Vector Spaces

Definition 1. *A* **vector space** *is an Abelian group* $(\mathbf{V}, +)$ *(vectors), a field* F *(scalars), and a binary operation* $\cdot : F \times \mathbf{V} \to \mathbf{V}$ *(scalar multiplication) satisfying the following properties for all scalars* $a, b \in F$ *and vectors* $\mathbf{x}, \mathbf{y} \in \mathbf{V}$*:*

(i) $a \cdot (\mathbf{x} + \mathbf{y}) = a \cdot \mathbf{x} + a \cdot \mathbf{y}$*;*

(ii) $(a + b) \cdot \mathbf{x} = a \cdot \mathbf{x} + b \cdot \mathbf{x}$*;*

(iii) $(ab) \cdot \mathbf{x} = a \cdot (b \cdot \mathbf{x})$*;*

(iv) $1 \cdot \mathbf{x} = \mathbf{x}$*.*

We say that $\mathbf{V}$ is a vector space over F, or simply that $\mathbf{V}$ is a vector space.

Remark. The above properties are listed on page 7 of the textbook without using the terminology of groups. You can identify them as follows: VS 1='Abelian', VS 2,3,4 are the group axioms, VS 7,8,6,5 are listed here as (i)-(iv).

Notations: Addition in F and in $\mathbf{V}$ are both denoted by $+$, multiplication in F and the scalar multiplication are both denoted by $\cdot$ (the context makes it clear which one is used). Similarly, 0 stands for the identity for $+$ and $-x$ for the (additive) inverse of x both in F and in $\mathbf{V}$ (although in this handout I use bold face $\mathbf{0}$ for the 'zero vector'). 1 is the multiplicative identity in F, and x^{-1} is the multiplicative inverse of the non-zero element $x \in F$. We often write ab and $c\mathbf{x}$ instead of $a \cdot b$ and $c \cdot \mathbf{x}$.

The following **properties** are easily seen. They express that most standard rules of high-school algebra are valid for vector spaces:

For all $a, b \in F$ and $\mathbf{x}, \mathbf{y} \in \mathbf{V}$:

- (a) $a \cdot \mathbf{0} = \mathbf{0}$
- (b) $0 \cdot \mathbf{x} = \mathbf{0}$ (Note the two different 0s!)
- (c) $(-a) \cdot \mathbf{x} = -(a \cdot \mathbf{x}) = a \cdot (-\mathbf{x})$
- (d) $(-a) \cdot (-\mathbf{x}) = a \cdot \mathbf{x}$
- (e) $(-1) \cdot \mathbf{x} = -\mathbf{x}$
- (f) $a \cdot (\mathbf{x} - \mathbf{y}) = a \cdot \mathbf{x} - a \cdot \mathbf{y}$
- (g) $(a - b) \cdot \mathbf{x} = a \cdot \mathbf{x} - b \cdot \mathbf{x}$
- (h) $a \cdot \mathbf{x} = \mathbf{0}$ **iff** either $a = 0$ or $\mathbf{x} = \mathbf{0}$

Remark: Very **formally** (too formally perhaps) a vector space is a six-tuple $(\mathbf{V}, F, \oplus, \odot, +, \cdot)$, where $\oplus : F \times F \to F$, $\odot : F \times F \to F$, $+ : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$, $\cdot : F \times \mathbf{V} \to \mathbf{V}$, and the operations satisfy 17 properties: the 9 properties expressing that $(F, \oplus, \odot)$ is a field, the 4 properties expressing that $(\mathbf{V}, +)$ is an Abelian group, and the above 4 properties connecting the scalar multiplication $\cdot$ with the other operations:

(i) $c \cdot (\mathbf{x} + \mathbf{y}) = c \cdot \mathbf{x} + c \cdot \mathbf{y}$;

(ii) $(a \oplus b) \cdot \mathbf{x} = a \cdot \mathbf{x} + b \cdot \mathbf{x}$;

(iii) $(a \odot b) \cdot \mathbf{x} = a \cdot (b \cdot \mathbf{x})$;

(iv) $1 \cdot \mathbf{x} = \mathbf{x}$.

We will use the first (sensible) notation, but keep in mind the dual roles of $+, -, \cdot$, and 0 (and also that we did not define multiplication of vectors).

A cautionary example:

Let $\mathbf{V} = \mathbb{Z}$ and $F = \mathbb{Z}_2$. Define (vector-)addititION on $\mathbb{Z}$ as ordinary addition of integers, and define scalar multiplication by the natural rules: $0 \cdot x = 0$ and $1 \cdot x = x$ for all $x \in \mathbb{Z}$.
Is $\mathbf{V}$ a vector space over F ? The answer is NO.
Proof (indirect): Assume it is. Then we would have

$$10 = 5 + 5 = 1 \cdot 5 + 1 \cdot 5 = (1 + 1) \cdot 5 = 0 \cdot 5 = 0,$$

a contradiction. $\square$

If you are confused about this, it would help a little to distinguish the different components: Let us write 0 and 1 for the elements of F, but $\underline{n}$ for the "vectors" $n \in \mathbf{V} = \mathbb{Z}$. Also, as above, we write $\oplus$ and $\odot$ for operations in F (mod 2 operations) and $+$ for addition in $\mathbf{V}$ (which is *not* mod 2, so 5+5 is 10 and not 0), and $\cdot$ for scalar multiplication.
Then the above line will read as:

$$\underline{10} = \underline{5} + \underline{5} = 1 \cdot \underline{5} + 1 \cdot \underline{5} = (1 \oplus 1) \cdot \underline{5} = 0 \cdot \underline{5} = \underline{0}\,, \qquad \text{a contradiction.}$$

The following general conclusion can be derived from this argument:

Theorem. *Let $\mathbf{V}$ be a non-trivial vector space over a field F. Then every non-zero vector in $\mathbf{V}$ has the same order. The characteristic of $\mathbf{V}$ can be defined as this common order if it is finite and 0 if the common order is infinite. With this definition, $char(\mathbf{V}) = char(F)$.*