Platonic Solids

Excerpt from http://en.wikipedia.org/wiki/Platonic_solid

Topological proof

A purely topological proof can be made using only combinatorial information about the solids. The key is Euler’s observation that \( v - e + f = 2 \), and the fact that \( pf = 2e = qv \). Combining these equations one obtains the equation

\[
\frac{2e}{q} - e + \frac{2e}{p} = 2.
\]

Simple algebraic manipulation then gives

\[
\frac{1}{q} + \frac{1}{p} = \frac{1}{2} + \frac{1}{e},
\]

Since \( e \) is strictly positive we must have

\[
\frac{1}{q} + \frac{1}{p} > \frac{1}{2}.
\]

Using the fact that \( p \) and \( q \) must both be at least 3, one can easily see that there are only five possibilities for \((p, q)\):

\[(3, 3), (4, 3), (3, 4), (5, 3), (3, 5).\]

Remark: These are, of course, the parameters for the five Platonic solids, but one needs geometric arguments to see that each one of these options determine a solid uniquely (up to dilation).