Math 250–Section #4 Hourly #2

Name: ________________

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1. [15 pts]

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 5 & 2 \\ 3 & 2 & 2 \end{bmatrix} \quad \text{find:}$$

(a) its characteristic polynomial
(b) its eigenvalues
(c) corresponding eigenvectors
(d) the matrix P such that $P^{-1}AP$ is a diagonal matrix

Answer: (a) We must find the determinant of the matrix $\lambda I - A$. We should try to keep the factors by expanding along the first row:

$$\begin{aligned} \begin{vmatrix} \lambda-3 & 0 & 0 \\ -1 & \lambda-5 & -2 \\ -3 & -2 & \lambda-2 \end{vmatrix} &= (\lambda-3)\begin{vmatrix} \lambda-5 & -2 \\ -2 & \lambda-2 \end{vmatrix} \\ &= (\lambda-3)((\lambda-5)(\lambda-2)-4) = (\lambda-3)((\lambda^2-7\lambda+6)). \end{aligned}$$

Factoring the quadratic [same as in Quiz 6] we get

$$|\lambda I - A| = (\lambda-3)(\lambda-6)(\lambda-1).$$

(b) $\lambda = 1, 3, 6$ are the eigenvalues.

(c) For each of these values we find the nullspace of the matrix $\lambda I - A$, whose bases give us the desired eigenvectors:

$$\lambda = 1 \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}; \quad \lambda = 3 \leftrightarrow \begin{bmatrix} -6 \\ -5 \\ 8 \end{bmatrix}; \quad \lambda = 6 \leftrightarrow \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

(d) The matrix P that effects the diagonalization has for columns the eigenvectors:

$$P = \begin{bmatrix} 0 & -6 & 0 \\ 1 & -5 & 2 \\ -2 & 8 & 1 \end{bmatrix}$$

2. [15 pts] Given the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 3 \\ 7 & -8 & 3 \\ 5 & -7 & 0 \end{bmatrix}$$

Find:
(a) basis for its row space
(b) basis for its column space
(c) basis for its nullspace

Answer: (a) Reduced row reduction leads to a basis for the row space

$$\begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 3 \\ 7 & -8 & 3 \\ 5 & -7 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The rows with the pivots give a basis of the row space:
$\{(1, -2, -1), (0, 3, 5)\}$.

(b) The columns of the original matrix A with the pivots give a basis of the column space (note that there is no need to process A^T):
$\{(1, 2, 7, 5), (-2, -1, -8, -7)\}$

(c) To find the nullspace we use the row reduced matrix [to solve for $Ax = 0$] to get that the solution set is made up of the multiples [the basis] of $(-13/3, -5/3, 1)$

3. [10 pts] Find a basis for the vector space of all 2×3 matrices.

Answer: All these matrices

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

are unique linear combinations of the 6 particular 2×3 matrices

$$a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}+b\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}+c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}+d\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}+e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}+f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This shows that these 6 matrices form a basis for the space.

4. [12 pts] Let A be a 3×3 matrix with eigenvalues $1/2$, 3 and 4.
(a) Show that A is diagonalizable.
(b) Show that $\det A \neq 0$.
(c) What are the eigenvalues of A^{-1} [explain]?
(d) Explain why the eigenvalues of A^2 are $1/4$, 9 and 16.

Answer: (a) A is 3×3 with distinct eigenvalues, so it is diagonalizable by class discussion.

(b) No eigenvalue is 0 so the nullspace of A is and therefore it is nonsingular. More precisely, since A is diagonalizable, there exists an invertible matrix P such that

$$P^{-1}AP = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

from which we get

$$\det(A) = \det(P^{-1}AP) = \det(P^{-1}) \det A \det P = 1/2 \times 3 \times 4 = 6.$$

(c) The other parts would follow from (b) but it is better seen as follows: If v is an eigenvector of eigenvalue λ

$$Av = \lambda v,$$

since A is invertible, we get by multiplying the equation by A^{-1}

$$A^{-1}Av = v = \lambda A^{-1} v,$$

and therefore

$$A^{-1}v = \lambda^{-1} v.$$

(d) Similarly, multiplying the eigenvector equation by A, we get

$$A(Av) = A(\lambda v) = \lambda(Av) = \lambda \cdot \lambda v = \lambda^2 v.$$

5. [14 pts] (a) Find the basis for the vector subspace W of P_3 consisting of all vectors of the form $at^3 + bt^2 + ct + d$, where $a = b$ and $c = d$.
(b) Determine $W^{\perp}$, the orthogonal complement of W.

Answer: In this problem, only part (a) will count [for the full score] as we did not discuss in class a notion of 'dot product' for the space P_3.
The polynomials specified above are simply those of the form

$$at^3 + at^2 + ct + d = a(t^3 + t^2) + c(t + 1),$$

that is the set of linear combinators of $t^3 + t^2$ and $t + 1$. Since they are linear independent, these polynomials form a basis for W.

6. [12 pts] Show that the following two matrices are NOT diagonalizable:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Answer: The characteristic polynomial for the first matrix is

$$\begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2.$$

Thus $\lambda = 1$ is the only eigenvalue. Let us check whether we have enough eigenvectors: If nullspace of $\lambda I - A$ has just one equation $x_2 = 0$, which gives the eigenvector $(1, 0)$. This means that we don't have a basis for the space R^2 made up of eigenvectors, so A is not diagonalizable.

For the other matrix, the characteristic polynomial

$$\begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1.$$

Its roots are $\pm\sqrt{-1}$, so A does not have real eigenvalues [and therefore cannot be diagonalizable in real space].

7. [12 pts] Let W be the subspace of R^4 with basis $(1,1,0,1)$, $(0,1,1,0)$. Find the projection of the vector $v=(2,1,3,0)$ onto W.

Answer: We must first use the Gram–Schmidt process to get an orthonormal basis for W. According to the given formula, we set $u_1=(1,1,0,1)$ and

$$\begin{aligned} u_2 &= (0,1,1,0)-\frac{(0,1,1,0)\cdot(1,1,0,1)}{(1,1,0,1)\cdot(1,1,0,1)}(1,1,0,1) \\ &= (0,1,1,0)-1/3(1,1,0,1)=(-1/3,2/3,1,-1/3) \end{aligned}$$

We now normalize u_1 and u_2 to obtain the desired basis w_1 and w_2:

$$w_1=\frac{u_1}{||u_1||}=1/\sqrt{3}(1,1,0,1)$$

$$w_2=\frac{u_2}{||u_2||}=\sqrt{1/15}(-1,2,3,-1)$$

Finally we use the projection formula

$$\mathrm{proj}_W(v)=(v\cdot w_1)w_1+(v\cdot w_2)w_2=1/5(2,11,9,2).$$

8. [10 pts] Give a solid explanation of the following fact: If A is a $m \times n$ matrix then the dimension of its row space is equal to the dimension of its column space. Begin by explaining the terms ‘row space’ and ‘column space’ of a matrix.

Answer: The *row space* of the matrix A is the *subspace* spanned by the row vectors of the matrix, that is, it is the set of all linear combinations of the row vectors $r_1, \ldots, r_m$ of A .[Note that this is much more that just the row vectors of the matrix.] Note that all the row vectors of all the matrices that occur when we carry out row reduction in A are linear combinators of the rows of A. Row reduction provides a set of vectors that is a basis for the row space.
The there a similar description of the column space.
[**Important point:**] As we carry out row reduction, the columns are changing with the new columns that may not lie in the original column space. **However**, the relationships between the original and the new columns have not changed [after all, the relationships are expressed as the solutions of the corresponding systems–and the solutions are still the same]. At the exit point of row reduction, it becomes clear that the columns with the pivots form a basis of the new column space, so that any additional column is a combination of the pivot columns. Consequently, the column of A in the same position is the ‘same’ linear combination of the columns of A designated by the pivot positions.

The routine to obtain a basis that is orthogonal from another basis [Gram–Schmidt process]: Input basis $S = \{u_1, \ldots, u_n\}$
Step 1: Set $v_1 = u_1$
Step 2: Compute $v_2, \ldots, v_n$ successively, one at a time, by

$$v_i = u_i - (\frac{u_i \cdot v_1}{v_1 \cdot v_1})v_1 - (\frac{u_i \cdot v_2}{v_2 \cdot v_2})v_2 - \cdots - (\frac{u_i \cdot v_{i-1}}{v_{i-1} \cdot v_{i-1}})v_{i-1}$$

Step 3: Set

$$w_i = \frac{v_i}{||v_i||}$$

Then $T = \{w_1, \ldots, w_n\}$ is an orthonormal basis.