Linear transformations

Throughout this note, V, W, and Z are vector spaces over the **same** field F.

Definition. *A* **linear transformation** *from V to W is a function $T : V \to W$ that preserves linear combinations:*

$$T(c_1\alpha_1 + c_2\alpha_2) = c_1T(\alpha_1) + c_2T(\alpha_2) \quad \textit{for all} \;\; \alpha_1, \alpha_2 \in V \;\; \textit{and} \;\; c_1, c_2 \in F.$$

We usually perform two separate tests:
T should preserve vector-addition: $T(\alpha + \beta) = T(\alpha) + T(\beta)$ for all $\alpha, \beta \in V$, and
T should preserve scalar multiplication: $T(c \cdot \alpha) = c \cdot T(\alpha)$ for all $\alpha \in V$ and $c \in F$.

Theorem 1 (Preservation of subspaces). *Let $T : V \to W$ be linear, and let A be a subspace of V. Then $T(A) := \{T(x) : x \in A\}$ is a subspace of W.*
Let B be a subspace of W. Then $T^{-1}(B) := \{x \in V : T(x) \in B\}$ is a subspace of V.

Definition. *The* **range** *of T is the set $R(T) = Range(T) = Im(T) = \{T(x) : x \in V\}$. The* **rank** *of T is the dimension of the range of T. The* **null space** *(or kernel) of T is the set $N(T) = Null(T) = Ker(T) = \{x \in V : T(x) = 0\}$. The* **nullity** *of T is the dimension of $N(T)$. (In view of the theorem above, the range $R(T) = T(V)$ is a subspace of W and the null space $N(T) = T^{-1}(\{\mathbf{0}\})$ is a subspace of V, so they have dimensions.)*

Here are some easy but important facts. For every *linear* transformation $T : V \to W$,

- $T(\mathbf{0}) = \mathbf{0}$ (note the two different 0-vectors: one is in V, one in W).
- Spanning sets of V (e.g., bases of V) are mapped into spanning sets of $R(T)$. Hence $\text{rank}(T) \leq \dim(V)$. (Clearly, $\text{rank}(T) \leq \dim(W)$ also holds.)
- In general, for any subspace U of V, spanning sets of U are mapped into spanning sets of $T(U)$. Hence $\dim(T(U)) \leq \dim(U)$: "linear transformations cannot increase dimensions!"
- **T can be defined arbitrarily on a basis of V, and then it is uniquely determined on the whole domain V.** Consequently, if T and T' are linear transformations from V to W, and they agree on a basis of V, then they agree everywhere on V.
- If $f : V \to W$ and $g : W \to Z$ are linear transformations, their **composition** $g \circ f : V \to Z$, defined as $(g \circ f)(\alpha) = g(f(\alpha))$ ($\alpha \in V$), is also a linear transformation.

The Fundamental Theorem of Linear Algebra

Theorem 2 (FTLA). *Let T be a linear transformation from V to W. Then V has a basis that can be written as $A \cup B$, where A and B are disjoint, A is a basis for the null space $N(T)$, and $T(B)$ is a basis for the range $R(T)$. Hence, if V is finite-dimensional, then*

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$

Theorem 3 (FTLA – matrix form). *Let M be an $m \times n$ matrix over the field F. The row space of M is orthogonal to the null space of M (of course), and their dimensions add up to n. The row space and the column space have the same dimension (the rank of M).*

Invertible linear transformations

Throughout this page, T is a linear transformation from V to W.

Definition. *A function $f : V \to W$ is* **invertible** *if there exists a function $g : W \to V$ such that $g \circ f = I_V$ and $f \circ g = I_W$. (I_V is the identity transformation on V.)*

It is easy to see that f is invertible if and only if it is one-to-one and onto (bijection). It is also easy to see that when such g exists, it is unique. We usually write f^{-1} for this unique inverse g. When f is an invertible linear transformation, then so is f^{-1} (from W to V).

Definition. *The linear transformation $T : V \to W$ is* **non-singular** *if $Null(T) = \{\mathbf{0}\}$. (That is, $\text{nullity}(T) = 0$, or $\alpha \neq \mathbf{0}$ implies $T(\alpha) \neq \mathbf{0}$, or $T(\alpha) = \mathbf{0}$ implies $\alpha = \mathbf{0}$.)*

Theorem 4. *Let $T : V \to W$ be a linear transformation.*

The Following Are Equivalent

- *T is non-singular.*
- *T is one-to-one.*
- *T maps linearly independent vectors into linearly independent vectors.*
- *For every finite-dimensional subspace U of V, $\dim(T(U)) = \dim(U)$.*

Theorem 5. *Let $T : V \to W$ be a linear transformation, and assume that V has a basis.*

The Following Are Equivalent

- *T is one-to-one.*
- *T maps every basis into linearly independent vectors.*
- *V has a basis which T maps into linearly independent vectors.*

Theorem 6. *If $\dim(V) < \infty$, then T is one-to-one if and only if $\text{rank}(T) = \dim(V)$. If $\dim(W) < \infty$, then T is onto if and only if $\text{rank}(T) = \dim(W)$.*

Corollary. *Let V and W be* **finite-dimensional**, *and assume $\dim(V) = \dim(W)$.*

The Following Are Equivalent

- *T is one-to-one.*
- *T is onto.*
- *T is invertible.*
- *T maps some basis of V into a basis of W.*
- *T maps every basis of V into a basis of W.*

Both assumptions in the corollary are important. Without them, the equivalences may be false even in the case $V = W$ (T is a linear operator on V), as the following example shows:
Example: Let $V = W = F^{\mathbb{N}} = \mathcal{F}(\mathbb{N}, F)$, the space of all F-sequences. Let LS and RS be the left-shift and right-shift operators on V:

$$LS(x_1, x_2, \ldots) = (x_2, x_3, \ldots), \quad \text{and} \quad RS(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots).$$

Clearly, LS is onto but not one-to-one, while RS is one-to-one but not onto.