Finite Projective Planes

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1 Introduction

The following axioms are among Euclid's axioms for traditional plane geometry.

Axiom 0. *Given any two lines, there is* at most *one point incident to both of them.*

Axiom 1. *Given any two points, there is a unique line incident to both of them.*

Note: In this class, two lines/points means two *distinct* lines/points.

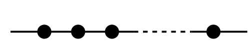
The phrase "at most" in Axiom 0 allows for parallel lines. That is, there may be two lines that don't intersect. If we forbid parallel lines, we get **projective planes**. That is, projective planes satisfy Axiom 1 and

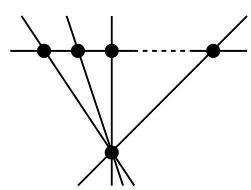
Axiom 2. *Given any two lines, there is* exactly *one point incident to both of them.*

In class, we discussed how "imaginary points" and an "imaginary line" can be added to the real plane to get the real projective plane. These notes address the finite planes that satisfy these axioms.

2 Examples

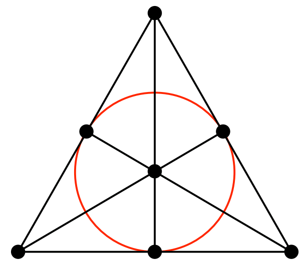
Consider the following examples.





Both of these are, in some sense, degenerate. In the first, we place any number of points on a single line. In the second, we simply add another point outside the first line and then draw connecting lines. There are also other degenerate examples, some of which have lines which don't even contain two points!

But there are nontrivial examples as well. Consider the Fano plane, pictured below. This finite projective plane consists of 7 lines and 7 points. (Note that one of these lines is drawn as a red circle. Each place where two lines cross only forms a point if it is indicated by a black dot.)



3 A New Axiom

To rule out the degenerate examples from the previous section, we add a new axiom.

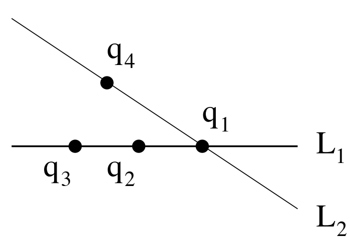
Axiom 3. *There exist four points so that no three of them are collinear.*

Axiom 3 clearly implies that there is more than one line. To see that it rules out the other degenerate examples, we prove the following useful lemma about planes satisfying Axioms 1, 2 and 3.

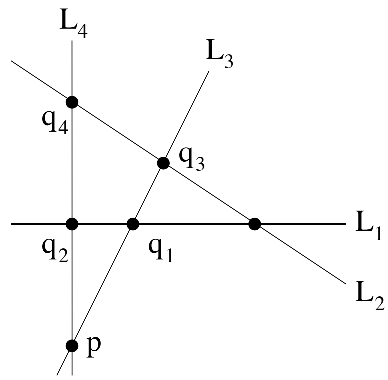
Lemma 1. *No two lines can cover all points.*

Indirect proof. Suppose that lines L_1 and L_2 cover all points. Let $Q = \{q_1, q_2, q_3, q_4\}$ be points so that no three are collinear. Q is guaranteed to exist by Axiom 3. By Axiom 2, there is a unique point incident to both L_1 and L_2.

Case 1: Suppose that this point is an element of Q, say q_1. Then each of the other three points of Q lie on either L_1 or L_2. By the pigeon-hole principle, at least two of these three points, say q_2 and q_3, lie on the same line. But then the "intersection point" of L_1 and L_2 are collinear with these two points. Contradiction.



Case 2: Suppose that the intersection point is not an element of Q. Then each element of Q lies on either L_1 or L_2 but not both. By assumption, no three of them lie on the same line. Hence two are incident to L_1 and two are incident to L_2. Without loss of generality, suppose that q_1 and q_2 are incident to L_1, and q_3 and q_4 are incident to L_2.



Axiom 1 guarantees the existence of a (unique) line L_3 incident to q_1 and q_3. Similarly, there must be a unique line L_4 incident to q_2 and q_4. But by Axiom 2, there is a unique point p which is incident to both L_3 and L_4. The point p cannot be an element of Q, as this would give a set of three collinear elements in Q. Furthermore, it cannot be on L_1 or L_2 - Axiom 2 says that the elements of Q form the *unique* intersection points between L_1 or L_2 with L_3 or L_4. Therefore p is not covered by L_1 and L_2. Contradiction.

□

4 Duality

The Fano plane is not the only (non-degenerate) finite projective plane. In fact, there are infinitely many (as we will see in later sections). All of them are very symmetric. One type of symmetry they exhibit is a *duality* between points and lines. The images we've seen above are just illustrations of plane geometries. Technically, each plane geometry is just a set of points $\mathcal{P}$, a set of lines $\mathcal{L}$, and a relation called *incidence* $\mathcal{I}$ which consists of point-line pairs. Axioms 1, 2 and 3 are restrictions on $\mathcal{I}$. We claim here that the geometry consisting of points $\mathcal{L}$, lines $\mathcal{P}$ and incidence $\mathcal{I}$ also satisfies Axioms 1, 2 and 3. To see this, it is enough to show that the "dual" of Axiom 3 is true. Axioms 1 and 2 are already dual to one another.

Exercise 1. *Show that the dual of Axiom 3 is true. That is, in any finite projective plane, there are four lines so that no three of them are incident to a single point.*

Then we can apply duality to get a second version of every theorem we prove. In particular,

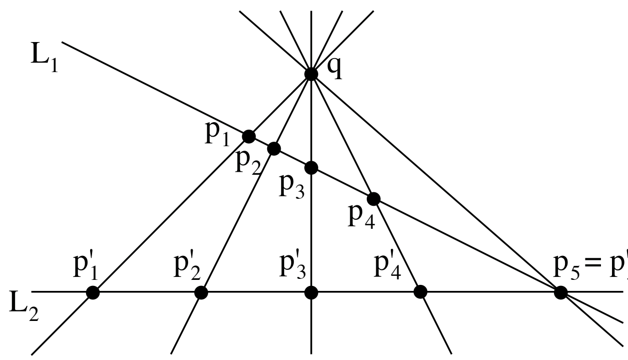
Corollary 1. *No two points cover all lines.*

Proof. We do not claim that every finite projective plane is the same as the one obtained by swapping points and lines. However, the dual of any particular finite projective plane is also a finite projective plane. So in the dual, no two lines cover all points. Then applying duality again to the dual gives back the original plane. So, in the original plane, no two points cover all lines. □

5 Classifying finite projective planes

Theorem 1. *Any two lines have the same number of points. We write $n+1$ for this number.*

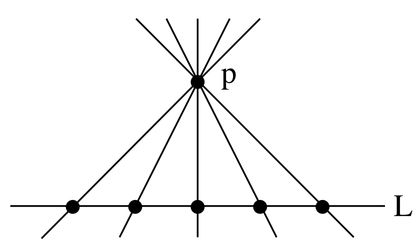
Proof. Let L_1 and L_2 be any two lines. By Lemma 1, there is a point q which is incident to neither L_1 nor L_2. Suppose L_1 is incident to $n+1$ points called $p_1, p_2, \ldots, p_{n+1}$. Then Axiom 1 gives $n+1$ lines incident to q and each p_i. Call them $L(q, p_i)$. Then for each $1 \leq i \leq n+1$, Axiom 2 gives a point p_i' which is incident to both L_2 and $L(q, p_i)$.



By Axiom 1, all of the lines $L(q, p_i)$ are distinct. (For any $i \neq j$, the unique line containing p_i and p_j is L_1, which does not contain q.) By Axiom 2, all of the points p_i' are all distinct. (For any $i \neq j$, the unique point incident to both $L(q, p_i)$ and $L(q, p_j)$, is q.) Furthermore, by Axioms 1 and 2, these are all of the points incident to L_2. (Another such point would form a new line with q, and this line would have to intersect L_1 somewhere.) Hence L_2 is incident to exactly $n+1$ points. $\square$

Theorem 2. *Every point is incident to exactly $n+1$ lines.*

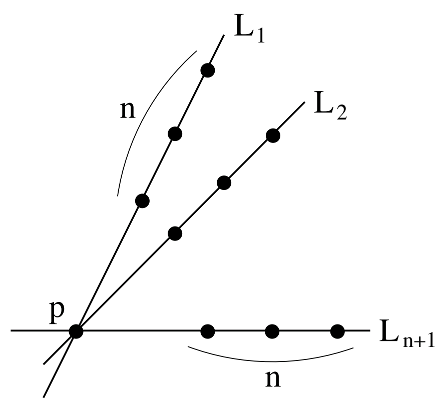
Proof. Let p be any point. By Lemma 1, there is a line L which is not incident to p. We claim that the points incident to L correspond to all the lines incident to p. The details of this proof are analogous to those of Theorem 1. We provide the illustration and leave the rest to the reader.



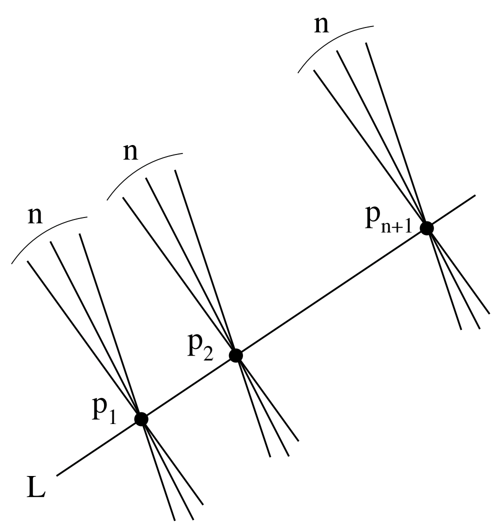
$\square$

Theorem 3. *There are exactly $n^2 + n + 1$ points. There are exactly $n^2 + n + 1$ lines.*

Proof. Let p be any point. There are $n+1$ lines incident to p and each contains an additional n points. This gives $n(n+1)+1$ points in total. We leave it to the reader to see that these points are distinct and that they're *all* of the points.



The second result can be obtained by a similar argument (see below), or by applying duality.



□

Therefore we can classify each finite projective plane according to the parameter n. One natural question is "for which n does such a plane exist?" Some more are "can we construct them?" and "are they unique?" We will not address the last question, but the first two can be (partially) answered using abstract algebra!

6 Finite fields

A detailed definition of fields will be presented in later talks. For now, think of a field $\mathbb{F}$ as a set of numbers that you can add, subtract, multiply and divide (by non-zero elements). One important field is given by modular arithmetic. Pick a prime p and add, subtract, and multiply the numbers $\{0, 1, \ldots, p-1\}$ normally. However, when your calculations result in a number outside $\{0, 1, \ldots, p-1\}$, add or subtract p as many times as needed. For clarity, we will write equality under this system as $\equiv_p$. For example, $5 \times 3 = 15$ but for $p = 7$

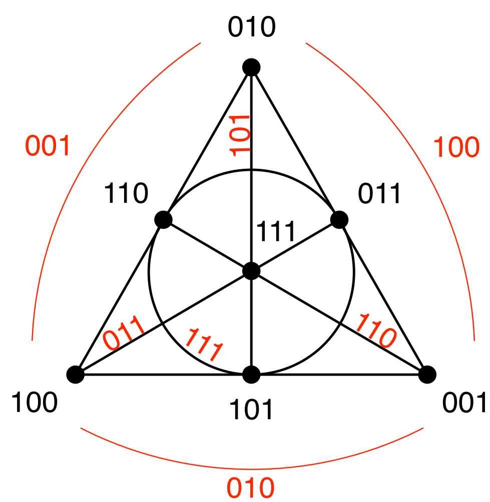
$$5 \times 3 \equiv_p 1 \quad \text{since } 15 - 7 - 7 = 1.$$

Hence 3 acts as the multiplicative inverse of 5. In other words, in this magical world of modulo 7 arithmetic, dividing by 5 always gives an integer - the same integer resulting from multiplying with 3.

We claim (but don't prove here) that, for any prime p, the numbers $\{1, 2, \ldots, p-1\}$ are invertible *modulo* p. In other words, we can divide by anything other than 0. Therefore $\{0, 1, \ldots, p-1\}$ is a field with with finitely many elements. This field is denoted $\mathbb{F}_p$. It is possible to construct fields with q elements if and only if q is a prime power.

7 The Fano plane and finite fields

Let's look at the Fano plane again, but now with labels. In the figure below, black labels correspond to points and red labels correspond to lines.



Each label consists of three 0's and 1's. In fact *all* triples of 0's and 1's are used except 000. These labels reflect an underlying rule for incidence: A point xyz is incident to the line abc if and only if

$$ax + by + cz \equiv_2 0.$$

A simple consequence of this rule is the fact that any two points on the same line add up (coordinate-wise, modulo 2) to the third point on that line. For example, the points 010, 111, and 101 all lie on the central vertical line. Remember that $1 + 1 \equiv_2 0$. Therefore the (coordinate-wise, modulo 2) sum of 010 and 111 is 101.

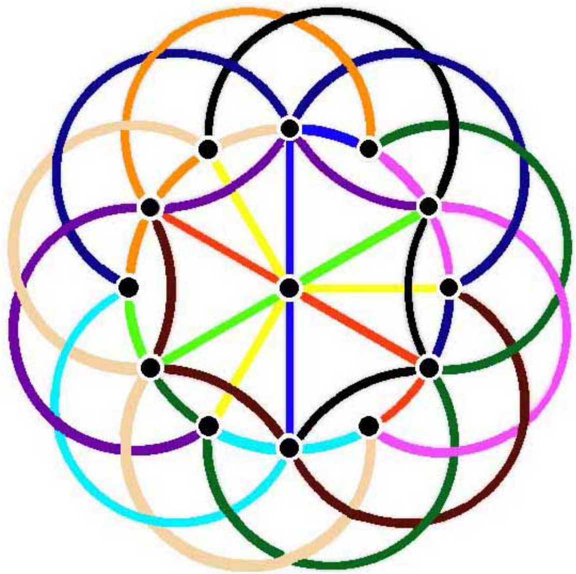
This observation is actually a recipe for building the Fano plane using $\mathbb{F}_2$. Step 1: Make a point and a line for each nonzero triple of integers in $\mathbb{F}_2$. Step 2: Say point xyz is incident to line abc iff $ax + by + cz \equiv_2 0$. The drawing above is just a representation of this rule.

We can do the same for $\mathbb{F}_p$ for any prime p with a small tweak. Underlying everything here is a deep result about linear algebra over finite fields. Points and lines correspond to subspaces of $\mathbb{F}_p^3$. A subspace of dimension 1 (a point here) consists of all scalar multiples of a single vector. In simpler terms, points aren't really vectors, they're *directions*. This is similar to the "points at infinity" we discussed in class for the real projective plane - we add one in each direction. A subspace of dimension 2 (a line here) consists of all linear combinations of two vectors. This is why labels of points on a line are closed under addition.

The bottom line: we need to ignore scalar multiples. We didn't need this idea for $\mathbb{F}_2$ because the only scalar multiples in this field are 0 and 1. In $\mathbb{F}_3$ we have a nontrivial scalar. Since $2 \times 2 \equiv_3 1$, the vectors 120 and 210 represent the same direction. In fact there are 13 distinct, nonzero directions:

001=002	010=020	100=200
011=022	101=202	110=220
012=021	102=201	120=210
112=221	121=212	211=122
	111=222	

In general, for $\mathbb{F}_p$, there are $(p^3-1)/(p-1)$ directions. This is because there are p^3-1 nonzero vectors and $p-1$ scalar multiples (times $1, 2, \ldots, p-1$) of each vector that "point in the same direction." Take a point p and a line L. Pick some vectors xyz and abc representing their respective directions. Then we define p to be incident to L exactly when $ax + by + cz \equiv_p 0$. The figure below shows the resulting plane for $\mathbb{F}_3$.



Each resulting plane satisfies Axioms 1, 2, and 3. This is a nontrivial fact but it can be proven using elementary facts about numbers. For example, it is easy to see that no three of the points 001, 010, 100, and 111 can lie on a single line. You can also show that each point is incident to $p+1$ lines, and each line is incident to $p+1$ points. Finally, the number of points (and lines) is $p^3 - 1/p - 1$. But simple polynomial division shows us that

$$\frac{p^3 - 1}{p - 1} = p^2 + p + 1.$$

So we have a concrete model for at least one finite projective plane for each prime n in Theorems 1, 2, and 3. You can extend this to prime powers n by using the (more involved) finite fields of prime power order. It is unknown whether other finite projective planes exist. Using number theory and computer simulations, researchers have shown that such planes do not exist for $n = 6$ and $n = 10$. Already the case $n = 12$ is open.