

**Math 250–Section #5 Hourly #1 Review**

**Name:** \_\_\_\_\_

Note: The total points in this review add to more than 100 [in the test it will total 100]. You SHOULD also review all the material in the quizzes, including Quiz # 3, whose answers will be posted by Thursday [if not before]. Some of the problems below have written answers, the others will be discussed in class on Tuesday, before the hourly, or at office hours.

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Answer:

1. [12 pts] Given the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & 1 \\ 1 & -1 & -2 & 2 \\ -4 & 2 & 3 & 1 \\ 1 & -1 & -1 & 3 \end{bmatrix}$$

- (a) Find its reduced echelon form  $R$  of  $A$ .
- (b) What are the rank and the nullity of  $A$ .
- (c) Argue that the rows of  $R$  with pivots are linearly independent.
- (d) Argue that the columns of  $A$  with pivots are linearly independent.

Answer:

2. [10 pts] Determine a value of  $r$  for which the set of vectors

$$\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ r \end{bmatrix} \right\}$$

is linearly dependent.

3. [12 pts] Find all the values for  $t$  for which the resulting system of equations (a) has no solution, (b) a unique solution, and (c) infinitely many solutions.

$$\begin{array}{rccccrcr} x & + & y & - & & z & = & 2 \\ x & + & 2y & + & & z & = & 3 \\ x & + & y & + & (t^2 - 5)z & = & t \end{array}$$

Answer: Let us Gauss–Jordan’s on this system

$$\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & t^2 - 5 & t \end{array}$$

Using the 1 in the position  $(1, 1)$  as a pivot, we get

$$\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & t^2 - 4 & t - 2 \end{array}$$

We are now ready: If  $t \neq \pm 2$ , we get a unique solution since we have 3 pivots. If  $t = 2$ , we have an infinite number of solutions. If  $t = -2$ , it is impossible.

4. [10 pts] Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Answer: Using either of the methods we know [Gauss-Jordan algorithm or the formula involving the adjoint matrix] you should get:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

5. [12 pts] (a) Explain what is a *linear combination* of vectors.  
(b) Find out whether the vector  $(1, 0, 1, 2)$  is a linear combination of  $(2, 1, 3, 0)$ ,  $(7, 3, 1, -1)$  and  $(0, 1, 4, 3)$ .

Answer: (a) A linear combination of the *vectors*  $v_1, \dots, v_n$  is a vector

$$v = c_1v_1 + \cdots + c_nv_n,$$

where  $c_1, \dots, c_n$  are scalars.

(b): To answer this item, you must check whether it is possible to solve for  $c_1, c_2$  and  $c_3$  the equation

$$(1, 0, 1, 2) = c_1(2, 1, 3, 0) + c_2(7, 3, 1, -1) + c_3(0, 1, 4, 3).$$

Using Gaussian elimination, it turns out to be impossible.

6. [10 pts] Answer (a) **or** (b):

(a) Let  $A$  and  $C$  be square matrices of the same sizes.

(i) Find examples such that  $AC = 0$  but  $CA \neq 0$ .

(ii) Argue that this cannot occur if one of the matrices (either one) is invertible.

(b) If a square matrix  $A$  satisfies the relation

$$A^3 - A + I = 0$$

show that it is invertible.

Answer:

$$(a) \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}. \quad \text{Then:}$$

$$AC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad CA = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

If  $A$  is invertible,  $A^{-1}AC = IC = 0$  implies  $C = 0$  implies  $CA = 0$ .

$$(b) \quad A(A^2 - I) = -I. \quad \text{Thus } A^{-1} = I - A^2.$$

7. [10 pts] Given 5 countries, assume each maintains diplomatic relations with SOME of the others. To organize these relationships we use a  $5 \times 5$  matrix  $A$ , here we set  $a_{ii} = 0$  and  $a_{ij} = 1$  if the countries  $i$  and  $j$  have diplomatic relation; if not we set  $a_{ij} = 0$ :

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

- (a) Which pairs of countries maintain diplomatic relations with one another.
- (b) How many countries link country 1 with country 3?
- (c) Give an interpretation of the  $(1, 4)$ -entry of  $A^3$ .

8. [10 pts] Let  $B$  and  $C$  be two  $3 \times 3$  matrices. Argue that the columns of  $BC$  are linear combinations of the columns of  $B$ . Can you say the same with regard to the rows of  $C$  vis-à-vis the rows of  $BC$ ? Can we argue from this that  $\text{rank } B \geq \text{rank } BC$  (and similarly for  $C$ )?

9. [10 pts] Prove that the inverse of a lower triangular matrix  $A$  (if it exists) is lower .

10. [10 pts] If  $A$  and  $B$  are  $2 \times 2$  symmetric matrices [what are these anyway?] show that  $AB$  may not be symmetric.

11. [ $\infty$  pts] What is your favorite matrix? (Warning: this is a deep psychological inquiry...)

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