Math 250–Section #3 Quiz #6

Name: ____________________

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1. (4 pts) Given the matrix

$$A = \begin{bmatrix} -7 & 5 & 4 \\ 0 & -3 & 0 \\ -8 & 9 & 5 \end{bmatrix}$$

(a) Find its characteristic npolynomial.
(b) Find the eigenvalues.
(c) Determine a basis for each eigenspace.

Answer: (a) The characteristic polynomial is

$$\begin{aligned} \det(A - tI) = \det \begin{bmatrix} -7-t & 5 & 4 \\ 0 & -3-t & 0 \\ -8 & 9 & 5-t \end{bmatrix} &= (-3-t)((-7-t)(5-t)+32) \\ &= (-3-t)(t^2+2t-3) = (-3-t)(t+3)(t-1). \end{aligned}$$

(b) The eigenvalues are: -3 (double) and 1 (simple).
(c) Let us the basis for the eigenspace of $\lambda = -3$, that is the nullspace of the matrix $A + 3I$. We use the usual setup

$$\left[\begin{array}{ccc|c} -7+3 & 5 & 4 & 0 \\ 0 & -3+3 & 0 & 0 \\ -8 & 9 & 5+3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -4 & 5 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

so the nullspace is given by the vectors with coordinates $x_2 = 0$ and $x_1 = x_3$ which has for a basis

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

For the eigenvalue $\lambda = 1$ we do a similar computation.

2. (4 pts) For the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$ (i.e. A is diagonalizable).

Answer: (We argued in class that 2×2 real symmetric matrices are always diagonalizable.) First we compute the characteristic polynomial and find its roots:

$$\det(A - tI) = \det \begin{bmatrix} 1-t & 2 \\ 2 & -2-t \end{bmatrix} = (1-t)(-2-t) - 4 = t^2 + t - 6 = (t-2)(t+3)$$

Now we find eigenvalues corresponding to eigenvalues (2 and -3):

$$\lambda = 2 \leftrightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \lambda = -3 \leftrightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

According to the theory we discussed,

$$P = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

will do—or we can check.

3. (2 pts) (a) Verify that if A is an invertible matrix with an eigenvalue λ then A^{-1} has an eigenvalue equal to $1/\lambda$.
(b) If A and B are square matrices such then $AB = BA$ and v is an eigenvector of A then if $Bv \neq 0$ then Bv is also an eigenvector of A. (It helps if you begin by writing very clearly the meaning of an eigenvector of A.)

Answer: (a) An eigenvector for A means a nonzero vector v

$$Av = \lambda v$$

Apply to both sides the inverse of A

$$A^{-1}Av = A^{-}(\lambda v) = \lambda A^{-1}v$$

Since $A^{-1}A = I$, and $Iv = v$, we have

$$v = \lambda A^{-1}v$$

and finally

$$A^{-1}v = \lambda^{-1}v,$$

as we want to show.

(b) Let us check that the nonzero vector Bv is indeed an eigenvector of A: (We are going to use that $AB = BA$)

$$ABv = B(Av) = B(\lambda v) = \lambda Bv.$$