1 Euclidean Constructible Numbers

**Theorem:** The set of numbers, $E$, that can be constructed using Euclidean construction rules forms a field.

1.1 Introduction to Fields

Reminder: Let $*$ be an associative binary operation on a nonempty set $G$. We call $(G, *)$ a group if $G$ has an identity and each element of $G$ has an inverse.

**Definition:** A field is a set $F$ with two commutative binary operations $+$ and $*$ (addition and multiplication) such that:

(i) $F$ is a group under $+$
(ii) $F \setminus \{0\}$ is a group under $*$
(iii) distributive law: $(\forall a, b, c \in F) \ a * (b + c) = a * b + a * c$

Which are fields under the usual addition and multiplication operations?

- $\mathbb{Q}$
- $\mathbb{R}$
- $\mathbb{Z}_5$
- $\mathbb{C}$
- $\mathbb{Z}$
- $\mathbb{Z}_6$

- More generally, for which $n$ is $\mathbb{Z}_n$ a field?

**Definition:** Suppose $F, K$ are fields. Then $F \preceq K$ means $F$ is a subfield of $K$, that is, $F \subseteq K$ and $F$ is a field with the same $+$ and $*$ of $K$.

- $\mathbb{Q} \preceq \mathbb{R} \preceq \mathbb{C}$
- $\mathbb{Z}_5 \not\preceq \mathbb{Z}_7$
- $\mathbb{Z}_5 \not\preceq \mathbb{R}$
1.2 Some facts about Euclidean Geometry

- A collapsing compass is equivalent to the modern or noncollapsing compass. That is, even with the collapsing compass we can copy given distances to other locations.
- Given a line segment $\overline{AB}$ along line $L$, we can construct a perpendicular bisector.
- We can construct a circle with a given diameter.
- Given a line $L$ and a point $P$, we can construct a line through $P$ that is perpendicular to $L$. (two cases: $P$ may or may not be on $L$)
- Given a line $L$ and a point $P$ not on $L$, we can construct a line through $P$ that is parallel to $L$.

1.3 Constructible Numbers

Once some distances $x$ and $y$ are constructed, we can construct new ones from them:

- $(\forall x, y \in \mathbb{E}) x + y \in \mathbb{E}$
- $(\forall x, y \in \mathbb{E}) x - y \in \mathbb{E}$
- $(\forall x, y \in \mathbb{E}) x \cdot y \in \mathbb{E}$
- $(\forall x, y \in \mathbb{E})$ if $y \neq 0$, then $x/y \in \mathbb{E}$

These four properties show that $\mathbb{E}$ is a field, containing all elements of $\mathbb{Q}$:

$$\mathbb{Q} \leq \mathbb{E} \leq \mathbb{R} \leq \mathbb{C}$$

However, there is an extra property of $\mathbb{E}$ that shows that it is larger than $\mathbb{Q}$, namely that $\mathbb{E}$ is closed under taking square root:

- $(\forall x \in \mathbb{E})$ if $x \geq 0$, then $\sqrt{x} \in \mathbb{E}$

Thus, $\mathbb{E}$, the set of constructible numbers, is strictly larger than the set of rational numbers since $\mathbb{E}$ is also closed under taking square root (of positive elements); an example of a number which is in $\mathbb{E}$ but not in $\mathbb{Q}$ is $\sqrt{2}$. 
