The set of all permutations of \( \{1, 2, \ldots, n\} \) forms a group under composition. It is called the **symmetric group** of degree \( n \) and is denoted by \( S_n \). The order of \( S_n \) is \( n! \).

Some standard facts (we will always assume that \( n \geq 2 \)):

- **\( S_n \) is indeed a group** with respect to composition. (Check the four group properties.)

- **Every permutation is a product of a unique set of disjoint cycles.** (Easy.)

  (Introduce cycle notation!)

  Remark: (pairwise) disjoint cycles commute.

  Remark: the order of a permutation is the least common multiple of its cycle lengths.

- **Every permutation in \( S_n \) is the product of at most \( n - 1 \) transpositions.** (Easy.)

  Example: \((12345) = (15)(14)(13)(12)\)

  [Remember: compositions are read from right to left!]

- While a given permutation can be factored into transpositions in many different ways, the number of factors in them is **always even or always odd**.

  This is not trivial at all! One natural proof is to show that composing a permutation with one more transposition always changes the parity of the number of inversions. In general, parity adds up modulo 2 when we compose permutations. The book proves this by using the so-called Vandermonde polynomial.

  If a permutation factors into an even number of transpositions, we say that the permutation is **even** (has parity 0), otherwise it’s **odd** (has parity 1). The sign (or signature) of a permutation is defined as +1 if the permutation is even and -1 if it’s odd.

The set of all even permutations in \( S_n \) is called the **alternating group** of degree \( n \) and is denoted by \( A_n \).

Some standard facts (still assuming \( n \geq 2 \)):

- **\( A_n \) is a subgroup of \( S_n \).** (For the proof, use the simple fact that the inverse of a product of transpositions is the same product read backwards.)

- The order of \( A_n \) is half the order of \( S_n \), that is \( o(A_n) = n!/2 \).

  [Hence, \( A_n \) is a normal subgroup of \( S_n \).]

  (For a proof, show that the mapping \( f : S_n \rightarrow S_n : \pi \mapsto (12)\pi \) is a bijection on \( S_n \) that maps even permutations and odd permutations to each other.)

- If \( n \geq 3 \), then \( A_n \) is **generated by the 3-cycles**.

  (Just as \( S_n \) was generated by the 2-cycles.)

  Proof. It’s enough to show that the composition of two 2-cycles splits into 3-cycles. Indeed, \((cd)(ab) = (acb)(acd)\), and \((ac)(ab) = (abc)\), and \((ab)(ab) = e = (cba)(abc)\).