6th homework, due 4/9/09.

PROBLEM 1. Let A and B be nonempty subsets of $\mathbb{R}$

(a) Formulate the negation of the following statement

$$(P) \qquad \exists x \in A \;:\; \forall y \in B \quad |x - y| < 1 \; .$$

(b) Construct an example of sets A and B such that (P) holds.

(c) Construct an example of sets A and B such that the negation of (P) holds.

PROBLEM 2. Let $\mathbb{R}$ denote the set of real numbers. For each of the following two statements give a proof of its truthfulness, or provide a counterexample.

(a) $\forall \lambda \in \mathbb{R} \quad \lambda > 0 \implies 1 \leq \frac{1}{\lambda} + \lambda$,

(b) $\forall \lambda \in \mathbb{R} \quad \lambda > 0 \implies \frac{1}{\lambda} + \lambda \leq 10$.

PROBLEM 3. Let A and B be two nonempty subsets of a set S, and let $\mathfrak{P}(A)$ and $\mathfrak{P}(B)$ denote the power sets of A and B, respectively. Explain (with proofs) how each of the sets $\mathfrak{P}(A \cap B)$, $\mathfrak{P}(A \cup B)$, and $\mathfrak{P}(A \setminus B)$ is related to $\mathfrak{P}(A)$ and $\mathfrak{P}(B)$.

PROBLEM 4. For any pair of non-negative integers k and n, with $0 \leq k \leq n$, we recall that the number "n choose k" is defined by

$$\binom{n}{k} = \frac{n!}{(n-k)!\; k!} \; .$$

Here $k!$ denotes the product of all the natural numbers between 1 and k (both included), with the additional convention that $0! = 1$.

(a) Prove that for any pair of integers, with $1 \leq k \leq n$,

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k} \; .$$

(b) Prove by induction that for any integer, $n \geq 0$,

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

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PROBLEM 5. Prove, using complete induction, that any natural number m may be written as a sum of powers of 2, *i.e.*, as

$$m = 2^{k_1} + 2^{k_2} + \ \ldots \ + 2^{k_N} \ ,$$

for some integers $0 \leq k_1 < k_2 < \ \ldots \ < k_N$. (Hint: when you verify the "induction step" it may be helpful to distinguish between the case of even and odd). Is the same statement true if "powers of 2" is replaced by "powers of 4"?

PROBLEM 6. Let $\equiv$ denote the relation defined on $\mathbb{Z}$ by

$$n \ \equiv \ m \quad \Longleftrightarrow \quad n - m \text{ is even} \quad .$$

(a) Prove that $\equiv$ is reflexive and transitive.
(b) Is $\equiv$ symmetric? If you believe it has this property then give a proof.
(c) How many different equivalence classes does $\equiv$ partition $\mathbb{Z}$ into? Write down these equivalence classes.