

Math 250–Section #3 Review Final

Final Place & Time: SEC-206, Friday, December 17, 4:00–7:00 PM

As preparation, you should **also** review the two hourlies & their reviews [their solutions are posted], and your own quizzes.

This exam will have 11-12 questions for a total of 100 points. [The spreadsheet will scale it to 200 points.] Here we have more questions for variety. One question just suggests a list of notions to look up in the textbook—they may occur in the questions so you should be familiar of their meanings. We will be in office daily, usually from 10–2 [M–Th]. There will be a blank page at the end for scratch [it may contain needed formulas, as individual formula sheets are not allowed]. For proper credit, answers must be explained. Good luck.

-1. You should be familiar with the key notions of the subject. As we have been doing in hourlies and quizzes, some questions require familiarity of basic concepts to the point of providing examples.

- vector space, vector subspace, subspace test
- linearly combination of vectors, linear dependence, linear independence
- span of a set of vectors
- basis of a vector space, dimension of a vector space
- $m \times n$ matrix
- if A is an $m \times n$ matrix, which function between \mathbb{R}^n and \mathbb{R}^m it defines?
- what is the nullspace of this function?
- what is the range of this function?
- what is the rank of a matrix and what are some of the ways it can be expressed?
- what is the inverse [resp. transpose] of a matrix?

- what is the determinant of a square matrix?
- Cramer's rule
- eigenvectors and eigenvalues of a matrix, eigenspace
- characteristic polynomial
- algebraic and geometric multiplicities of an eigenvalue
- diagonalizable matrix [what is needed]
- linear independence of eigenvectors: for example, suppose u, v and w are eigenvectors of a matrix A corresponding to the eigenvalues 2, 3 and 6; why are these vectors linearly independent?
- if you have trouble, try with u and v only
- inner product [dot product]
- length of vectors, orthogonality
- distance between vectors
- Gram-Schmidt
- least squares, projection
- orthogonal complement and nullspace
- orthogonal matrices
- symmetric matrices and the spectral theorem
- equations of plane conics

0. Given that the 5×5 matrix $A = [c_1|c_2|c_3|c_4|c_5]$ has determinant 3, find the determinant of the matrix

$$B = [c_2 + c_3|c_3 + c_4|c_4 + c_5|c_5 + c_1|c_1 + c_2].$$

Answer: We carry out elementary column operations on B until we get some matrix more closely related to A :

$$B = [c_2 + c_3|c_3 + c_4|c_4 + c_5|c_5 + c_1|c_1 + c_2] \rightarrow$$

$$[(c_2+c_3)-(c_3+c_4)+(c_4+c_5)-(c_5+c_1)+(c_1+c_2) = 2c_2|c_3+c_4|c_4+c_5|c_5+c_1|c_1+c_2]$$

$$\rightarrow [2c_2|c_3 + c_4|c_4 + c_5|c_5 + c_1|c_1] \rightarrow$$

$$[2c_2|c_3|c_4|c_5|c_1] \rightarrow 4 \text{ transpositions } [c_1|2c_2|c_3|c_4|c_5]$$

so the determinant of B is

$$2 \times \det(A) \times (-1)^4 = 6$$

1. Given the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} :$$

- (a) Verify that the vector $[1, 1, 1]$ is an eigenvector;
- (b) find its characteristic equation;
- (c) find its eigenvalues;
- (d) find bases of the eigenspaces.
- (e) Show that A is diagonalizable.

Answer: Done several times—look up in one of the hourlies and their reviews.

1. Given the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} :$$

(a) Without computing the characteristic polynomial [which is hard to find], show that the vector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

is an eigenvector.

- (b) What is the corresponding eigenvalue?
- (c) Argue that there is an eigenvalue = 0.

2. Let

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -3 \end{bmatrix}$$

- (a) Find a basis of its nullspace.
- (b) Find a basis of its row space.
- (c) Is $[0, -2, 2]$ a vector in the left nullspace of A ? [What is the left nullspace anyway?]
- (d) Is there a vector v so that

$$Av = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}?$$

- (e) Could v be found using Cramer's rule?

Answer:

3. Find the FULL set of solutions of the system of equations

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 7 & 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.$$

Answer: By the FULL solution, we mean to find one of the three possible outcomes—no solution, the unique solution, the solution set with infinitely many solutions. In this case the third option holds. You done this for first hourly.

4. Let V be the set of all 2×2 matrices.

(a) Explain why V is a vector space, describe one of its basis and find the dimension of V .

(b) Let S be the set of all matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

with $a + b + c = 0$ and $a + d = 0$. Show that S is a subspace of V and find a basis for S .

Answer: Part (a) we have done before. For part (b), notice that $d = -a$ and $c = -a - b$ so S consists of matrices of the form

$$\begin{bmatrix} a & b \\ -a - b & -a \end{bmatrix}.$$

Adding two such matrices,

$$\begin{bmatrix} a & b \\ -a - b & -a \end{bmatrix} + \begin{bmatrix} a' & b' \\ -a' - b' & -a' \end{bmatrix} = \begin{bmatrix} a + a' & b + b' \\ -a - a' - b - b' & -a - a' \end{bmatrix},$$

which has the same pattern. The same will happen if we multiply one of these matrices by a scalar r . This shows that S is a subspace.

(b) Note that

$$\begin{bmatrix} a & b \\ -a - b & -a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

It shows that the two numerical matrices are in S (fit the pattern), span S and are independent. They form a basis of S , so the dimension of S is two (dimension is the number of vectors in a basis).

5. (a) Give a reason why the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

are linearly independent,

(b) but the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

cannot be linearly independent for any choice of a, b, c, d .

Answer: For part (a), you must check the rank of the matrix formed from the 3 vectors. For part (b), you have 5 vectors in the space \mathbb{R}^4 , which has dimension 4, so your vectors are not linearly independent.

6. Find an orthogonal matrix S such that $S^{-1}AS$ is diagonal, where

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}.$$

Answer: Done several times.

6.5. Find the general solution of the system of differential equations

$$\begin{aligned}x_1' &= -2x_1 + 2x_2 \\x_2' &= 2x_1 + x_2\end{aligned}$$

Answer: See page 294

7. (a) What is an orthogonal matrix Q ? Give an example.
 (b) If A is a symmetric matrix and Q is an orthogonal matrix of the same size, show that $Q^{-1}AQ$ is symmetric.
 (c) If Q is an orthogonal matrix and \mathbf{v} is an eigenvector for the eigenvalue λ , explain why λ can only be 1 or -1 .

Answer: (a) Orthogonal matrix means $Q^T Q = I$. Lots of them in our experience.

- (b) If Q is orthogonal,

$$Q^{-1}AQ = Q^T AQ$$

and therefore

$$(Q^T AQ)^T = Q^T A^T (Q^T)^T = Q^T AQ,$$

since the transpose of a product is the product of the transposes in the reverse order. This calculation shows that $Q^T AQ$ is its own transpose, that is it is a symmetric matrix.

- (c) If v is an eigenvector of the orthogonal matrix Q , $Qv = \lambda v$, look at the dot product [this problem was done in class]

$$v \cdot Qv = v \cdot \lambda v = \lambda \|v\|^2,$$

but we can also write this number as a matrix product

$$v^T Qv = (v^T Qv)^T = v^T Q^T v = v^T Q^{-1}v = \lambda^{-1} \|v\|^2,$$

that shows

$$\lambda = \lambda^{-1},$$

so $\lambda = \pm 1$.

8. Let A be a 3×3 matrix with 3 nonzero entries of 2, 3 and 6. The other 6 entries are 0. Find and explain all the possible values for the determinant such matrices.

Answer:

8. If A and B are 3×3 with $\det A = 3$, $\det B = 1$, find

$$\det((2A)^2(3B)).$$

[Be watchful of the size of the matrices and the parentheses.]

9. Let A be a 3×3 matrix whose columns are the vectors v_1, v_2 and v_3 .

(a) If a matrix B has for columns the vectors $2v_2 + v_3, 3v_3 + v_1$ and v_1 , respectively, how are the determinants of A and B related?

(b) Suppose further that v_1, v_2, v_3 are perpendicular to each other and satisfy

$$v_1 \cdot v_1 = 2, \quad v_2 \cdot v_2 = 6, \quad v_3 \cdot v_3 = 3.$$

Argue that the determinant of A is ± 6 . (Hint: multiply A by its transpose and take determinants.)

10. If A is a 3×3 matrix and $\det A = 2$, find the determinant of B if

(a) $B = 2A^2$ (careful, this is not $(2A)^2$)

(b) B is derived from A as follows: The first row of A is moved to the second row, the second row to the third row and the third row to the first row.

(c) $B = A^T \cdot A^{-1}$.

Answer:

11. Suppose a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ satisfies

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(a) Show that the three vectors of \mathbb{R}^3 are linearly independent.

(b) Find $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$.

Answer:

11. If A is a square matrix with an eigenvalue $\lambda = 2$, explain why the matrix $A^3 + A^2 + I$ has an eigenvalue equal to 13.

Answer:

11. Find the determinant of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Answer:

11. Suppose that A is a 4×4 matrix, written as four columns $A = [v_1|v_2|v_3|v_4]$ and assume that $\det(A) = 3$. Find:
- (a) $\det(\text{adj } A)$
 - (b) $\det[v_1 + v_2|v_2 + v_4|v_1|v_3]$
 - (c) $\det[2v_1 + v_2|v_3 + v_4|v_1 - v_3|2v_2]$

Answer:

11. Explain why the nullspace of a matrix is perpendicular to its row space.

or

(10 pts) If W is a subspace of \mathbb{R}^n , explain what is its orthogonal complement W^\perp .

Answer:

12. (a) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 7 & t & t \end{bmatrix}$$

(b) Determine the values of t for which the matrix is invertible.

Answer: (a) $t + 14 - 2t + 7 - 4t - t = 21 - 6t$

(b) A square matrix is invertible [non-singular being another name] precisely when its determinant is nonzero. Here this means that $t \neq 21/6 = 7/2$

12. Explain and give a [nontrivial] example of each of the following notions:

(a) Eigenvector of a matrix A

(b) Linearly dependent set of vectors v_1, v_2 and v_3 ;

(c) Adjoint of a matrix A

(d) Inverse of a matrix A [Refresh also the algorithms to find inverses]

(e) If v_1, v_2, v_3 are nonzero vectors that are perpendicular to one another then they are linearly independent

Answer: Look up these in the textbook.

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