**Theorem.** Any finite subgroup of $SO(3)$ is isomorphic to a cyclic group $C_n$, or a dihedral group $D_n$, or one of groups $A_4, S_4, A_5$ (the symmetry groups of the five Platonic solids).

**Sketch of proof**

Let $G$ be a finite subgroup of $SO(3)$ of order $n > 1$. We will write $G' = G \setminus \{e\}$.

Let $S$ denote the unit sphere centered at the origin. For a rotation $g \in G'$, the two points where the axis of $g$ pierce $S$ are called poles. Clearly, a point $p \in \mathbb{R}^3$ is a pole if and only if $p \in S$ and $p$ is fixed by at least one rotation in $G'$: $(\exists g \in G')g(p) = p$.

We will write $F$ for the set of all poles (of rotations in $G'$). Clearly $F$ is finite (since $|F| \leq 2|G'|$). Given a point $p \in F$, the orbit $Gp = \{g(p) : g \in G\}$ of $p$ with respect to $G$ is the set of points where the rotations in $G$ move the point $p$.

**Lemma 1.** The actions of elements of $G$ move poles to poles:

$$(\forall p \in F)(\forall g \in G) g(p) \in F.$$  

Given $p, q \in F$, we write $p \sim q$ if $q \in Gp$. Using that $G$ is a group, it is easy to see that $\sim$ is an equivalence relation. The equivalence classes are called orbits. Lemma 1 implies that the orbits partition $F$, that is, $F$ is the disjoint union of orbits.

A pole $p \in F$ is a fixed point for several rotations in $G'$ (all about the same axis), these rotations together with the identity form a cyclic subgroup of $G$; we write $m(p)$ for the order of that group, and call it the degree of the pole $p$. Clearly, for any $p \in F$,

$$m(p) = |\{g \in G : g(p) = p\}| = \max\{o(g) : g \in G, g(p) = p\}.$$  

$$(\star)$$

**Lemma 2.** Within an orbit all poles have the same degree. Consequently, an orbit of degree $m$ has exactly $n/m$ points. (Why?)

Note that, by the first sentence of the last lemma, it makes sense to talk about $m(\varphi)$, the degree of orbit $\varphi$ (being equal to the degree of any pole in $\varphi$).

Note also that a pole $p$ and its polar opposite $p'$ on the sphere are fixed points for the exact same $m(p) - 1$ rotations in $G'$. Thus, since every $g \in G'$ fixes exactly two poles,

$$2|G'| = 2(n - 1) = \sum_{p \in F} (m(p) - 1)$$

Thus, writing $F/G$ for the set of orbits in $F$, we have

$$2(n - 1) = \sum_{\varphi \in F/G} \frac{n}{m(\varphi)} (m(\varphi) - 1)$$

that is,

$$2 \left(1 - \frac{1}{n}\right) = \sum_{\varphi \in F/G} \left(1 - \frac{1}{m(\varphi)}\right)$$

Since each $m(\varphi) \geq 2$, we get $2 \leq |F/G| \leq 3$, that is: a non-trivial finite subgroup of $SO(3)$ can only have two or three orbits in $F$.

The rest of the proof is Cataloguing.
Proof of Lemmas 1 and 2. Let \( p \in F \), let \( h \in G \) be arbitrary, and write \( q = h(p) \). Choose a \( g \in G' \) for which \( g(p) = p \) and \( m(p) = o(g) \). We need to show that \( q \in F \), that is, that there exists a \( g' \in G' \) such that \( g'(q) = q \) (since \( q \in S \) follows from the facts that \( p \in F \) and \( g \) is an isomorphism fixing the origin).

Indeed, let \( g' = hgh^{-1} \) (thus, \( g' \) is a conjugate of \( g \)). Since \( g \neq e \), so \( g' \neq e \), and hence \( g' \in G' \). Writing (as customary in group theory) composition as a product,

\[
g'(q) = (hgh^{-1})(q) = (hgh^{-1})(h(p)) = (hg)(h^{-1}h)(p) = (hg)(p) = h(g(p)) = h(p) = q,
\]
proving Lemma 1.

To prove Lemma 2, that is to show that \( m(q) = m(p) \), it is enough to prove the inequality \( m(q) = m(h(p)) \geq m(p) \), since the symmetry of the relation \( p \sim q \) then implies equality. Indeed, in virtue of (*), \( m(q) \geq o(g') \), since \( g' \) fixes \( q \). But \( o(g') = o(g) = m(p) \), since conjugate elements have the same order. (Why?)

For more information see
http://www-history.mcs.st-and.ac.uk/~john/geometry/Lectures/L11.html