Math 250–Section #4 Review Final

Place & Time: ARC-105 [usual classroom], Thursday, December 20, 12–3:00 PM

As preparation, students should also review the hourlies and quizzes [their solutions are posted].

This exam has 11-12 questions for a total of 100 points. [The spreadsheet will scale it to 200 points.] Here we have more questions for variety. They will be solved in class at review time. There will be a blank page at the end for scratch [it may contain needed formulas, as individual formula sheets are not allowed]. For proper credit, answers must be explained. Good luck.

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1. (15 pts) Given the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}:$$

(a) Verify that the vector $[1, 1, 1]$ is an eigenvector;
(b) find its characteristic equation;
(c) find its eigenvalues;
(d) find bases of the eigenspaces.
(e) Show that A is diagonalizable.

Answer:

1. (15 pts) Given the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} :$$

(a) Without computing the characteristic polynomial [which is hard to find], show that the vector

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

is an eigenvector.

(b) What is the corresponding eigenvalue?

(c) Argue that there is an eigenvalue $= 0$.

2. (12 pts) Let

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -3 \end{bmatrix}$$

(a) Find a basis of its nullspace.

(b) Find a basis of its row space.

(c) Is $[0, -2, 2]$ a vector in the left nullspace of A? [What is the left nullspace anyway?]

(d) Is there a vector v so that

$$Av = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}?$$

(e) Could v be found using Cramer's rule?

Answer:

3. (12 pts) Find the FULL set of solutions of the system of equations

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 7 & 8 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -7 \end{bmatrix}.$$

Answer:

4. (10 pts) Let V be the set of all 2×2 matrices.

(a) Explain why V is a vector space, describe one of its basis and find the dimension of V.

(b) Let S be the set of all matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

with $a + b + c = 0$ and $a + d = 0$. Show that S is a subspace of V and find a basis for S.

Answer:

5. (8 pts) (a) Give a reason why the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

are linearly independent,

(b) but the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

cannot be linearly independent for any choice of a, b, c, d.

Answer:

6. (6 pts) Find an orthogonal matrix S such that $S^{-1}AS$ is diagonal, where

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}.$$

Answer:

7. (6 pts) (a) What is an orthogonal matrix Q? Give an example.

(b) If A is a symmetric matrix and Q is an orthogonal matrix of the same size, show that $Q^{-1}AQ$ is symmetric.

(c) If Q is an orthogonal matrix and $\mathbf{v}$ is an eigenvector for the eigenvalue λ, explain why λ can only be 1 or -1.

Answer:

8. (6 pts) Let A be a 3×3 matrix with 3 nonzero entries of 2, 3 and 6. The other 6 entries are 0. Find and explain all the possible values for the determinant such matrices.

Answer:

8. (6 pts) If A and B are 3×3 with $\det A = 3$, $\det B = 1$, find

$$\det((2A)^2(3B)).$$

[Be watchful of the size of the matrices and the parentheses.]

9. (8 pts) Let A be a 3×3 matrix whose columns are the vectors v_1, v_2 and v_3.

(a) If a matrix B has for columns the vectors $2v_2 + v_3$, $3v_3 + v_1$ and v_1, respectively, how are the determinants of A and B related?

(b) Suppose further that v_1, v_2, v_3 are perpendicular to each other and satisfy

$$v_1 \cdot v_1 = 2, \quad v_2 \cdot v_2 = 6, \quad v_3 \cdot v_3 = 3.$$

Argue that the determinant of A is ± 6. (Hint: multiply A by its transpose and take determinants.)

10. (9 pts) If A is a 3×3 matrix and $\det A = 2$, find the determinant of B if

(a) $B = 2A^2$ (careful, this is not $(2A)^2$)

(b) B is derived from A as follows: The first row of A is moved to the second row, the second row to the third row and the third row to the first row.

(c) $B = A^T \cdot A^{-1}$.

Answer:

11. (8 pts) Suppose a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ satisfies

$$T(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad T(\begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(a) Show that the three vectors of $\mathbb{R}^3$ are linearly independent.

(b) Find $T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$.

Answer:

11. (10 pts) If A is a square matrix with an eigenvalue $\lambda = 2$, explain why the matrix $A^2 + I$ has an eigenvalue equal to 5.

Answer:

11. (10 pts) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Answer:

11. (10 points) Suppose that A is a 4×4 matrix, written as four columns $A = [v_1|v_2|v_3|v_4]$ and assume that $\det(A) = 3$. Find:

(a) $\det(adj\ A)$

(b) $\det[v_1 + v_2|v_2 + v_4|v_1|v_3]$

(c) $\det[2v_1 + v_2|v_3 + v_4|v_1 - v_3|2v_2]$

Answer:

11. (10 pts) Explain why the nullspace of a matrix is perpendicular to its row space.

or

(10 pts) If W is a subspace of $\mathbb{R}^n$, explain what is its orthogonal complement $W^{\perp}$.

Answer:

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