*)$

$$\left(\frac{H(t_1, R)}{2\sqrt{t^*}} - C_q \right) h > 2\tilde{C}_{\alpha,R} h^{\alpha+1/2} \tag{51}$$

for all R sufficiently large (with $t_R \geq t_1$) and all h sufficiently small. Given the properties of H, the coefficient of h on the left side of (51) is clearly uniformly positive if R is chosen sufficiently large. Since the right side of (51) is $O(h^{\beta})$ with $\beta = \alpha + \frac{1}{2}$ we now see that (51) holds for all sufficiently small h. This establishes inequalities (50), (49), and (44), and the lemma. The fact that the constant $\tilde{C}_{\alpha,R}$ is uniformly bounded for R in any compact subinterval of $(\inf \phi, \infty)$ immediately implies that we may use a common $h_R = h_I^*$ for R sufficiently large, but in a bounded interval I.

As a consequence of the previous lemma we obtain

Lemma A.2 *Let the notation be as in Lemma A.1. There exists* R_0 *with the property that for any* $R \geq R_0$ *one may find* $\tilde{h}_R > 0$ *so that* ϕ *is strictly increasing on the interval* $[t_R, t_R + \tilde{h}_R]$.

Proof: For a given sufficiently large R, let h_R be as asserted by Lemma A.1. Let I be the compact interval $\phi([t_R, t_R + h_R])$, and let $\tilde{h}_R = h_I \leq h_R$ be the common increment that may be used for all values in I, according to Lemma A.1. We proceed to show that ϕ is strictly increasing on $[t_R, t_R + \tilde{h}_R]$. Suppose $t, \tilde{t} \in [t_R, t_R + \tilde{h}_R]$ with $t < \tilde{t}$. Define $T = \phi(t)$; since $\phi(s) < R = \phi(t_R) \leq \phi(t)$ for $s < t_R$ it follows immediately that $t_T \in [t_R, t_R + \tilde{h}_R]$. We also have $t_T \leq t < \tilde{t}$. From Lemma A.1, and the fact that the same $\tilde{h}_R$ may be used for all values in I (and thus for T) it now follows that

$$\phi(t_T) < \phi(s) \quad \text{for all} \quad s \in (t_T, t_T + \tilde{h}_R] \ .$$

In particular, since $t_T < \tilde{t} \leq t_R + \tilde{h}_R \leq t_T + \tilde{h}_R$, it follows that

$$\phi(t) = \phi(t_T) < \phi(\tilde{t}) \ ,$$

which completes the proof.

Using Lemma A.2 it is now fairly simple to give the proof of Proposition 4.1.

Proof of Proposition 4.1 : We shall show that ϕ is strictly increasing on the interval $[t_{R_0}, t^*)$, where R_0 and t_{R_0} are as in Lemma A.2. To this end we define

$$\tilde{t}^* = \sup\{ \ t \ : \ t_{R_0} \leq t < t^* \ , \text{ and } \phi \text{ is strictly increasing on the interval } [t_{R_0}, t] \ \} \ .$$

It is clear that $\tilde{t}^* \leq t^*$, and from Lemma A.2 we know that $t_{R_0} < \tilde{t}^*$. The function ϕ is strictly increasing on the interval $[t_{R_0}, t^*)$ if and only if $\tilde{t}^* = t^*$. We proceed by contradiction: Suppose $t_{R_0} < \tilde{t}^* < t^*$. By continuity we know that ϕ is strictly increasing on the interval $[t_{R_0}, \tilde{t}^*]$. If we define $T = \phi(\tilde{t}^*)$ then we have $R_0 < T$ and $t_T = \tilde{t}^*$, and so by Lemma A.2 we conclude that ϕ is strictly increasing on some interval $[t_T, t_T + h_T] = [\tilde{t}^*, \tilde{t}^* + h_T]$, $h_T > 0$. By combination with the strict monotonicity on $[t_{R_0}, \tilde{t}^*]$ this yields that ϕ is strictly increasing on the interval $[t_{R_0}, \tilde{t}^* + h_T]$, in contradiction to the definition of $\tilde{t}^*$. We therefore conclude that $\tilde{t}^* = t^*$, and this completes the proof of Proposition 4.1.

Remark 7 We note that a result entirely similar to that of Proposition 4.1 holds if H is non-increasing (in z) with $\lim_{z \to -\infty} H(s, z) = \infty$, P is strictly decreasing, and $\limsup_{t \to t^*} \phi(t) = -\infty$. The appropriate conclusion is then that $\phi(t)$ is strictly decreasing on some interval $[t_R, t^*)$ for R sufficiently negative.

B Appendix: An integral estimate

In this appendix we establish the following estimate, which was used for the verification of the upper bound in Section 4.2.

Lemma B.1 *Let $F \in C^2(\mathbb{R})$ with $F(u) > 0$, $u > 0$, be superlinear in the sense of (A2) and suppose $F'(u)/F(u) > 0$ is non-increasing for $u \geq M > 0$ as required by assumption (A3). For any $d > 1$ there exist constants $0 < \beta \leq 1$ and C such that*

$$\int_{p/d}^{\infty} \frac{y}{F^2(y)}\, dy \leq C \left(\frac{p^2}{F^2(p)} \right)^{\beta} \quad \text{for all } p > 1 \quad . \tag{52}$$

Proof: As noted in Section 2, the condition (A2) together with the fact that $F(u) > 0$ for $u > 0$ implies that $F(u)/u^{1+\delta}$, $u \geq M$, is strictly increasing for any $\delta < \delta_1$. Therefore

$$\frac{F(y)}{y^{1+\delta}} \geq \frac{F(p/d)}{(p/d)^{1+\delta}} \quad \text{for} \quad y \geq p/d \geq M \quad ,$$

so that $F(y) \geq \frac{F(p/d)}{(p/d)^{1+\delta}} y^{1+\delta}$, or

$$\frac{1}{F^2(y)} \leq \frac{(p/d)^{2(1+\delta)}}{F^2(p/d)} \frac{1}{y^{2(1+\delta)}} \quad .$$

As a result

$$\begin{aligned} \int_{p/d}^{\infty} \frac{y}{F^2(y)}\, dy &\leq \frac{(p/d)^{2(1+\delta)}}{F^2(p/d)} \int_{p/d}^{\infty} y^{-1-2\delta}\, dy \\ &= \frac{p^{2+2\delta}}{2\delta d^{2+2\delta} F^2(p/d)} \left(\frac{d}{p} \right)^{2\delta} \\ &= \frac{p^2}{2\delta d^2 F^2(p/d)}, \quad p/d \geq M \ . \end{aligned}$$

We thus have

$$\begin{aligned} \limsup_{p\to\infty} \left(\frac{F^2(p)}{p^2} \right)^{\beta} \int_{p/d}^{\infty} \frac{y}{F^2(y)}\, dy &\leq \limsup_{p\to\infty} \left(\frac{F^2(p)}{p^2} \right)^{\beta} \frac{p^2}{2\delta d^2 F^2(p/d)} \\ &= \frac{1}{2\delta d^2} \limsup_{p\to\infty} p^{2(1-\beta)} \frac{F^{2\beta}(p)}{F^2(p/d)}. \end{aligned} \tag{53}$$

We shall now show that the lim sup on the right in (53) is finite. Since the expression

$$\left(\frac{F^2(p)}{p^2}\right)^\beta \int_{p/d}^\infty \frac{y}{F^2(y)}\,dy$$

is bounded for p in any bounded interval $(1, N)$, the boundedness of the lim sup is sufficient to verify the estimate of this lemma. Let $H(u) = F'(u)/F(u)$. For a fixed $d > 1$ choose

$$\beta = \frac{K_1 - 1}{dK_1 - 1} \tag{54}$$

where $K_1 = 1 + \delta_1 > 1$ and δ_1 is the constant in the superlinearity assumption (A2). Clearly $0 < \beta < 1$. Note that this choice for β yields

$$\frac{1}{d\beta} - \frac{1-\beta}{dK_1\beta} = 1 \quad .$$

Since $H(u)$, $u \geq M$ is positive and non-increasing we have $H(u) \leq H(u/d)$ for $u \geq Md$, and so with β given by (54)

$$\begin{aligned} \frac{H(u)}{H(u/d)} &\leq 1 \\ &= \frac{1}{d\beta} - \frac{1-\beta}{dK_1\beta} \\ &= \frac{1}{d\beta} - \frac{1-\beta}{\beta u}\frac{1}{dK_1/u} \\ &\leq \frac{1}{d\beta} - \frac{1-\beta}{\beta u}\frac{1}{H(u/d)} \quad , \end{aligned} \tag{55}$$

for $u \geq Md$. The last inequality in (55) follows from (A2) in the form $H(u/d) \geq dK_1/u$. We multiply both sides of (55) by $\beta H(u/d)$ to obtain

$$\beta H(u) \leq \frac{1}{d}H(u/d) - \frac{1-\beta}{u} \qquad \text{for } u \geq Md \quad . \tag{56}$$

Integration of both sides of (56) from $u = Md$ to $u = p$ (note that $H(u) = F'(u)/F(u)$) and simplification yields

$$F^\beta(p) \leq \tilde{C}F(p/d)p^{\beta-1}$$

with $\tilde{C} = \frac{F^\beta(Md)(Md)^{1-\beta}}{F(M)}$. The boundedness of the lim sup in (53) follows, and this proves the lemma.

Remark 8 Note that with β given by (54) β will approach 1 as d approaches 1. Moreover, the constant $\tilde{C}$ in the proof of Lemma B.1 also approaches 1 as $d \to 1$. The constant C in the estimate (52) can be taken as

$$C = \frac{1}{2\delta d^2}\tilde{C}^2 = \frac{F^{2\beta}(Md)M^{2(1-\beta)}}{2\delta d^{2\beta}F^2(M)}$$

where β is given by (54). As a consequence a possible constant C that may be used in (52) will approach $\frac{1}{2\delta}$ as $d \to 1$. Here δ may be picked arbitrarily close to δ_1, the constant from the superlinearity assumption (A2).

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References

[1] C. Bandle and H. Brunner, *Blowup in diffusion equations: A survey*, Journal of Computational and Applied Mathematics, **97**, 1998, pp. 3-22.

[2] K. Bryan and M.S. Vogelius, *Transient behavior of solutions to a class of nonlinear boundary value problems*, to appear in the Quarterly of Applied Math.

[3] K. Deng, *The blow up of the heat equation with Neumann boundary conditions*, J. Math Anal Appl, **188**, 1994, pp. 641-650.

[4] K. Deng and M. Xu, *Remarks on the blow up behavior for a nonlinear diffusion with Neumann boundary conditions*, Proc. Amer. Math. Soc., **127**, 1999, pp. 167-172.

[5] M. Fila and J. Filo, *Blow up on the boundary: A survey*, pp. 67-78 in Singularities and Differential Equations, S. Janeczko et al. (eds), Banach Center Publ. **33**, Polish Acad. Sciences, Warsaw, 1996.

[6] M. Fila and J. Guo, *Complete Blow up and incomplete quenching for the heat equation with a nonlinear boundary condition*, Nonlinear Analysis, **48**, 2002, pp. 995-1002.

[7] S. Fu, J. Guo and J. Tsai *Blow-up behavior for a semilinear heat equation with a nonlinear boundary condition*, Tohoku Math. J. (2), **55**, 2003, pp. 565-581.

[8] S. Kichenassamy, *Recent Progress on Boundary Blow-up*, pp. 329-341 in "Elliptic and Parabolic Problems", Volume 63 of the Book Series "Progress in Nonlinear Differential Equations and Their Applications", Birkhäuser, Basel, 2005.

[9] H. Levine and L. Payne, *Nonexistence theorems for the heat equation with nonlinear boundary conditions and for the porous medium equation backward in time*, J. Differential Equations, **16**, 1974, pp. 319-334.

[10] Z. Lin and M. Wang, *The blow up properties of solutions to semilinear heat equations with nonlinear boundary conditions*, Z Angew Math Phys, **50**, 1999, pp. 361-374.

[11] F. Quiros, J.D. Rossi and J.L. Vazquez, *Complete blow-up and thermal avalanche for heat equations with nonlinear boundary conditions*, Comm in PDE, **27**, 2002, pp. 395-424.

[12] C. Roberts, *Recent results on blow-up and quenching for nonlinear Volterra equations*, J of Comp and Applied Math, **205**, 2007, pp. 736-743.

[13] W. Walter, *On existence and nonexistence in the large of solutions of parabolic differential equations with a nonlinear boundary condition*, SIAM J Math Anal, **6**, 1975, pp. 85-90.