# The Bernstein-Schröder Theorem

**Definitions.** Let $A$ and $B$ be arbitrary nonempty sets. We say that $A$ has fewer elements than $B$, and write $A \preceq B$, if there exists a one-to-one function (injection) $f : A \to B$ from $A$ to $B$. We say that $A$ is **equinumerous** to $B$, and write $A \sim B$, if there exists a bijection $f : A \to B$ from $A$ to $B$.

Since the composition of two injections is an injection and the composition of two bijections is a bijection, both relations $\preceq$ and $\sim$ are transitive; it is also obvious that both are reflexive.

Since the inverse $f^{-1} : B \to A$ of a bijection $f : A \to B$ always exists and is a bijection itself, the relation of being equinumerous is also symmetric; hence it's an equivalence relation. Thus, $\sim$ splits the universe of all sets (a very problematic notion, but we choose not to make this more precise now!) into equivalence classes, called **cardinalities**. So formally speaking: the cardinality of $A$ is defined to be the equivalence class containing the set $A$.

What about the relation $\preceq$ ? Common sense suggests that it is an order relation.

The following theorem of Felix Bernstein and Ernst Schröder only states that $\preceq$ is a partial order (reflexive, antisymmetric, and transitive). More precisely, it says that the relation $\preceq$ is antisymmetric on cardinalities: if $A$ and $B$ are two sets with $A \preceq B$ and $B \preceq A$, then $A$ and $B$ are equinumerous. That it is actually an order relation (not only a partial order: there are no incomparable sets) cannot be proved without the Axiom of Choice.

**Theorem.** *Let $A$ and $B$ be two sets. If there exist two injections $f : A \to B$ and $g : B \to A$, then there is a bijection $h : A \to B$. (If $A \preceq B$ and $B \preceq A$, then $A \sim B$.)*

*In fact, there are two partitions $A = A_1 \uplus A_2$ and $B = B_1 \uplus B_2$ such that $f$ is a bijection between $A_1$ and $B_1$ and $g$ is a bijection between $B_2$ and $A_2$, and hence the function*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_1 \\ g^{-1}(x) & \text{if } x \in A_2 \end{cases}$$

*is a bijection between $A$ and $B$.*

For comments on the proof, see the other side of this page.

The proof of the B-S theorem can be found in the LBB (page 58), but it can best be understood by solving Exercise 10:28 (page 59):

> Let $A$ be a square and $B$ a disc. Prove that there are disjoint decompositions $A = A_1 \uplus A_2$ and $B = B_1 \uplus B_2$ such that $A_1$ is similar to $B_1$ and $A_2$ is similar to $B_2$.

While this exercise has nothing to do with cardinalities, an application of the detailed form of the Bernstein-Schröder theorem would give a simple formal solution. It is illuminating however – and it could serve as a way to discover the proof for the B-S theorem – to solve this exercise *by hand*; to start with two dilations: $f$ mapping the square *inside* the disk and $g$ mapping the disk *inside* the square, and then to find a function $h$ as above that only uses the original dilation $f$ for some points of the square and the dilation $g^{-1}$ for the rest.

Starting with points near the corners of $A$, one cannot use $g^{-1}$ since these points are outside the range of $g$; we must use $f$ there. Similarly, some point near the boundary of $B$ must use $g$ for a bijection. Peeling $A$ and $B$ region by region like two onions, we discover the desired regions $A_1, A_2, B_1, B_2$.

The resulting picture (courtesy of Chris Ross, U-alumnus in Spring '02 and Spring '03) can be seen by clicking on the name of Schröder in math.rutgers.edu/∼useminar/math.htm
The white regions there form $A_1$ and $B_1$, clearly similar, and the black regions $A_2$ and $B_2$ are also similar.