Math 250–Section #2 Hourly #2

Name: ____________________

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1. [20 pts] Let

$$A = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \quad \text{find:}$$

(a) Verify that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A
(b) Find its characteristic polynomial
(c) Find its eigenvalues
(d) Find corresponding eigenvectors
(e) Find the matrix P such that $P^{-1}AP$ is a diagonal matrix

Answer: (a) Just observe that

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

which shows that -2 is an eigenvalue and that the given vector is an eigenvector. (This item was placed here to tell you that -2 would be a root of the characteristic polynomial, in case you had trouble with the factorization.)

(b)

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = \lambda^3 - 3\lambda + 2.$$

(c) We already know one root (-2); in any event you should first look for roots in the divisors of the constant term: ± 1, ± 2. 1 and -2 work out:

$$\lambda^3 - 3\lambda + 2 = (\lambda + 2)(\lambda - 1)(\lambda - 1).$$

(d) The eigenvector for $\lambda = -2$ we already know from part (a); easy to do again but saves time. For $\lambda = 1$ we substitute and find the nullspace of $I - A$: There is just one equation, $x + y + z = 0$, out of which we get two independent eigenvectors, $(-1, 0, 1)$ and $(-1, 1, 0)$.

(e) We use the three vectors as the columns of the matrix P such that $P^{-1}AP$ is diagonal.

2. [16 pts] Given the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ 7 & -8 & -1 \end{bmatrix}$$

Find:
(a) basis for its row space
(b) basis for its column space
(c) basis for its nullspace
(d) basis for its left nullspace
(e) explain the following fact [it happens in this example]: If the row vectors of a square matrix are linearly dependent then the column vectors are also linearly dependent.

Answer: Done lots & lots of time. (Please ask in office if you still have difficulty with these problems.)
Summary answer: You should get two row vectors after row reduction as the basis of the row space. The columns marked by the pivots will form a basis of the col space.
The rank of a matrix is the dimension of its row space [or column space]. When the rows are dependent the rank is smaller than the size of the [square] matrix, so the columns will also be dependent.

3. [16 pts] (a) Show that the set W of all 2×2 matrices of the form [notice the pattern]

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

is a vector space
(b) Find a basis of W
(c) What is the dimension W [explain]?
(d) If A and B are matrices of this format, verify that AB has the same format.

Answer: (a) If you add two such matrices,

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a' & b' \\ -b' & a' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ -b-b' & a+a' \end{bmatrix}$$

you get a matrix in the same format. Same if you multiply A by a scalar c. These two properties make W a subspace.

(b) Note that

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

which shows that the two matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

of W span the subspace and are independent: therefore they form a basis.

(c) The dimension of a subspace is the number of elements in a basis: 2 in this case.

(d) If you multiply two such matrices,

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a' & b' \\ -b' & a' \end{bmatrix} = \begin{bmatrix} aa'-bb' & ab'+ba' \\ -(ab'+ba') & aa'-bb' \end{bmatrix},$$

same pattern.

4. [12 pts] Let $u = (1, 0, 2, 2)$, $v = (2, 1, 2, 1)$ and $w = (1, 2, 1, 2)$ be vectors of $\mathbb{R}^4$.
(a) Let P be the plane spanned by u and v. Find an orthonormal basis for P.
(b) Find the projection of w onto P
(c) Find the distance from w to P

Answer: I will put more detail, time permitting. You can always come to office.) (Note that you only have to use $G - S$ to u and v (not u, v, w!)

5. [12 pts] Recall that a square matrix Q is orthogonal if $Q^TQ = I$.
(a) Which of the two matrices

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

is orthogonal?
(b) Complete the matrix so that it becomes orthogonal

$$\begin{bmatrix} 2/\sqrt{5} & a \\ 1/\sqrt{5} & b \end{bmatrix}$$

(c) Show that the determinant of any orthogonal matrix Q is always ± 1.

Answer: (a) If you check, you see that the first matrix is not but the second satisfies the condition.

(b) We must choose a and b so that the column is perpendicular to he first column and of magnitude 1: get $a = 1/\sqrt{2}$, $b = -2/\sqrt{2}$ (or reversed signs).

(c) (Done in class.)

$$\det(Q^TQ) = \det(Q^T)\det Q = \det(Q)\det(Q) = 1,$$

since $\det(Q^T) = \det(Q)$ for any matrix. It shows that $\det Q = \pm 1$.

6. [12 pts] For the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

(a) Find the eigenvalues and corresponding eigenvectors
(b) Diagonalize it (i.e. find P so that $P^{-1}AP$ is diagonal

Answer: (a) (This is a symmetric real matrix so it will be diagonalizable.) The characteristic polynomial is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - \lambda - 1.$$

We use the (supplied) quadratic formula to find the eigenvalues:

$$\lambda_1 = \frac{1 + \sqrt{5}}{2}, \quad \lambda_2 = \frac{1 - \sqrt{5}}{2}$$

The eigenvectors are $(\lambda_1, 1)$ and $(\lambda_2, 1)$, and the matrix that diagonalizes A has the eigenvectors as column vectors.

7. [12 pts] If A is a 3×3 matrix has eigenvalues equal to $0, 1, 2$
(a) Show that the matrix $A^2 + I$ has eigenvalues $1, 2, 5$ (I is the identity matrix of the same size as A).
(b) Explain why $\det(A) = 0$ but $\det(A^2 + I) = 10$.

Answer:
(a) (Like in the review exam.) If v is an eigenvector for A, say $Av = \lambda v$, then $(A^2 + I)v = A^2 v + Iv = (\lambda^2 + 1)v$. Thus if A has eigenvalues $0, 1, 2$, then $A^2 + I$ has corresponding eigenvalues $0^2 + 1, 1^2 + 1, 2^2 + 1$.

(b) The determinant of a matrix is the product of its eigenvalues always.

The routine to obtain a basis that is orthogonal from another basis [Gram–Schmidt process]: Input basis $S = \{u_1, \dots, u_n\}$
Step 1: Set $v_1 = u_1$
Step 2: Compute $v_2, \dots, v_n$ successively, one at a time, by

$$v_i = u_i - (\frac{u_i \cdot v_1}{v_1 \cdot v_1})v_1 - (\frac{u_i \cdot v_2}{v_2 \cdot v_2})v_2 - \cdots - (\frac{u_i \cdot v_{i-1}}{v_{i-1} \cdot v_{i-1}})v_{i-1}$$

Step 3: Set

$$w_i = \frac{v_i}{||v_i||}$$

Then $T = \{w_1, \dots, w_n\}$ is an orthonormal basis.