Images and Inverse Images

Let A and B be non-empty sets, and let $f : A \to B$ be a function from A to B. Recall that A is called the **domain** of f and B is the **codomain** of f (no, it's *not* the range!).

For $X \subseteq A$ and $Y \subseteq B$, we define the **image** of X (under f) as

$$f(X) := \{f(x) : x \in X\} = \{y \in B\colon\ \exists x \in X \text{ such that } f(x) = y\},$$

and the **inverse image** of Y (under f) as

$$f^{-1}(Y) := \{x \in A\colon f(x) \in Y\}.$$

Remarks. Images (under f) are sometimes called "direct images." Note that f^{-1} here does not denote a function; the argument of f^{-1} is not an element of B but a subset of B!

An important example is the image of the whole domain A, called the **range** of f:

$$\mathit{Range}(f) := f(A) = \{f(x) : x \in A\} = \{y \in B\colon\ \exists x \in A \text{ such that } f(x) = y\}.$$

The following identities are easy consequences of the definition: let U and V be arbitrary subsets of B. Then,

(a) $f^{-1}(B) = A \quad \text{and} \quad f^{-1}(\emptyset) = \emptyset$,

(b) $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$,

(c) $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$,

(d) $f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$,

(e) $f^{-1}(\overline{U}) = \overline{f^{-1}(U)}$.

[Overline stands for "complement": $\overline{U} := B \setminus U$, and $\overline{f^{-1}(U)} := A \setminus f^{-1}(U)$.]

Hence, in general, for any Boolean expression φ, $f^{-1}(\varphi(U, V, \ldots)) = \varphi(f^{-1}(U), f^{-1}(V), \ldots)$.

Note: While (b) also holds with f replacing f^{-1} [in which case, of course, U and V would be subsets of A], (c), (d) and (e) **do not!** In short: inverse images - unlike direct images - are easy to work with; they satisfy most natural identities.

More differences between inverse and direct images

Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. Then, f is continuous if and only if f^{-1} maps open sets to open sets. (In general topological spaces, this is the definition of continuity.)

BUT: a continuous function doesn't necessarily map open sets to open sets.