

Math 250–Section #4 Review Questions for Hourly # 2

The syllabus for this exam are the sections discussed in class since Hourly #1: 3.2, 4.1, 4.2, 4.3, 5.1, 5.2, 5.3, 5.5, 6.1&6.2. Look up also the quizzes since the First Exam. Written below is an old exam, it has extra questions. The actual exam will only have 7–8 questions.

General suggestions:

1. Make sure you understand all the key concepts [from *determinant* to *orthogonal projection*].
2. Review parts of Exercise #1 of each Section.
3. Practice the following exercises:
3.2.33, 4.1.33, 4.2.33, 4.2.36 [typical B problem], 4.3.31, 5.1.40,
5.2.25, 5.3.17, 5.5.27, 6.1.21, 6.2.9

1. Supply a third column vector so that the matrix

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & a \\ 1/\sqrt{3} & 0 & b \\ 1/\sqrt{3} & -1/\sqrt{2} & c \end{bmatrix}$$

is orthogonal. Begin by explaining what is an orthogonal matrix.

Answer: A matrix Q is orthogonal if $Q^T Q = I$. This means that the column vectors have magnitude 1 and are orthogonal (perpendicular) to one another.

For our matrix Q , the first two columns are perpendicular to each other and have magnitude 1. If they are to be perpendicular to the third column, we set their dot products to zero

$$\begin{aligned} a/\sqrt{3} + b/\sqrt{3} + c/\sqrt{3} &= 0 \\ a/\sqrt{2} - c/\sqrt{2} &= 0 \end{aligned}$$

or more simply $a + b + c = 0$ and $a - c = 0$. Thus $c = a$ and $b = -2a$. But the third column must have magnitude 1,

$$a^2 + b^2 + c^2 = a^2 + (-2a)^2 + a^2 = 6a^2 = 1.$$

Thus $a = 1/\sqrt{6}$, and the third vector is

$$\begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}.$$

2. (a) Find an orthonormal basis for the subspace S of \mathbb{R}^4 spanned by $\{[1, 2, 0, 2], [1, 0, 1, 1], [2, 1, 1, 1]\}$. (b) Use it to find the projection of the vector $[1, -1, 1, -1]$ onto S .

Answer: You must use the Gram-Schmidt process to obtain an orthonormal basis. That done [i.e. q_1, q_2, q_3 found] the book shows how to obtain the projection very quickly. There is another question of this kind with solution written out.

3. Consider the matrix

$$A = \begin{bmatrix} -1 & 0 & 1 \\ -7 & 2 & 3 \\ 3 & 0 & 1 \end{bmatrix}.$$

- (a) Find its characteristic polynomial.
- (b) Find its eigenvalues and bases of the corresponding eigenspaces.
- (c) Find out whether A is diagonalizable.

Answer:

(a): To find the characteristic polynomial expand the determinant of $A - \lambda I$ along the second column

$$\det \begin{bmatrix} -1 - \lambda & 0 & 1 \\ -7 & 2 - \lambda & 3 \\ 3 & 0 & 1 - \lambda \end{bmatrix} = (2 - \lambda)((-1 - \lambda)(1 - \lambda) - 1 \cdot 3)$$

$$= (2 - \lambda)(\lambda^2 - 4) = (2 - \lambda)(\lambda - 2)(\lambda + 2).$$

(b) The eigenvalues are -2 (single) and 2 (double). To find their eigenspaces, we plug them in the matrix $A - \lambda I$ and find the corresponding nullspaces. A simple calculation will show that for $\lambda = -2$, the eigenspace are the multiples of $(2, 5, -2)$, and for $\lambda = 2$ the multiples of $(0, 1, 0)$. In particular the eigenvalue $\lambda = 2$ has algebraic multiplicity 2 but geometric multiplicity 1.

(c) Part (b) shows that the eigenspaces only span a subspace of dimension 2, not enough to form a basis of R^3 : A is NOT diagonalizable.

4. If A is a matrix with an eigenvalue equal to 2, then the matrix $A^2 + I$ has an eigenvalue equal to 5. (I is the identity matrix of the same size as A ; it might help if you begin by saying what an eigenvector is.)

Answer: Note that an eigenvalue of A is a number λ for which there is a nonzero vector v such that

$$Av = \lambda v.$$

Note that

$$A^2v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda\lambda v = \lambda^2v,$$

showing that λ^2 is an eigenvalue of A^2 .

Now consider our matrix ($\lambda = 2$),

$$(A^2 + I)v = A^2v + Iv = 4v + v = 5v.$$

5. Explain very carefully the following facts about a 6×5 matrix A :

$$\begin{aligned}\text{dimension of row space} + \text{dimension of nullspace} &= 5 \\ \text{dimension of column space} + \text{dimension of left nullspace} &= 6\end{aligned}$$

Answer: Look up in chapter 6 the rank equation.

6. Consider the symmetric matrix

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

- (a) Find the characteristic polynomial of A .
- (b) Verify that the eigenvalues of A are always real numbers.
- (c) Show that A is diagonalizable.

Answer: (a) The characteristic equation (or polynomial) is

$$\begin{vmatrix} a - \lambda & b \\ b & c - \lambda \end{vmatrix} = (a - \lambda)(c - \lambda) - b^2 = \lambda^2 - (a + c)\lambda + ac - b^2.$$

(b) We use the quadratic formula to check the roots of this polynomial:

$$\begin{aligned}\lambda &= \frac{a + c \pm \sqrt{(a + c)^2 - 4(ac - b^2)}}{2} \\ &= \frac{a + c \pm \sqrt{(a - c)^2 + 4b^2}}{2}.\end{aligned}$$

Note that the number in the square root function is never negative, so we have either (i) two distinct roots or (ii) a double root if $b = 0$ and $c = a$.

(c) In both cases the matrix is diagonalizable (add details).

7. Given the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 5 & 2 \\ 3 & 2 & 2 \end{bmatrix} \quad \text{find:}$$

- (a) its characteristic polynomial
- (b) its eigenvalues
- (c) corresponding eigenvectors

(d) the matrix P such that $P^{-1}AP$ is a diagonal matrix

Answer: (a) We must find the determinant of the matrix $\lambda I - A$. We should try to keep the factors by expanding along the first row:

$$\begin{aligned} \begin{vmatrix} \lambda - 3 & 0 & 0 \\ -3 & \lambda - 5 & -2 \\ -3 & -2 & \lambda - 2 \end{vmatrix} &= (\lambda - 3) \begin{vmatrix} \lambda - 5 & -2 \\ -2 & \lambda - 2 \end{vmatrix} \\ &= (\lambda - 3)((\lambda - 5)(\lambda - 2) - 4) = (\lambda - 3)((\lambda^2 - 7\lambda + 6)). \end{aligned}$$

Factoring the quadratic [same as in Quiz 6] we get

$$|\lambda I - A| = (\lambda - 3)(\lambda - 6)(\lambda - 1).$$

(b) $\lambda = 1, 3, 6$ are the eigenvalues.

(c) For each of these values we find the nullspace of the matrix $\lambda I - A$, whose bases give us the desired eigenvectors:

$$\lambda = 1 \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}; \quad \lambda = 3 \leftrightarrow \begin{bmatrix} -6 \\ -5 \\ 8 \end{bmatrix}; \quad \lambda = 6 \leftrightarrow \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

(d) The matrix P that effects the diagonalization has for columns the eigenvectors:

$$P = \begin{bmatrix} 0 & -6 & 0 \\ 1 & -5 & 2 \\ -2 & 8 & 1 \end{bmatrix}$$

8. Given the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 3 \\ 7 & -8 & 3 \\ 5 & -7 & 0 \end{bmatrix}$$

Find:

- (a) basis for its row space
- (b) basis for its column space
- (c) basis for its nullspace

Answer: (a) Reduced row reduction leads to a basis for the row space

$$\begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 3 \\ 7 & -8 & 3 \\ 5 & -7 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The rows with the pivots give a basis of the row space: $\{(1, -2, -1), (0, 3, 5)\}$.

(b) The columns of the original matrix A with the pivots give a basis of the column space (note that there is no need to process A^T): $\{(1, 2, 7, 5), (-2, -1, -8, -7)\}$

(c) To find the nullspace we use the row reduced matrix [to solve for $Ax = 0$] to get that the solution set is made up of the multiples [the basis] of $(-13/3, -5/3, 1)$

9. Find a basis for the vector space of all 2×3 matrices.

Answer: All these matrices

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

are unique linear combinations of the 6 particular 2×3 matrices

$$a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This shows that these 6 matrices form a basis for the space.

10. Let A be a 3×3 matrix with eigenvalues $1/2$, 3 and 4 .

(a) Show that A is diagonalizable.

(b) Show that $\det A \neq 0$.

(c) What are the eigenvalues of A^{-1} [explain]?

(d) Explain why the eigenvalues of A^2 are $1/4$, 9 and 16 .

Answer: (a) A is 3×3 with distinct eigenvalues, so it is diagonalizable by class discussion.

(b) No eigenvalue is 0 so the nullspace of A is and therefore it is non-singular. More precisely, since A is diagonalizable, there exists an invertible matrix P such that

$$P^{-1}AP = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

from which we get

$$\det(A) = \det(P^{-1}AP) = \det(P^{-1}) \det A \det P = 1/2 \times 3 \times 4 = 6.$$

(c) The other parts would follow from (b) but it is better seen as follows: If v is an eigenvector of eigenvalue λ

$$Av = \lambda v,$$

since A is invertible, we get by multiplying the equation by A^{-1}

$$A^{-1}Av = v = \lambda A^{-1}v,$$

and therefore

$$A^{-1}v = \lambda^{-1}v.$$

(d) Similarly, multiplying the eigenvector equation by A , we get

$$A(Av) = A(\lambda v) = \lambda(Av) = \lambda \cdot \lambda v = \lambda^2 v.$$

11. Show that the following two matrices are NOT diagonalizable:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Answer: The characteristic polynomial for the first matrix is

$$\begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2.$$

Thus $\lambda = 1$ is the only eigenvalue. Let us check whether we have enough eigenvectors: If nullspace of $\lambda I - A$ has just one equation $x_2 = 0$, which gives the eigenvector $(1, 0)$. This means that we don't have a basis for the space \mathbb{R}^2 made up of eigenvectors, so A is not diagonalizable.

For the other matrix, the characteristic polynomial

$$\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1.$$

Its roots are $\pm\sqrt{-1}$, so A does not have real eigenvalues [and therefore cannot be diagonalizable in real space].

12. Let W be the subspace of R^4 with basis $(1, 1, 0, 1)$, $(0, 1, 1, 0)$. Find the projection of the vector $v = (2, 1, 3, 0)$ onto W .

Answer: We must first use the Gram–Schmidt process to get an orthonormal basis for W . According to the given formula, we set $u_1 = (1, 1, 0, 1)$ and

$$\begin{aligned} u_2 &= (0, 1, 1, 0) - \frac{(0, 1, 1, 0) \cdot (1, 1, 0, 1)}{(1, 1, 0, 1) \cdot (1, 1, 0, 1)}(1, 1, 0, 1) \\ &= (0, 1, 1, 0) - 1/3(1, 1, 0, 1) = (-1/3, 2/3, 1, -1/3) \end{aligned}$$

We now normalize u_1 and u_2 to obtain the desired basis w_1 and w_2 :

$$w_1 = \frac{u_1}{\|u_1\|} = 1/\sqrt{3}(1, 1, 0, 1)$$

$$w_2 = \frac{u_2}{\|u_2\|} = \sqrt{1/15}(-1, 2, 3, -1)$$

Finally we use the projection formula

$$\text{proj}_W(v) = (v \cdot w_1)w_1 + (v \cdot w_2)w_2 = 1/5(2, 11, 9, 2).$$

13. Give a solid explanation of the following fact: If A is a $m \times n$ matrix then the dimension of its row space is equal to the dimension of its column space. Begin by explaining the terms ‘row space’ and ‘column space’ of a matrix.

Answer: The *row space* of the matrix A is the *subspace* spanned by the row vectors of the matrix, that is, it is the set of all linear combinations of the row vectors r_1, \dots, r_m of A . [Note that this is much more than just the row vectors of the matrix.] Note that all the row vectors of all the matrices that occur when we carry out row reduction in A are linear combinations of the rows of A . Row reduction provides a set of vectors that is a basis for the row space.

There is a similar description of the column space.

[Important point:] As we carry out row reduction, the columns are changing with the new columns that may not lie in the original column space. **However**, the relationships between the original and the new columns have not changed [after all, the relationships are expressed as the solutions of the corresponding systems—and the solutions are still the same]. At the end

point of row reduction, it becomes clear that the columns with the pivots form a basis of the new column space, so that any additional column is a combination of the pivot columns. Consequently, the column of A in the same position is the 'same' linear combination of the columns of A designated by the pivot positions.

The routine to obtain a basis that is orthogonal from another basis
[Gram–Schmidt process]: Input basis $S = \{u_1, \dots, u_n\}$

Step 1: Set $v_1 = u_1$

Step 2: Compute v_2, \dots, v_n successively, one at a time, by

$$v_i = u_i - \left(\frac{u_i \cdot v_1}{v_1 \cdot v_1}\right)v_1 - \left(\frac{u_i \cdot v_2}{v_2 \cdot v_2}\right)v_2 - \dots - \left(\frac{u_i \cdot v_{i-1}}{v_{i-1} \cdot v_{i-1}}\right)v_{i-1}$$

Step 3: Set

$$w_i = \frac{v_i}{\|v_i\|}$$

Then $T = \{w_1, \dots, w_n\}$ is an orthonormal basis.