Math 250–Section #3 Final

This exam has 12 questions, for 100 points. [The spreadsheet will scale to 200 points.] Answers must be explained to be acceptable–part credit to be given for good effort. There is a page with some formulas at the back and a scratch page. Good luck!

Name: ____________________

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1. [10 pts] Consider the following 4×3 matrix A and its reduced row echelon form $\mathrm{rref}(A)$:

$$A = \begin{bmatrix} 1 & 1 & 7 \\ 1 & 2 & 10 \\ 1 & 3 & 13 \\ 1 & 4 & 16 \end{bmatrix}, \quad \mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) $\dim \mathrm{Col}(A) =$ 2. Give a set of column vectors in $\mathbf{R}^4$ that is a basis for $\mathrm{Col}(A)$.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

(b) Write each column of A not in your basis as a linear combination of the basis vectors.

$$\begin{bmatrix} 7 \\ 10 \\ 13 \\ 16 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

(c) $\dim \mathrm{Null}(A) =$ 1. Give a set of column vectors in $\mathbf{R}^3$ that is a basis for $\mathrm{Null}(A)$.

$$\begin{bmatrix} -4 \\ -3 \\ 1 \end{bmatrix}$$

(d) $\dim \text{Row}(A) =$ 2 and $\dim \text{Null}(A^T) =$ 2.

(e) Suppose $\mathbf{y} \in \text{Col}(A)$ and $\mathbf{z} \in \text{Null}(A^T)$. Prove that $\mathbf{y}^T\mathbf{z} = 0$.

$z \in Null(A^T)$ means

$$A^T . z = 0$$

so each row of A^T is $\perp$ to z. But ~~cols~~ rows of A^T are columns of A

2. [10 pts] Let

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -3 \end{bmatrix}$$

(a) Find a basis of its nullspace.
(b) Find a basis of its row space.
(c) Is

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

a vector in the nullspace of A?
(d) Is there a vector v so that

$$Av = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}?$$

(e) Could v be found using Cramer's rule?

Answer:

$$\begin{matrix} 1 & -1 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -3 \end{matrix} \rightarrow \begin{matrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{matrix}$$

a) $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

b) $(1,0,0), (0,1,1)$

c) not multiple of a)

d) No, b/c $\det A = 0$

3. [10 pts] The matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}$ has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 6$, and $\lambda_3 = 3$. Let $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ be corresponding eigenvectors (normalized to have length one).

(a) Since A is symmetric and λ_1, λ_2, and λ_3 are all different, it follows that

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = \underline{\ 0\ }$$

$u_i \perp u_j$, $i \neq j$

(b) The 3×3 matrix $P = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3]$ satisfies

$$P^T P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (fill in the entries of this 3×3 matrix).

(c) Let P be the matrix of normalized eigenvectors from **(b)**. Then $A = PDP^{\mathrm{T}}$, where

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ (fill in the entries of this 3×3 matrix).

(d) Verify that the characteristic polynomial of A is $-t(t-3)(t-6)$ by using a cofactor expansion of an appropriate determinant. Give details.

$$\begin{vmatrix} 2-t & 2 & 0 \\ 2 & 3-t & 2 \\ 0 & 2 & 4-t \end{vmatrix} =$$ do it!

(Problem 3 continues on the next page)

(Continuation of Problem 3)

(e) Calculate eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ for A corresponding to the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 6$, and $\lambda_3 = 3$. (you don't have to normalize these vectors). After calculating each eigenvector $\mathbf{v}_i$, verify that $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$.
Recall that

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

nullspace of $A - \lambda I$

$\lambda = 0$

$$\left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 2 & 3 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$u_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

4. [8 pts] Let $u = [2, 1, 1]$, $v = [1, 0, 1]$ and $w = [0, 1, 1]$ be vectors of $\mathbb{R}^3$.
(a) What does it mean to say that a set $u_1, \ldots, u_n$ of vectors is linearly independent?
(b) Are u, v, w above linearly independent?
(c) If the vector $\alpha = [3, 1, -1]$ lies in the span of u, v, w, express it as a linear combination of these vectors.

Answer:

(a) Look up def'n in text

(b)

$$\begin{matrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{matrix} \rightarrow \begin{matrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

∴ lin. ind.

(c) Solve

$$\left.\begin{matrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{matrix}\;\right|\;\begin{matrix} 3 \\ 1 \\ -1 \end{matrix}$$

5. [8 pts] Let $A\mathbf{x} = \mathbf{b}$ be a system of linear equations.
(a) What does it mean to say that the **system is CONSISTENT**?

(b) Are all systems consistent? [If not, give example.]

(c) Explain the following fact: The system is consistent only if the matrix A and the extended matrix $[A|\mathbf{b}]$ have the same rank.

Answer:

Look up in textbook

6. [8 pts] (a) Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 1 \\ 3 & t & t \end{bmatrix}$$

(b) Determine the values of t for which the matrix is invertible.

Answer: (a) cofactor exp along row #1

$$\det A = 1 \begin{vmatrix} 2 & 1 \\ t & t \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 3 & t \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 3 & t \end{vmatrix}$$

$$= t - 2(2t-3) - 2t + 6 = -5t + 12$$

A inv $\Leftrightarrow$ $\det A \neq 0$

$-5t + 12 \neq 0$

$\therefore t \neq \frac{12}{5}$

7. [8 pts] Given the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix},$$

(a) Find its inverse.
(b) Find the determinant of $2(A)^2$.
(c) Find the determinant of AA^TA^{-1}.

Answer:

(a) Do algorithm

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right]$$

(b) $\det A = -1 \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$

$= -6 + 4 = -2$

$\det 2A^2 = 2^3 \cdot \det(A)^2$

$= 32$

(c) $\det AA^TA^{-1} = \det(A) \det(A^T) \cdot \det A^{-1}$ (cancel)

$\det A^T = \det A$

$\therefore = -2$

8. [9 pts] Let S be a subset of $\mathbf{R}^5$. Explain:
(a) When is S a subspace?
(b) What is a basis of S.
(c) Why the dimension of S cannot be larger than 5?
(d) If S is the solution set of the system of equations

$$A \cdot \mathbf{x} = \mathbf{b},$$

when is S a subspace?

Answer: (a), (b), (c) Look up in text

(d) If S is a subspace and x_1 & x_2 are solns, then $x_1 + x_2$ must be a solution

$A(x_1 + x_2) = b$?

$Ax_1 + Ax_2 = b$?

$b + b = b$ ∴ only happens when $b = 0$

9. [8 pts] (a) Consider the matrix $\mathbf{u} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$. Find the matrix $A = \mathbf{u}\mathbf{u}^T$ and show that it has rank 1.

(b) Consider the nonzero matrix $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Find the matrix $B = \mathbf{v}\mathbf{v}^T$ and show that it has rank 1.

(c) If $\mathbf{v} \neq 0$, show that $\mathbf{v}^T\mathbf{v}$ has always rank 1.

Answer:

(a) $$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

colums are multis of first

(b) Same in general

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} [a, b, c] = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$$

If, say, $a \neq 0$, 2nd & 3rd cols are multis of first.

Same argument if $b \neq 0$ or $c \neq 0$

10. [8 pts] (a) Find an orthogonal matrix S such that $S^{-1}AS$ is diagonal, where

$$A = \begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix}.$$

(b) Use Part (a) to decide whether the equation $10x^2 + 6xy + 2y^2 = 18$ represents an ellipse, a parabola or a hyperbola.

Answer:

Done offline

11. [8 pts] Let W be the subspace of $\mathbf{R}^3$ spanned by the vector $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(a) Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ Find the orthogonal projection $\mathbf{w}$ of $\mathbf{v}$ onto W.

$$\text{proj}_W v = \frac{v \cdot u}{u \cdot u} u = \frac{4}{6} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

(b) Suppose $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in $W^{\perp}$. Write down the equation satisfied by x_1, x_2, x_3. Use this to find a basis for $W^{\perp}$.

Answer:

free var

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = x_1 + 2x_2 + x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

basis of $W^{\perp}$

12. [5 pts] Explain the following fact: For ANY square matrix A, the matrix $B = AA^T$ is diagonalizable.

Answer:

$B = AA^T$ is always symmetric

$$B^T = (AA^T)^T = (A^T)^T A^T$$

$$= AA^T$$

used properties: transpose of a product is product of transposes in reverse order

Scratch page

The routine to obtain a basis that is orthogonal from another basis [Gram–Schmidt process]: Input basis $S = \{u_1, \ldots, u_n\}$
Step 1: Set $v_1 = u_1$
Step 2: Compute $v_2, \ldots, v_n$ successively, one at a time, by

$$v_i = u_i - (\frac{u_i \cdot v_1}{v_1 \cdot v_1})v_1 - (\frac{u_i \cdot v_2}{v_2 \cdot v_2})v_2 - \cdots - (\frac{u_i \cdot v_{i-1}}{v_{i-1} \cdot v_{i-1}})v_{i-1}$$

Step 3: Set

$$w_i = \frac{v_i}{||v_i||}$$

Then $T = \{w_1, \ldots, w_n\}$ is an orthonormal basis.

Quadratic Equation: The roots of

$$ax^2 + bx + c = 0, \quad a \neq 0$$

are:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$