Math 250–Section #4 Hourly #1

The exam has 8 questions and 1 blank page. Note: Correct answers with justification required! Recall: no notes or calculator!

Name: ____________________

---

1. (15 pts) (a) Determine if the system of equations

$$\begin{array}{rcrcrcrcr} x & + & 2y & + & z & + & w & = & -1 \\ -2x & - & 6y & - & z & & & = & 5 \\ x & + & 3y & + & 2z & + & 2w & = & 2 \end{array}$$

is consistent.

(b) If it is, find the general solution in vector format.

Answer: Row reduction will yield

$$\left.\begin{array}{cccr} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 4/3 \end{array}\right| \begin{array}{r} -4 \\ 0 \\ 3 \end{array}$$

from whih you conclude:

(a) system is consistent

(b)

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 3 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1/3 \\ 1/3 \\ -4/3 \\ 1 \end{bmatrix}$$

2. (16 pts) Given the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ -4 & 1 & 2 & 3 \\ 1 & 0 & -1 & -1 \\ 1 & 3 & -1 & -1 \\ 2 & 1 & -1 & -2 \end{bmatrix}$$

(a) Find its reduced echelon form R of A.
(b) What are the rank and the nullity of A. Explain why these two numbers always add to the number of columns.
(c) Argue that the rows of R with pivots are linearly independent.
(d) Argue that the columns of A with pivots are linearly independent.

Answer: Done in class in review, basically.

3. (15 pts) (3) (a) Explain what is a *linear combination* of vectors.

(3) (b) Explain what is the *span* of a set $\{v_1, \ldots, v_m\}$ of vectors.

(9) (c) Find out whether the vector $(1, 3, 5, 1)$ is a linear combination of $(2, 1, 1, -1)$, $(7, 3, 0, -1)$ and $(0, 1, 4, 3)$.

Answer: Look up definitions in book for (a) and (b). For (c), we test whether

$$(1, 3, 5, 1) = a(2, 1, 1, -1) + b(7, 3, 0, -1) + c(0, 1, 4, 3)$$

is possible. Solving for a, b, c we see that the system of 4 equations is not consistent–so answer is no.

4. (15 pts) Explain the following basic facts about systems of linear equations

$$A \cdot \mathbf{x} = \mathbf{b}$$

(a) If the ranks of the matrices

$$A, \quad [A \mid \mathbf{b}]$$

are different, then the system does not have solution.

(b) If the system is consistent, then $\mathbf{b}$ is a linear combination of the columns of A.

(c) If the system has 2 solutions, then it must have many more.

Answer: (a) and (b) have been discussed often. For (c): the event can only occur if we have a consistent system with at least one free variable x_i. For each of its (infinitely many) values we would have distinct solutions.

5. (12 pts) Given the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

(a) Find the inverse of A.
(b) If we have the matrix equality

$$ABA = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

show how to solve for B. [You can leave the last calculation indicated.]

Answer: (a)

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -4 & 3 & -1 \\ 2 & -2 & 1 \end{bmatrix}$$

(b) Multiplying the equality by $A^(- 1$ on the left and on the right gives

$$B = A^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} A^{-1}$$

6. (12 pts) (a) Compute the determinant of the matrix

$$\begin{bmatrix} 4 & 2 & 2 & -3 \\ 6 & -1 & -1 & 5 \\ 0 & -3 & 0 & 0 \\ 2 & -5 & 0 & 0 \end{bmatrix}$$

using cofactor expansion along a row or column of your choice.

(b) Prove that, for any 2×2 matrices A and B, $\det(AB) = \det(A)\det(B)$.

Answer: (a) Expanding along row 3 (perhaps quickest) gives det = 42

Substitute

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for A, and

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

for B and compute.

7. (10 pts) Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors of $\mathbb{R}^3$. Prove that if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent then $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{w}\}$ is linearly independent.

Answer: If the three sums were not linearly independent, we would have an equation

$$a(\mathbf{u}+\mathbf{v}) + b(\mathbf{v}+\mathbf{w}) + c(\mathbf{u}+\mathbf{w}) = 0,$$

where one of the 3 scalars is not zero. But this equation can also be written

$$(a+c)\mathbf{u} + (a+b)\mathbf{v} + (b+c)\mathbf{w} = 0.$$

Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, we have

$$\begin{aligned} a+c &= 0 \\ a+b &= 0 \\ b+c &= 0, \end{aligned}$$

which solving gives us $a = b = c = 0$, as desired.

8. (5 pts) Prove that for ANY square matrix A, the matrices $A + A^T$ and AA^T are symmetric.

Answer: We use the following properties of taking the transpose of a matrix: $(A + B)^T = A^T + B^T$ and $(AB)^T = B^T A^T$ (notice the order).

$(A + A^T)^T = A^T + (A^T)^T = A^T + A$, so $A + A^T$ is symmetric

$(AA^T)^T = (A^T)^T A^T = AA^T$, so AA^T is symmetric

This page is blank. Use it for scratch.