Math 250–Section #C2   Selected Problems

Name: ____________________

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1. (4 pts) (a) Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$

(b) Use the answer to (a) to solve the system of equations

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 6 \end{bmatrix}.$$

Answer: (a) Using the Gaussian algorithm, we reduce

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

into

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7/3 & 2/3 & 1 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 8/3 & -1/3 & -1 \end{array}\right] = \left[I \mid A^{-1}\right]$$

(b) Multiply both sides of the equation by A^{-1}

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 9 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -13 \\ -12 \\ 17 \end{bmatrix}.$$

2. (3 pts) [Give explanations using 3×3 matrices]
(a) What are upper triangular matrices?
(b) If A and B are upper triangular, show that AB is also upper triangular.
(c) If A is upper triangular and invertible, show that A^{-1} is also upper triangular.

Answer: Will discuss in class.

3. (3 pts) Find the LU decomposition of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 4 & -1 & 4 \\ -2 & 1 & 5 \end{bmatrix}$$

Answer:

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 4 & -1 & 4 \\ -2 & 1 & 5 \end{bmatrix} \xrightarrow{-2r_1+r_2} E_1 A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ -2 & 1 & 5 \end{bmatrix} \xrightarrow{r_1+r_3} E_2 E_1 A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{bmatrix} = U$$

Now we compute $L = (E_2 E_1)^{-1} = E_1^{-1} E_2^{-1}$ (see the order?)

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-r_1+r_3} E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{2r_1+r_2} E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = L$$

A direct check shows

$$A = LU.$$

Observe the reverse sequence of the related elementary transformations applied to A to get U, and applied to I to obtain L.

2. (3 pts)
(a) Give an example (use 2×2 matrices) that a sum $A + B$ of two invertible matrices may not be invertible.
(b) Argue that if A, B, C, D are 2×2 matrices such that

$$ABC = D,$$

and A, C, D invertible, then B is also invertible.
(c) Explain why if A is invertible then A^3 is also invertible.

Answer: (a) If

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

they are both invertible matrices, but $A + B = O_2$, which obviously is not invertible.

(b) We solve for B: multiply

$$ABC = D$$

by A^{-1} on the LEFT

$$A^{-1}ABC = BC = A^{-1}D,$$

and now multiply both sides on the RIGHT by C^{-1}

$$BCC^{-1} = B = A^{-1}DC^{-1},$$

which shows that B is the product of 3 invertible matrices [so invertible].

(c) $A^3 = AAA$, so it is the product of 3 invertible matrices. Its inverse is obviously $(A^{-1})^3$.

3. (3 pts) Given

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 5 & 6 & 7 \\ 5 & 7 & 8 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 2 & 2 & 3 \\ 5 & 6 & 7 \\ 1 & 3 & 2 \end{bmatrix},$$

find an elementary matrix E such that

$$EA = B.$$

Begin by explaining what is an *elementary* matrix.

Answer: An $n \times n$ matrix E is called an elementay matrix if E can be obtained from I_n by a single row operation.
If

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix},$$

E is elementary ("add to row 3 (-2) times row 1") and

$$EA = B.$$

Find all the values for t for which the resulting system of equations (a) has no solution, (b) a unique solution, and (c) infinitely many solutions.

$$\begin{array}{rcrcrcl} x & + & y & - & z & = & 2 \\ x & + & 2y & + & z & = & 3 \\ x & + & y & + & (t^2-5)z & = & t \end{array}$$

Answer: Let us use Gauss–Jordan's on this system

$$\left.\begin{array}{ccr} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & t^2-5 \end{array}\right| \begin{array}{c} 2 \\ 3 \\ t \end{array}$$

Using the 1 in the position $(1,1)$ as a pivot, we get

$$\left.\begin{array}{ccr} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & t^2-4 \end{array}\right| \begin{array}{r} 2 \\ 1 \\ t-2 \end{array}$$

We are now ready:
If $t \neq \pm 2$, we get a unique solution since we have 3 pivots.

If $t = 2$, we have an infinite number of solutions, since the last variable is free.

If $t = -2$, it is not consistent.