Math 250–Section #5 Hourly #1

Note: Correct answers with justification required!

Name: ____________________

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1. [16 pts] Given the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 3 \\ -4 & 2 & 3 & 1 \\ 2 & -1 & -2 & 1 \\ 1 & -1 & -2 & 2 \end{bmatrix}$$

(a) Find its reduced echelon form R of A.
(b) What are the rank and the nullity of A. Explain why these two numbers always add to the number of columns.
(c) Argue that the rows of R with pivots are linearly independent.
(d) Argue that the columns of A with pivots are linearly independent.

Answer: (a) The reduced echelon form of A is

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) From R we get: rank $A = 4$ (number of pivots), nullity $A = 0$ (number of free variables). That we always have rank + nullity = number of columns is obvious since rank counts the number of columns with pivots and nullity counts the number of columns without pivots, so together they count all columns.
(c) The 4 rows of R with pivots, v_1, v_2, v_3, v_4 have the property that a linear combination

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = (c_1, c_2, c_3, c_4),$$

which is only zero if $c_1 = c_2 = c_3 = c_4 = 0$. So the v_i's are linearly independent.

(d) If u_1, u_2, u_3, u_4 are the columns of A then any linear relation

$$x_1u_1 + x_2u_2 + x_3u_3 + x_4u_4 = 0$$

gives rise to a similar relation for the matrix R (as the two systems have the same solutions). But note that when we use the columns of R we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

so $x_1 = x_2 = x_3 = x_4 = 0$ and the u_i's are linearly independent.

2. [12 pts] Write the parametric representation of the general solution of the system of linear equations

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & - & x_3 & + & 3x_4 & = & 0 \\ 2x_1 & - & 4x_2 & + & 3x_3 & + & 7x_4 & = & 0 \\ -2x_1 & + & 4x_2 & + & x_3 & + & 5x_4 & = & 0 \end{array}$$

Answer: You should find that the system has nullity 1, that just x_2 is the free variable. The parametric representation of the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

3. [10 pts] Solve the system of equation and show that it has a unique solution for any value of t:

$$\begin{array}{ccccccc} x & + & y & - & tz & = & 2 \\ x & + & 2y & + & z & = & 3 \\ x & + & 3y & + & tz & = & t \end{array}$$

Answer:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t^2 + \frac{7}{2}t + 3 \\ -\frac{1}{2}t^2 - \frac{3}{2}t - 1 \\ 2 - \frac{1}{2}t \end{bmatrix}$$

4. [12 pts] Find the inverses of the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 5 & 6 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & -5 & -29 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix}$$

5. [10 pts] (a) Give a reason why the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

are linearly independent,
(b) but the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

cannot be linearly independent for any choice of a, b, c, d.

Answer: (a) When you row reduce the matrix formed by the 3 column vectors, you get 3 pivots–so the vectors are linearly independent.
(b) Processing the 5 vectors [they form a 4×5 matrix] would give at most 4 pivots, which means that (regardless of how the constants are chosen) the vectors would be linearly dependent.

6. [10 pts] (a) Explain what is a *linear combination* of vectors.
(b) Find out whether the vector $(9, 6, 12, 5)$ is a linear combination of $(2, 1, 3, 0)$, $(7, 3, 1, -1)$ and $(0, 1, 4, 3)$.

Answer: (a) A linear combination of the vectors $v_1, v_2, \dots, v_n$ is a vector

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n,$$

where the c_i are numbers.
(b) To see whether $(9, 6, 12, 5)$ is a linear combination of the other vectors, one checks whether the system of equations

$$(9, 6, 12, 5) = x_1(2, 1, 3, 0) + x_2(7, 3, 1, -1) + x_3(0, 1, 4, 3)$$

is consistent. Solving we get $x_1 = 1$, $x_2 = 1$, $x_3 = 2$, and the answer is yes.

7. [10 pts] Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$.
(a) Verify that $A^2 - 4A + I_2 = 0$.
(b) Use the result of (a) to show that A is invertible and that $A^{-1} = 4I_2 - A$.

Answer: (a) Just verification.
(b) From the equation, we get that

$$A(A - 4I_2) = -I_2,$$

and therefore

$$A(4I_2 - A) = I_2$$

so $4I_2 - A$ is the inverse of A.

8. [10 pts] Given a group of 4 persons, associate a 4×4 matrix A defined by

$$a_{ij} = \begin{cases} 1 & \text{if } i \neq j \text{ and person } i \text{ likes person } j \\ 0 & \text{otherwise} \end{cases}$$

We say that i and j are friends if they like each other; that is, $a_{ij} = a_{ji} = 1$. Suppose that A is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

(a) List all pairs of friends.
(b) Given an interpretation of the entries of A^2.
(c) Let B be the 4×4 matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{if persons } i \text{ and } j \text{ are friends} \\ 0 & \text{otherwise} \end{cases}$$

Determine B. Is B a symmetric matrix?
Answer: (a) i and j are friends when $a_{ij} = a_{ji} = 1$. There are 4 pairs that meet this conditions: $(1\&2), (1\&4), (2\&3), (3\&4)$
(b) Calculate A^2 to get

$$\begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 1 & 2 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

Note that its entries, say at cell (i, j), are

$$\sum_{k=1}^{4} a_{ik} a_{kj}.$$

The term $a_{ik}a_{kj}$ is 1 or 0 depending on whether i likes k who likes j or not. Note also that A^2 responds to a question we did not ask: the elements in the diagonal are the number of friends of the corresponding person.
(c)

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

which is symmetric [naturally, from the nature of the relationship it expresses and of how we set up the matrix].

9. [10 pts] Find the LU decomposition of the matrix

$$\begin{bmatrix} 2 & 3 & 4 \\ 6 & 8 & 10 \\ -2 & -4 & -5 \end{bmatrix}$$

Answer: You should get [and check] that

$$\begin{bmatrix} 2 & 3 & 4 \\ 6 & 8 & 10 \\ -2 & -4 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

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